Topics in Probability: Problem Sheet 7

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Here are some extra practice questions before the final exam. Unless otherwise specified $(\Omega, \mathcal{F}, \mathbb{P})$ is a generic probability space and $\mathcal{G}$ is a sub-σ algebra of $\mathcal{F}$.

1. Show that if $X_n \geq 0$ for all $n \in \mathbb{N}$ and $N_t = \sup\{n \in \mathbb{N} \colon \sum_{k=1}^n \leq t\}$ then $N_t + 1$ is a stopping time.

2. Prove the Parseval relation: for probability measures μ, ν on $\mathbb{R}^d$ with characteristic functions ϕ_μ, ϕ_ν,
$$\int \phi_\nu(t)\mu(dt) = \int \phi_\mu(t)\nu(dt). \tag{1}$$

3. In this exercise we prove that if S_* is a recurrent random walk in $\mathbb{R}^d$ then for $\delta > 0$
$$\sup_{r<1} \int_{(-\delta)} \mathrm{Re}\left(\frac{1}{1 - r\phi(y)}\right) dy. \tag{2}$$
Write $S_n = \sum_{k=1}^n$ and denote by ϕ the characteristic function of X_1.
   (a) Show that $1 - \cos(x) \geq \frac{x^2}{4}$ for $|x| \leq \frac{\pi}{3}$.
   (b) Using Parseval's relation and the measure with density $\frac{\delta(1-\cos(x/\delta))}{\pi x^2}$ show that
$$\mathbb{P}\left(\|S_n\|_1 < \frac{1}{\delta}\right) \geq \int \prod_{j=1}^d \frac{\delta\left(1 - \cos\left(t_j/\delta\right)\right)}{\pi t_j^2} \phi^n(t)\, dt \tag{3}$$
   (c) Deduce that for $r < 1$,
$$\sum_{n=1}^\infty r^n \mathbb{P}\left(\|S_n\|_1 < \frac{1}{\delta}\right) \geq \int \prod_{j=1}^d \frac{\delta\left(1 - \cos\left(t_j/\delta\right)\right)}{\pi t_j^2} \frac{1}{1 - r\phi(t)}\, dt \tag{4}$$
   (d) Deduce that (2) must hold.

4. Recall that for $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$ we defined $\mathbb{P}(A|B) = \frac{\mathbb{P}\{A\cap B\}}{\mathbb{P}\{B\}}$. Let $A \in \mathcal{G}$ and $B \in \mathcal{F}$ and show that
$$\mathbb{P}(A|B) = \frac{\mathbb{E}\left(\mathbb{P}(B|\mathcal{G})\, 1_G\right)}{\mathbb{E}\left(\mathbb{P}(B|\mathcal{G})\right)}.$$

5. Prove that Chebychev's, Jensen's and Hölder's inequalities hold for conditional expectation

6. For $X \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ define $\mathrm{var}(X|\mathcal{G}) = \mathbb{E}\left(X^2|\mathcal{F}\right) - \mathbb{E}\left(X|\mathcal{F}\right)^2$. Prove that

$$\mathrm{var}(X) = \mathbb{E}\left[\mathrm{var}\left(X|\mathcal{G}\right)\right] + \mathrm{var}\left[\mathbb{E}\left(X|\mathcal{G}\right)\right]. \tag{5}$$

7. Let $N \in \mathrm{L}^2$ be a random variables taking values in the natural numbers $\mathbb{N}$ such that and let $X_1, X_2, \ldots$ be i.i.d random variables, also independent of N, with $\mathbb{E}X_j = \mu$ and $\mathrm{var}X_j = \sigma^2$. Show that

$$\mathrm{var}\left(\sum_{k=1}^{N} X_k\right) = \sigma^2 \mathbb{E}N + \mu^2 \mathrm{var}\left(N\right).$$