Suggested Exercises for the Midterm

APM 384: PDEs

Autumn 2013

1 Questions

Below are some suggested problems you may want to attempt in preparation for the midterm. There are many excellent problems in Haberman for which you can find solutions in the back: those are marked with an asterisk.

1. **Definitions and basic classification of differential equations:** Go through Chapter 1 of Haberman. Whenever you see a PDE, make sure you can state its order and whether it is linear and/or homogeneous.

2. **The method of characteristics:** In class, the exercises and handouts we saw many examples of how to apply the method of characteristics to fist-order linear PDEs in two variables. Try to extend this method to three dimensions: i.e. try to solve the PDE

$$u_x + yu_y + zu_z = 0 \tag{1}$$

   subject to the boundary condition $u(0, x, y) = \sin(x)\cos(y)$.

3. **Separation of Variables:**

   (a) We only touched very briefly on solving Laplace's equation in a rectangle. It is a great exercise to solve this fully using the method of separation of variables and then Fourier series.

   (b) Do exercises 2.4.1-2.4.3 in Haberman

   (c) Whenever we solved an eigenvalue problem we chose a particular sign for the eigenvalues. Check that we did not lose any generality by showing that there is no eigenvalue for the other sign (you can do this with the various cases of the heat equation as well as Laplace's equation we discussed.)

   (d) Do the starred exercises in section 2.5 of Haberman:

4. **Complex differentiable and harmonic functions:**

   (a) Show that the following functions are harmonic:

$$u_1(x, y) = x - y$$
$$u_2(x, y) = e^x \sin(y).$$

(b) Check that the mean value theorem holds by computing

$$\int_0^{2\pi} u_1(x+\cos(\theta), y+\sin(\theta))\, d\theta$$

(c) Find minima and maxima for u_1 on the unit disk $\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$. Check the maximum principle by showing that they lie on the boundary.

5. (a) Find the Fourier series for the following functions $f_j \colon [-1,1] \longrightarrow \mathbb{R}$

   i. $f_1(x) = x$

   ii. $f_2(x) = \cos(x)$

   iii. $f_3(x) = e^x$

   (b) Find the Fourier sine series of $f \colon [0,L] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$.

   (c) On the problem sheet you found Fourier series for certain functions f. Check that the method of complex Fourier series works by finding coefficients c_n such that $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi x}{L}}$ using the method of today's lecture.

2 Answers

1. This should be straightforward.

2. We are looking for curves $\gamma(t)$ such that u is constant along $\gamma(t)$, i.e. $\frac{d}{dt}u(\gamma(t)) = 0$. Writing $\gamma(t) = (t, y(t), z(t))$ and using the chain rule (check the details using the equation!) this yields

$$\frac{d}{dt}y(t) = y(t) \quad \text{and} \quad \frac{d}{dt}z(t) = z(t) \tag{2}$$

i.e. any curve of the form $\gamma(t) = (t, y_0e^t, z_0e^t)$ has the property that u is constant along γ. It follows that for any $t > 0$ and any $y_0, z_0 \in \mathbb{R}$,

$$u(\gamma(t)) = u(\gamma(0)) \tag{3}$$

i.e.

$$u\left(t, y_0e^t, z_0e^t\right) = u(0, y_0, z_0). \tag{4}$$

Let now $(x,y,z) \in \mathbb{R}^3$. In order to find $u(x,y,z)$ we need to find values of $y_0, z_0 \in \mathbb{R}$ and $t \geq 0$ such that $(x,y,z) = \gamma(t) = (t, y_0e^t, z_0e^t)$. Clearly $t = x$ and then $y_0e^x = y$ and $z_0e^x = z$. Hence $y_0 = ye^{-x}$ and $z_0 = ze^{-x}$. By (4) it now follows that

$$u(x,y,z) = u(0, y_0, z_0) = u\left(0, ye^{-x}, ze^{-z}\right). \tag{5}$$

Thus, for any continuously differentiable function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$, the function u defined by $u(x, y, z) = f(ye^{-x}, ze^{-x})$ satisfies our PDE. The boundary condition tells us that $u(0, x, y) = \sin(x)\cos(y)$, i.e.

$$\sin(x)\cos(y) = f(x, y).$$

Hence the solution to the PDE with boundary condition is

$$u(x, y, z) = \sin\left(ye^{-x}\right)\cos\left(ze^{-x}\right) \tag{6}$$

3. (a) You can check your answers by looking at section 2.5.1 in Haberman.
   (b) These are solved in Haberman.
   (c) The answer is given in the question.
   (d) Solved in Haberman.

4. (a) Differentiating u_1 twice with respect to either variable gives zero, so the function is harmonic. For u_2 you have two choices: you can either compute directly or use the connection to complex analysis: the real and imaginary parts of a complex differentiable function are harmonic. Since u_2 is the imaginary part of $f(z) = e^z$ it is harmonic.
   (b) This is a straightforward computation:

$$\begin{aligned}\int_0^{2\pi} u_1(x + \cos(\theta), y + \sin(\theta))\, d\theta &= \int_0^{2\pi} [(x + \cos(\theta)) - (y + \sin(\theta))]\, d\theta \\ &= 2\pi(x - y) + \int_0^{2\pi} \cos(\theta)\, d\theta - \int_0^{2\pi} \sin(\theta)\, d\theta \\ &= 2\pi(x - y) = 2\pi u(x, y)\end{aligned}$$

   which by the mean value theorem is the right answer.
   (c) u_1 is maximised when $x = 1$ and $y = 0$ and minimised when $x = 0$ and $y = -1$. Thus, the maxima lie on the boundary.

5. (a) You can check if your answer is $A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{nx}{\pi}) + \sum_{n=1}^{\infty} B_n \sin(\frac{nx}{\pi})$ where the coefficients are given below:
      i. Clearly there are no cosine terms ($A_n = 0$ for all n)because x is an odd function. Moreover $B_n = \int_{-1}^{1} x \sin(\frac{nx}{\pi}) dx = (-1)^{n+1}\frac{2}{n\pi}$ (integration by parts!).
      ii. $\cos(x)$ is its own Fourier series: $A_1 = 1$ and $A_m = B_n = 0$ for all other n and all $m \in \mathbb{N}$.
      iii. We have $B_n = \int_{-1}^{1} e^x \sin(\frac{nx}{\pi}) dx$. Using integration by parts twice we get an expression involving B_n and constants that solves to $B_n = (-1)^{n+1}\frac{n\pi(e^2-1)}{e()1+n^2\pi^2}$. Clearly $A_0 = \frac{1}{2}\int_{-1}^{1} e^x = \sinh(1)$. Finally

$$A_n = \int_{-1}^{1} e^x \cos(\frac{nx}{\pi}) dx = (-1)^n \frac{e^2 - 1}{e(1 + n^2\pi^2)}.$$

(b) Because we are looking for the sine series we need to chose the odd extension of f, i.e. $\widehat{f}_o \colon [-L, L] \longrightarrow \mathbb{R}$ defined by

$$\widehat{f}_o(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0. \end{cases}$$

Since $f(0) = 0$ this gives a continuously differentiable function, however $\widehat{f}_o(-L) = -L^2 \neq L^2 = \widehat{f}_o(x)(L)$, so we do not expect the Fourier series to converge to f at the end points. In fact, $\widehat{f}_o(x)(L) + \widehat{f}_o(x)(-L) = 0$, so this is where the Fourier sine series will converge at $-L$ and L. Computing the coefficients, we get $A_0 = A_n = 0$ (because of odd-ness) and

$$B_n = \frac{2}{L} \int_0^L x^2 \sin\left(\frac{nLx}{\pi}\right) dx = (-1)^n \frac{4L^2}{n^2\pi^2}.$$

(c) Your answer is correct when you get the same result as in the problem sheet!