Suggested Exercises for the Final

APM 384: PDEs

Autumn 2013

1 Questions

Below are some suggested problems you may want to attempt in preparation for the final exam, corresponding to the material we discussed after the midterm. There are many excellent problems in Haberman for which you can find solutions in the back: those are marked with an asterisk.

1. **Sturm–Liouville problems:** Exercises 5.3.1, 5.3.2 5.3.4, 5.4.3, 5.8.7,5.8.10 in Haberman

2. **Green's function and the Fredholm alternative**

   (a) Recall the definition of δ_ϵ from Question 2 of PS 5. Define $\delta_\epsilon^{(k)} \colon \mathbb{R}^k \longrightarrow \mathbb{R}$ by

   $$\delta_\epsilon^{(k)} (\vec{x}) = \prod_{j=1}^{k} \delta_\epsilon (x_j) .$$

   Show that $\delta^{(x)} (\vec{x}) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ for all $\vec{x} \in \mathbb{R}^d \setminus \{0\}$ and that $\int_{\mathbb{R}^k} f (\vec{x}) \, \delta^{(k)} (\vec{x} - \vec{y}) \, d\vec{x} = f (\vec{y})$ for all $\vec{y} \in \mathbb{R}^k$.

   (b) Let $L = \frac{\partial}{\partial x} \left(p \frac{\partial}{\partial x} \right) + q(x)$. Show that if v satisfies $L(v) = \delta(x - x_0)$ for some $x_0 \in [a, b]$ then v still satisfies

   $$\int_{-a}^{a} [uLv - vLu] \, dx = p \big[uv' - u'v \big]_a^b \tag{1}$$

   for any differentiable $u \colon [a, b] \longrightarrow \mathbb{R}$

   (c) Exercises 9.3.5, 9.4.6, 9.4.10 in Haberman

3. **The Fourier transform**

   (a) Solve the non-homogeneous heat equation

   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(t, x) \tag{2}$$

   subject to the initial condition $u(0, x) = 0$

(b) Solve Laplace's equation in a semi-infinite strip:

$$\Delta u(x,y) = 0 \ \forall (x,y) \in (0,L) \times (0,\infty) \tag{3}$$
$$u(0,y) = g_1(y), \quad u(0,L) = g_2(y), \quad u(x,0) = f(x) \tag{4}$$

where $g_1(y)$ and $g_2(y)$ tend to zero as $y \longrightarrow \infty$.

(c) Show that if $u(t,x)$ satisfies the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on the whole of $\mathbb{R}$ and the initial condition $u(0,x) = f(x)$ (with $\int_{-\infty}^{\infty} |f| < \infty$) then

$$u(t,x) = \int_{-\infty}^{\infty} f(y) \frac{1}{4\pi t} e^{-(x-y)^2/4t} dy \tag{5}$$

and deduce that $\lim_{x\to\infty} u(t,x) = 0$ for all $t \in \mathbb{R}$.

(d) Exercises 10.3.6,10.3.7,10.3.9(a),10.3.10, 10.3.11 in Haberman

4. **Numerical methods**:

(a) Write down a finite difference scheme for the two-dimensional heat equation. Use Fourier–von Neumann stability analysis to show that we have guaranteed convergence if and only if $s \in \left(0, \frac{1}{4}\right)$.

(b) Show that homogeneity is not needed by setting up a finite difference scheme for the highly non-homogeneous IBVP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(t,x) \tag{6}$$
$$u(t,0) = A(t), \quad u(t,L) = B(t), \quad u(0,x) = f(x) \tag{7}$$

2 Answers

1. Answers can be found in the back of Haberman.

2. (a) The answer is in the question.

   (b) Compute the left-hand side by splitting the integral into three regions: $[a, x_0 - \epsilon]$, $[x_0 - \epsilon, x_0 + \epsilon]$ and $[x_0 + \epsilon, b]$. Then use the properties of the Dirac delta and integration techniques.

   (c) See Haberman

3. (a) The solution to (2) subject to zero initial condition is given by

$$u(t,x) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\, e^{-\frac{|x-y|^2}{4(t-s)}}\, q(s,y)\, dy\, ds \tag{8}$$

   (b) See Section 10.6.2 in Haberman.

   (c) The answer is in the question. However, notice the similarity between this answer and that of part a)!

   (d) See back of Haberman.

4. See Sections 6.3.7 and 6.4 in Haberman.