Topics in Probability: Problem Sheet 4

Janosch Ortmann

October 21, 2013

This is the second exercise sheet that will contribute to your final mark. Please hand in solutions before the lecture on **Wednesday October 30**. Unless otherwise specified $(\Omega, \mathcal{F}, \mathbb{P})$ is a generic probability space. Each question is worth 20 marks.

1. Show that $X_n \longrightarrow X$ a.s. if and only if for every $\epsilon > 0$

$$\mathbb{P}\left\{|X_n - X| \geq \epsilon \text{ i.o.}\right\} = 0. \tag{1}$$

2. Let $A_n \in \mathcal{F}$, $n \in \mathbb{N}$. Show that

   (a) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$

   (b) $\left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} A_n^c$

   (c)

$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) \geq \limsup_{n\to\infty} \mathbb{P}\left(A_n\right) \geq \liminf_{n\to\infty} \mathbb{P}\left(A_n\right) \geq \mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \tag{2}$$

3. Let X_n be random variables such that $0 \leq X_n \leq X_{n+1}$ for all $n \in \mathbb{N}$

   (a) Let further c_n be real numbers such that $0 \leq c_n \leq c_{n+1}$ for all $n \in \mathbb{N}$. Show that if there exists a sequence $(n_k \colon k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} \frac{X_{n_k}}{c_{n_k}} \longrightarrow 1 \tag{3}$$

   almost surely, as $n \longrightarrow \infty$, then in fact

$$\lim_{n\to\infty} \frac{X_n}{c_n} \longrightarrow 1. \tag{4}$$

   (b) Deduce that if additionally $\frac{n^{-\beta}}{a}\mathbb{E}X_n \longrightarrow 1$ for $a, \beta > 0$ and $\operatorname{var}(X_n) \leq Cn^{\gamma}$ for $B > 0$ and $\gamma < 2\beta$ then $n^{-\beta}X_n \longrightarrow a$ a.s.

4. Let X be a random variable.

   (a) Suppose that $G(x) = \int_{-\infty}^{x} g(y)\, dy$ for $g \colon \mathbb{R} \longrightarrow [0, \infty)$. Show that

$$\mathbb{E}\left[G(X)\right] = \int_{-\infty}^{\infty} g(y)\mathbb{P}(X > y)\, dy. \tag{5}$$

(b) Deduce a bound on $\mathbb{E}e^{aX}$ in terms of $\mathbb{P}(X > x)$

5. Let $X_n \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $n \in \mathbb{N}$ (not necessarily independent). Suppose $\mathbb{E}X_n = 0$ and that there exists a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ such that $a_n \longrightarrow 0$ as $n \longrightarrow 0$ and $\mathbb{E}X_nX_m \leq a_{m-n}$ for $n < m$. (The lack of absolute values on the left-hand side is not a typo.) Show that

$$\frac{1}{n}\sum_{k=1}^{n} X_k \longrightarrow 0 \qquad \text{in probability as } n \longrightarrow \infty. \tag{6}$$

6. Let $\mathcal{F}\colon [0,1] \longrightarrow \mathbb{R}$ be measurable and such that $\int_0^1 |f(y)|\, dy < \infty$ (for example, f could be continuous or, more generally, bounded).

   (a) Let U_n, $n \in \mathbb{N}$ be independent uniform random variables on $(0,1)$ (i.e. $\mathbb{P}(U_n < x) = x$ for $x \in (0,1)$). Show that

$$\frac{1}{n}\sum_{k=1}^{n} f(U_k) \longrightarrow \int_0^1 f(y)\, dy \tag{7}$$

   (b) If also $\int_0^1 |f(y)|^2\, dy < \infty$ (again, any bounded function satisfies this – why?) use Chebychev's inequality to get an estimate for

$$\mathbb{P}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} f(U_k) - \int_0^1 f(y)\, dy\right| > an^{-1/2}\right\} \tag{8}$$

   This gives a method for numerically evaluating integrals, called Monte-Carlo integration

7. Let $(X_n\colon n \in \mathbb{N})$ be independent random variables. Show that there exists $K > 0$ such that $X_n \leq K$ a.s. for all $n \in \mathbb{N}$ if and only if there exists $C > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > C) < \infty. \tag{9}$$

8. Let $X_n\colon n \in \mathbb{N}$ be independent and such that $\mathbb{P}(X_n = 0) = p_n$ and $\mathbb{P}(X_n = 1) = 1 - p_n$.

   (a) Show that $X_n \longrightarrow 0$ in probability if and only if $p_n \longrightarrow 0$

   (b) Show that $X_n \longrightarrow 0$ a.s. if and only if $\sum_n p_n < \infty$.