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Dynamic topological S5

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ABSTRACT

The topological semantics for modal logic interprets a standard modal propositional language in topological spaces rather than Kripke frames: the most general logic of topological spaces becomes S4. But other modal logics can be given a topological semantics by restricting attention to subclasses of topological spaces: in particular, S5 is logic of the class of *almost discrete* topological spaces, and also of *trivial* topological spaces. Dynamic Topological Logic (DTL) interprets a modal language enriched with two unary temporal connectives, *next* and *henceforth*. DTL interprets the extended language in *dynamic topological systems*: a DTS is a topological space together with a continuous function used to interpret the temporal connectives. In this paper, we axiomatize four conservative extensions of S5, and show them to be the logic of continuous functions on almost discrete spaces, of homeomorphisms on almost discrete spaces, of continuous functions on trivial spaces and of homeomorphisms on trivial spaces.

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1. Background

1.1. S5 *in the topological semantics*

Let $\mathcal{L}^{\Box}$ be a modal propositional language with a set $PV = \{p_1, \ldots, p_n, \ldots\}$ of propositional variables, parentheses, Boolean connectives & and ¬, and a unary modal connective □. We assume that the Boolean connectives ∨, ⊃ and ≡, and the unary modal connective ◇ are defined in the usual way. The McKinsey–Tarski topological semantics[1] interprets $\mathcal{L}^{\Box}$ in topological spaces, interpreting □ as topological interior. The resulting modal logic, S4, can thus be seen as the modal logic of topological spaces.

Formally, a *topological model* is an ordered pair $M = \langle X, V \rangle$, where X is a topological space and $V : PV \to \mathcal{P}(X)$. The function V is extended to all formulas of $\mathcal{L}^{\Box}$ as follows, where $Int(Y)$ is topological interior of Y, for any $Y \subseteq X$:

$$V(\neg A) = X - V(A)$$
$$V(A \,\&\, B) = V(A) \cap V(B)$$
$$V(\Box A) = Int(V(A)).$$

We define four validity relations, where $M = \langle X, A \rangle$ and where $\mathcal{T}$ is a class of topological spaces:

$$M \vDash A \quad \text{iff} \quad V(A) = X$$
$$X \vDash A \quad \text{iff} \quad \langle X, V \rangle \vDash A, \text{ for every } V : PV \to X$$
$$\mathcal{T} \vDash A \quad \text{iff} \quad X \vDash A, \text{ for every } X \in \mathcal{T}$$
$$\vDash A \quad \text{iff} \quad X \vDash A, \text{ for every topological space } X.$$

The main theorem of [9] is as follows: $\vDash A$ iff $A \in$ S4.

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[1] See [8,9]. This semantics predates the Kripke semantics of [6,7].

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Any class $\mathcal{T}$ of topological spaces determines a modal logic, namely $\{A : A$ is a formula of $\mathcal{L}^{\Box}$ and $\mathcal{T} \vDash A\}$. In particular, the modal logic S5 is determined in this way by the class $\mathcal{AD}$ of almost discrete spaces,[2] and the class $\mathcal{TR}$ of trivial spaces.[3] That is, for any formula A in the language $\mathcal{L}^{\Box}$, we have the following[4]:

$$A \in S5 \quad \text{iff} \quad \mathcal{AD} \vDash A$$
$$A \in S5 \quad \text{iff} \quad \mathcal{TR} \vDash A.$$

$\mathcal{AD}$ and $\mathcal{TR}$ are not the only classes of topological spaces that determine S5, but each of $\mathcal{AD}$ and $\mathcal{TR}$ is noteworthy:

$\mathcal{AD}$. The class $\mathcal{AD}$ is the *largest* class of topological spaces that determines S5: thus $\mathcal{AD}$ not only determines S5 but is, in a clear sense, determined by S5.

$\mathcal{TR}$. Suppose that for a topological model $M = \langle X, V\rangle$, we think of the points in X as possible worlds and of a formula A as *true in* the world x iff $x \in V(A)$. Then the trivial topological spaces are distinguished by the simplicity of the interpretation of the truth of $\Box A$ and $\Diamond A$ in a world x: $\Box A$ is true in x iff A is true in every world, and $\Diamond A$ is true in x iff A is true in some world.

Thus, below, the classes $\mathcal{AD}$ and $\mathcal{TR}$ will figure prominently.

1.2. Dynamic topological logic

The Dynamic Topological Logic (DTL) programme of [5] extends the language $\mathcal{L}^{\Box}$ to a modal-temporal language $\mathcal{L}$ with two additional connectives: the unary temporal connectives $\circ$ (next) and $*$ (henceforth).[5] This language is interpreted in *dynamic topological systems* rather than topological spaces: a *dynamic topological system* (DTS) is an ordered pair $\langle X, f\rangle$, where X is a topological space and f is a continuous function on X. We interpret the temporal connectives of the modal-temporal language $\mathcal{L}$ by means of the function f: $\circ A$ will be true at a world x iff A is true at fx; and $*A$ will be true at x iff A is true at each of $x, fx, ffx, fffx$, and so on. More precisely, a *dynamic topological model* is an ordered triple $M = \langle X, f, V\rangle$, where $\langle X, f\rangle$ is a topological space and $V : PV \to \mathcal{P}(X)$. The function V is extended to all formulas of $\mathcal{L}$ as follows:

$$\begin{aligned} V(\neg A) &= X - V(A) \\ V(A \,\&\, B) &= V(A) \cap V(B) \\ V(\Box A) &= Int(V(A)) \\ V(\circ A) &= f^{-1}(V(A)) \\ V(*A) &= \bigcap_{i \in \mathbb{N}} f^{-i} Int(V(A)). \end{aligned}$$

Here, for any set $Y \subseteq X$, the set $f^{-1}(Y)$ is the inverse image of Y, i.e. $f^{-1}(Y) = \{x \in X : f(x) \in Y\}$; we also define $f^0(Y) = Y$, and $f^{-(i+1)}(Y) = f^{-1}(f^{-i}(Y))$.

We define six validity relations, where $M = \langle X, f, V\rangle$, where $\mathcal{T}$ is a class of topological spaces, and where $\mathcal{F}$ is a class of continuous functions:

$M \vDash A$	iff	$V(A) = X$
$\langle X, f\rangle \vDash A$	iff	$\langle X, f, V\rangle \vDash A$, for every $V : PV \to X$
$X \vDash A$	iff	$\langle X, f\rangle \vDash A$, for every continuous f on X
$\mathcal{T} \vDash A$	iff	$X \vDash A$, for every $X \in \mathcal{T}$
$\mathcal{T}, \mathcal{F} \vDash A$	iff	$\langle X, f\rangle \vDash A$, for every $X \in \mathcal{T}$ and every continuous function f on X such that $f \in \mathcal{F}$
$\vDash A$	iff	$X \vDash A$, for every topological space X.

Our main project in the current paper is to axiomatize the following four logics, where $\mathcal{H}$ is the class of homeomorphisms[6]:

1. The logic of continuous functions on almost discrete spaces: $\{A : \mathcal{AD} \vDash A\}$.
2. The logic of homeomorphisms on almost discrete spaces: $\{A : \mathcal{AD}, \mathcal{H} \vDash A\}$.
3. The logic of continuous functions on trivial spaces[7]: $\{A : \mathcal{TR} \vDash A\}$.
4. The logic of homeomorphisms on trivial spaces: $\{A : \mathcal{TR}, \mathcal{H} \vDash A\}$.

Given our remarks in Section 1.1, each of these logics is a conservative extension of the logic S5 formulated in the language $\mathcal{L}^{\Box}$.

2 A topological space is *almost discrete* iff every open set is closed. An alternative definition: a topological space X is *almost discrete* iff there is a family $\mathcal{O}$ of pairwise disjoint nonempty open sets such that $X = \bigcup \mathcal{O}$. Note that this family forms a basis for the topology.

3 A topological space is *trivial* iff there are exactly two open sets: the empty set and the whole space.

4 The claims which follow are immediate consequences of the work of [6].

5 See [5] for some motivation of the DTL programme and for references. A similar programme was independently initiated by [1] and [2].

6 A function on a topological space is a *homeomorphism* iff it is a continuous bijection with a continuous inverse.

7 Of course, every function on a trivial space is continuous.

2. Four axiom systems

2.1. The systems

Suppose that we formulate a purely temporal logic in the purely temporal language $\mathcal{L}^{\circ *}$, i.e. the language $\mathcal{L}$ without the modal connective $\Box$. The function-based interpretation of $\circ$ and $*$ gives us the linear time logic LTL, determined by the following axioms and rules[8]:

Axioms:	Classical tautologies	
	S4 axioms for $*$:	$*(A \supset B) \supset (*A \supset *B)$
		$*A \supset A$
		$*A \supset **A$
	$\circ$ commutes with $\neg, \vee, *$:	$\circ\neg A \equiv \neg\circ A$
		$\circ(A \vee B) \equiv (\circ A \vee \circ B)$
		$\circ *A \equiv *\circ A$
	$*$ implies $\circ$:	$*A \supset \circ A$
	The induction axiom:	$A \,\&\, *(A \supset \circ A) \supset *A$
Rules:	Modus Ponens:	$(A \supset B), A/B$
	Necessitation for $*$:	$A/*A$.

We define S5C as the logic in the modal-temporal language $\mathcal{L}$ given by the following axioms and rules:

Axioms:	Classical tautologies	
	S5 axioms for $\Box$:	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
		$\Box A \supset A$
		$\Box A \supset \Box\Box A$
		$\Diamond A \supset \Box\Diamond A$
	LTL axioms for $\circ$ and $*$:	$*(A \supset B) \supset (*A \supset *B)$
		$*A \supset A$
		$*A \supset **A$
		$\circ\neg A \equiv \neg\circ A$
		$\circ(A \vee B) \equiv (\circ A \vee \circ B)$
		$\circ *A \equiv *\circ A$
		$*A \supset \circ A$
		$A \,\&\, *(A \supset \circ A) \supset *A$
	The continuity axiom:	$\circ\Box A \supset \Box\circ A$
Rules:	Modus Ponens:	$(A \supset B), A/B$
	Necessitation for $*$:	$A/*A$
	Necessitation for $\Box$:	$A/\Box A$.

If we take the three rules as given, we can think of S5C as follows:

$$\text{S5C} = \text{S5} + \text{LTL} + (\circ\Box A \supset \Box\circ A).$$

S5C is the logic of continuous functions on almost discrete spaces (see Section 3.5). The logic of continuous functions on trivial spaces, can be axiomatized in a similar way (see Section 3.2):

$$\text{S5Ct} = \text{S5} + \text{LTL} + (\Box A \supset \Box\circ A).$$

The logic of homeomorphisms on trivial spaces can be axiomatized by converting the distinctive conditional axiom of S5Ct into a biconditional (see Section 3.3):

$$\text{S5Ht} = \text{S5} + \text{LTL} + (\Box A \equiv \Box\circ A).$$

In order to axiomatize the logic of homeomorphisms on almost discrete spaces, we add an additional rule, the rule of *next removal*:

Next removal: $\circ A/A$.

And we define S5H as follows (see Section 3.6):

$$\text{S5H} = \text{S5} + \text{LTL} + (\circ\Box A \equiv \Box\circ A) + \circ A/A.$$

Note that each of S5C, S5Ct, S5H and S5Ht is a conservative extension of the logic S5 formulated in the language $\mathcal{L}^{\Box}$. To see this, given any formula A of $\mathcal{L}$, let A' be the result of deleting all occurrences of $\circ$ and $*$. Then for any formula A of $\mathcal{L}$, if A is a theorem of S5C [S5Ct, S5H, S5Ht], then A' is a theorem of S5. In particular, if A has no occurrences of $\circ$ or $*$, then if A is a theorem of S5C [S5Ct, S5H, S5Ht], then A itself is a theorem of S5. By a similar argument, each of S5C, S5Ct, S5H and S5Ht is a conservative extension of the logic LTL formulated in the purely temporal language $\mathcal{L}^{\circ *}$.

[8] See [10] for an introduction to and history of LTL.

Our main theorem is the following soundness and completeness theorem:

Theorem 2.1.1. *For every formula A,*

(1)	$A \in$ S5C	*iff*	$\mathcal{AD} \vDash A$
(2)	$A \in$ S5H	*iff*	$\mathcal{AD}, \mathcal{H} \vDash A$
(3)	$A \in$ S5Ct	*iff*	$\mathcal{TR} \vDash A$
(4)	$A \in$ S5Ht	*iff*	$\mathcal{TR}, \mathcal{H} \vDash A$.

The ($\Rightarrow$) directions of the biconditionals in Theorem 2.1.1 correspond to soundness, and are left to the reader. In Section 3 we prove the ($\Leftarrow$) directions of the biconditionals, i.e. completeness, for S5Ct, S5Ht, S5C and S5H, in that order. We will also prove the decidability of these four logics, despite the failure of the finite model property for S5H and S5Ht (see Sections 3.3 and 3.6).

Remark 2.1.2. After seeing a first draft of this paper, Frank Wolter noted that the logics considered here are closely related to the many-dimensional modal logics considered in [3]. In the notation and terminology of [3], S5C = LTL × S5, the *product* of LTL and S5. [3] considers a temporal logic PTL, slightly different from LTL: rather than our unary *henceforth* connective, PTL has a unary *always in the future* connective, a unary *always in the past* connective, and a binary *until* connective. Though both the motivation for the semantics and the perspective of [3] are quite different from ours, [3]'s semantics for its logic PTL × S5 is (more or less) a notational variant of our semantics in the special case where the class of topological spaces is the class of almost discrete spaces and the class of functions is the class of *all* continuous functions on almost discrete spaces. [3] presents an axiomatization of PTL × S5, together with completeness and decidability proofs. Our completeness and decidability results for S5C (Theorem 2.1.1, (1), and Corollary 3.5.5) follow from the results in [3], though our proofs are quite different.[9] Analogues of our our semantics in our other three cases (represented by Theorem 2.1.1, (2)–(4)) are not considered in [3], nor are the concomitant logics S5H, S5Ct and S5Ht: these logics do not spring as quickly to mind from the perspective in [3] as they do from the perspective of DTL.

2.2. Some useful facts

We use the facts proved in this subsection to establish further results. These facts also give a feel for the interaction between the topological modality and the temporal modalities in our logics.

Fact 2.2.1. *Suppose that* L *is one of* S5C, S5H, S5Ct *and* S5Ht. *And suppose that* $(A \supset \circ A) \in$ L. *Then* $(A \supset *A) \in$ L.

Proof. Given that $(A \supset \circ A) \in$ L, we also have $*(A \supset \circ A) \in$ L. Given the induction axiom of LTL, we have $((A \,\&\, *(A \supset \circ A)) \supset *A) \in$ L. Therefore $(*(A \supset \circ A) \supset (A \supset *A)) \in$ L. Thus $(A \supset *A) \in$ L, as desired. □

Fact 2.2.2. *Suppose that* L *is one of* S5C, S5H, S5Ct *and* S5Ht. *Then* $(\circ * A \,\&\, A \supset *A) \in$ L.

Proof.

(1)	$\circ A \supset (A \supset \circ A)$	$\in$	L	Axiom of L
(2)	$*\circ A \supset *(A \supset \circ A)$	$\in$	L	by (1)
(3)	$*\circ A \,\&\, A \supset A \,\&\, *(A \supset \circ A)$	$\in$	L	by (2)
(4)	$A \,\&\, *(A \supset \circ A) \supset *A$	$\in$	L	Axiom of L
(5)	$*\circ A \,\&\, A \supset *A$	$\in$	L	by (3), (4)
(6)	$\circ * A \,\&\, A \supset *A$	$\in$	L	$\circ$ commutes with $*$. □

Fact 2.2.3. $(\Box A \supset \circ\Box A) \in$ S5Ct.

Proof.

(1)	$(\Box A \supset \Box\Box A)$	$\in$	S5Ct	Axiom of S5Ct
(2)	$(\Box\Box A \supset \Box\circ\Box A)$	$\in$	S5Ct	Axiom of S5Ct
(3)	$(\Box\circ\Box A \supset \circ\Box A)$	$\in$	S5Ct	Axiom of S5Ct
(4)	$(\Box A \supset \circ\Box A)$	$\in$	S5Ct	by (1), (2), (3). □

Fact 2.2.4. $(\circ\Box A \supset \Box A) \in$ S5Ct.

Proof.

(1)	$(\Box\neg\Box A \supset \circ\Box\neg\Box A)$	$\in$	S5Ct	by Fact 2.2.3
(2)	$(\Box\neg\Box A \supset \neg\Box A)$	$\in$	S5Ct	Axiom of S5Ct
(3)	$(\circ\Box\neg\Box A \supset \circ\neg\Box A)$	$\in$	S5Ct	by (2)
(4)	$(\Box\neg\Box A \supset \circ\neg\Box A)$	$\in$	S5Ct	by (1), (3)
(5)	$(\Box\neg\Box A \supset \neg\circ\Box A)$	$\in$	S5Ct	$\circ$ commutes with $\neg$
(6)	$(\circ\Box A \supset \neg\Box\neg\Box A)$	$\in$	S5Ct	by (5)
(7)	$(\circ\Box A \supset \Diamond\Box A)$	$\in$	S5Ct	by (6)
(8)	$(\Diamond\Box A \supset \Box A)$	$\in$	S5Ct	Axiom of S5Ct
(9)	$(\circ\Box A \supset \Box A)$	$\in$	S5Ct	by (7), (8). □

9 [3] presents refined decidability results for PTL × S5, for example that its decision problem is EXPSPACE-complete (p. 268).

Fact 2.2.5. $(\circ\Box A \supset \Box\circ A) \in$ S5Ct.

Proof. See Fact 2.2.4 and the distinctive axiom of S5Ct. □

Fact 2.2.6. $(\Box\circ A \supset \circ\Box A) \in$ S5Ht.

Proof. Clearly, S5Ct ⊆ S5Ht. So $(\Box A \supset \circ\Box A) \in$ S5Ht, by Fact 2.2.3. Also, $(\Box A \equiv \Box\circ A)$ is an axiom of S5Ht. So $(\Box\circ A \supset \circ\Box A) \in$ S5Ht. □

2.3. More facts

The facts in the current subsection are stated and proved in order to give more of a feel for the interaction between the topological modality and the temporal modalities in our logics. We could have waited until completeness was proved for our four logics, and then given semantic proofs of the facts in this section. But we believe that it is instructive to give the syntactic proofs here.

Fact 2.3.1. $(\Diamond\circ A \supset \Diamond A) \in$ S5Ct.

Proof.

(1)	$(\Box\neg A \supset \Box\circ\neg A)$	$\in$	S5Ct	Axiom of S5Ct
(2)	$(\neg\Box\circ\neg A \supset \neg\Box\neg A)$	$\in$	S5Ct	by (1)
(3)	$(\neg\Box\neg\circ A \supset \neg\Box\neg A)$	$\in$	S5Ct	$\circ$ commutes with $\neg$
(4)	$(\Diamond\circ A \supset \Diamond A)$	$\in$	S5Ct	by (3). □

Fact 2.3.2. $(\Box A \supset *\Box A) \in$ S5Ct.

Proof. See Facts 2.2.3 and 2.2.1. □

Fact 2.3.3. $(\Box * A \supset *\Box A) \in$ S5H.

Proof.

(1)	$**A \supset \circ * A$	$\in$	S5H	Axiom of S5H
(2)	$*A \supset **A$	$\in$	S5H	Axiom of S5H
(3)	$*A \supset \circ * A$	$\in$	S5H	by (1), (2)
(4)	$\Box * A \supset \Box\circ * A$	$\in$	S5H	by (3)
(5)	$\circ\Box *\Box A \equiv \Box\circ *\Box A$	$\in$	S5H	Axiom of S5H
(6)	$\Box * A \supset \circ\Box * A$	$\in$	S5H	by (4), (5)
(7)	$\Box * A \supset *\Box * A$	$\in$	S5H	by Fact 2.2.1
(8)	$*A \supset A$	$\in$	S5H	Axiom of S5H
(9)	$\Box * A \supset \Box A$	$\in$	S5H	by (8)
(10)	$*\Box * A \supset *\Box A$	$\in$	S5H	by (9)
(11)	$\Box * A \supset *\Box A$	$\in$	S5H	by (7), (10). □

Fact 2.3.4. $(*\Box A \supset \Box * A) \in$ S5C.

Proof.

(1)	$*\Box A \supset \Box A$	$\in$	S5C	Axiom of S5C
(2)	$\Diamond *\Box A \supset \Diamond\Box A$	$\in$	S5C	by (1)
(3)	$\Diamond\Box A \supset \Box A$	$\in$	S5C	Axiom of S5C
(4)	$\Diamond *\Box A \supset \Box A$	$\in$	S5C	by (2), (3)
(5)	$\Box A \supset A$	$\in$	S5C	Axiom of S5C
(6)	$\Diamond *\Box A \supset A$	$\in$	S5C	by (4), (5)
(7)	$\circ * A \,\&\, A \supset * A$	$\in$	S5C	by Fact 2.2.2
(8)	$\Box\circ * A \,\&\, \Box A \supset \Box * A$	$\in$	S5C	by (7)
(9)	$\circ\Box * A \supset \Box\circ * A$	$\in$	S5C	Axiom of S5C
(10)	$\circ\Box * A \,\&\, \Box A \supset \Box * A$	$\in$	S5C	by (8), (9).
(11)	$\neg\Box * A \,\&\, \Box A \supset \neg\circ\Box * A$	$\in$	S5C	by (10)
(12)	$\neg\Box * A \,\&\, \Box A \supset \circ\neg\Box * A$	$\in$	S5C	$\circ$ commutes with $\neg$
(13)	$\neg\Box * A \,\&\, \Diamond *\Box A \supset \circ\neg\Box * A$	$\in$	S5C	by (4), (12)
(14)	$**\Box A \supset \circ *\Box A$	$\in$	S5C	Axiom of S5C
(15)	$*\Box A \supset **\Box A$	$\in$	S5C	Axiom of S5C
(16)	$*\Box A \supset \circ *\Box A$	$\in$	S5C	by (14), (15)
(17)	$\Diamond *\Box A \supset \Diamond\circ *\Box A$	$\in$	S5C	by (16)
(18)	$\Diamond *\Box A \supset \neg\Box\neg\circ *\Box A$	$\in$	S5C	by (17)
(19)	$\Diamond *\Box A \supset \neg\Box\circ\neg *\Box A$	$\in$	S5C	$\circ$ commutes with $\neg$
(20)	$\circ\Box\neg *\Box A \supset \Box\circ\neg *\Box A$	$\in$	S5C	Axiom of S5C

(21)	$\Diamond{*}\Box A \supset \neg\circ\Box\neg{*}\Box A$	$\in$	S5C	by (19), (20)
(22)	$\Diamond{*}\Box A \supset \circ\neg\Box\neg{*}\Box A$	$\in$	S5C	$\circ$ commutes with $\neg$
(23)	$\Diamond{*}\Box A \supset \circ\Diamond{*}\Box A$	$\in$	S5C	by (22)
(24)	$\neg\Box{*}A \,\&\, \Diamond{*}\Box A \supset \circ\neg\Box{*}A \,\&\, \circ\Diamond{*}\Box A$	$\in$	S5C	by (13), (23)
(25)	$\neg\Box{*}A \,\&\, \Diamond{*}\Box A \supset \circ(\neg\Box{*}A \,\&\, \Diamond{*}\Box A)$	$\in$	S5C	$\circ$ commutes with &
(26)	$\neg\Box{*}A \,\&\, \Diamond{*}\Box A \supset {*}(\neg\Box{*}A \,\&\, \Diamond{*}\Box A)$	$\in$	S5C	by Fact 2.2.1
(27)	$(\neg\Box{*}A \,\&\, \Diamond{*}\Box A) \supset \Diamond{*}\Box A$	$\in$	S5C	propositional tautology
(28)	${*}(\neg\Box{*}A \,\&\, \Diamond{*}\Box A) \supset {*}\Diamond{*}\Box A$	$\in$	S5C	by (27)
(29)	$\neg\Box{*}A \,\&\, \Diamond{*}\Box A \supset {*}\Diamond{*}\Box A$	$\in$	S5C	by (26), (28)
(30)	$\Diamond{*}\Box A \supset \Box{*}A \vee {*}\Diamond{*}\Box A$	$\in$	S5C	by (29)
(31)	$\Box{*}A \supset {*}A$	$\in$	S5C	Axiom of S5C
(32)	$\Diamond{*}\Box A \supset {*}A \vee {*}\Diamond{*}\Box A$	$\in$	S5C	by (29)
(33)	${*}\Diamond{*}\Box A \supset {*}A$	$\in$	S5C	by (6)
(34)	$\Diamond{*}\Box A \supset {*}A$	$\in$	S5C	by (32), (33)
(35)	$\Box\Diamond{*}\Box A \supset \Box{*}A$	$\in$	S5C	by (35)
(36)	${*}\Box A \supset \Box\Diamond{*}\Box A$	$\in$	S5C	Axiom of S5C
(37)	${*}\Box A \supset \Box{*}A$	$\in$	S5C	by (35), (36). $\Box$

2.4. *Relations among our logics*

The facts in the current subsection help spell out the relations among our four logics. Their proofs rely on the soundness claims in Theorem 2.1.1, which we are taking as proved.

Fact 2.4.1. $(\Box p \supset \circ\Box p) \in$ S5Ct – S5H.

Proof. Given the soundness of S5H for homeomorphisms on almost discrete spaces, it suffices to find a dynamic topological model $M = \langle X, f, V\rangle$, where X is almost discrete and f is a homeomorphism and $M \not\models (\Box p \supset \circ\Box p)$. Let $X = \{0, 1\}$ with open sets $\emptyset$, $\{0\}$, $\{1\}$ and X; $f(0) = 1$ and $f(1) = 0$; and $V(p) = \{0\}$. It is easy to check that $M \not\models (\Box p \supset \circ\Box p)$. $\Box$

Fact 2.4.2. $(\Box\circ p \supset \circ\Box p) \in$ S5H – S5Ct.

Proof. Given the soundness of S5Ct for trivial spaces, it suffices to find a dynamic topological model $M = \langle X, f, V\rangle$, where X is trivial and $M \not\models (\Box\circ p \supset \circ\Box p)$. Let $X = \{0, 1\}$ with open sets $\emptyset$ and X; $f(0) = f(1) = 1$; and $V(p) = \{1\}$. It is easy to check $M \not\models (\Box\circ p \supset \circ\Box p)$. $\Box$

Theorem 2.4.3. *Our four logics are related as follows:*

S5C $\subsetneq$ S5Ct $\subsetneq$ S5Ht
S5C $\subsetneq$ S5H $\subsetneq$ S5Ht
S5Ct $\not\subseteq$ S5H
S5H $\not\subseteq$ S5Ct.

Proof. S5C $\subseteq$ S5Ct, by Fact 2.2.5. And clearly S5Ct $\subseteq$ S5Ht, by Fact 2.2.5.

Clearly S5C $\subseteq$ S5H. By Fact 2.2.5, $(\circ\Box A \supset \Box\circ A) \in$ S5Ct $\subseteq$ S5Ht. And by Fact 2.2.6, $(\Box\circ A \supset \circ\Box A) \in$ S5Ht. So $(\circ\Box A \equiv \Box\circ A) \in$ S5Ht. So S5H $\subseteq$ S5Ht.

S5Ct $\not\subseteq$ S5H, by Fact 2.4.1. Thus S5Ct $\not\subseteq$ S5C and S5Ht $\not\subseteq$ S5H.

S5H $\not\subseteq$ S5Ct, by Fact 2.4.2. Thus S5H $\not\subseteq$ S5C and S5Ht $\not\subseteq$ S5Ct. $\Box$

2.5. *The rule* next removal

Next removal is a peculiar rule. Some basic facts concerning it are as follows:

1. *Next removal* is admissible in S5Ht (Fact 2.5.1).
2. *Next removal* is admissible in S5C (Theorem 3.5.6).
3. S5Ct + $\circ A/A$ = S5Ht (Fact 2.5.2).
4. *Next removal* is not admissible in S5Ct, since S5Ct $\subsetneq$ S5Ht (Theorem 2.4.3).

In this subsection, we give a syntactic proof of (1). It would be nice to have a syntactic proof of (2), but we do not know of one. Instead, we give a semantic proof after we prove completeness for S5C (Section 3.5, Theorem 3.5.6). We prove (3) in this subsection, from which (4) follows.

Fact 2.5.1. *If* $\circ A \in$ S5Ht *then* $A \in$ S5Ht.

Proof. Suppose that $\circ A \in$ S5Ht. Then we have the following:

(1)	$\Box\circ A$	$\in$	S5Ht	Necessitation for $\Box$
(2)	$(\Box A \equiv \Box\circ A)$	$\in$	S5Ht	Axiom of S5Ht
(3)	$\Box A$	$\in$	S5Ht	by (1), (2)
(4)	$(\Box A \supset A)$	$\in$	S5Ht	Axiom of S5Ht
(5)	A	$\in$	S5Ht	by (3), (4). $\Box$

Fact 2.5.2. S5Ct + $\circ A/A$ = S5Ht.

Proof. Clearly S5Ct $\subseteq$ S5Ht. Also, S5Ht is closed under the rule of *next removal*, by Fact 2.5.1. So S5Ct + $\circ A/A \subseteq$ S5Ht. To show that S5Ct + $\circ A/A$ = S5Ht, it suffices to show that $(\Box A \equiv \Box\circ A) \in$ S5Ct + $\circ A/A$. Given that $(\Box A \supset \Box\circ A) \in$ S5Ct, it suffices to show that $(\Box\circ A \supset \Box A) \in$ S5Ct + $\circ A/A$. Here goes:

(1)	$(\circ\Box\circ A \supset \Box\circ A)$	$\in$	S5Ct	by Fact 2.2.4
(2)	$(\Box\circ A \supset \circ A)$	$\in$	S5Ct	Axiom of S5Ct
(3)	$(\circ\Box\circ A \supset \circ A)$	$\in$	S5Ct	by (1), (2)
(4)	$\circ(\Box\circ A \supset A)$	$\in$	S5Ct	$\circ$ commutes with $\supset$
(5)	$\circ(\Box\circ A \supset A)$	$\in$	S5Ct + $\circ A/A$	by (4)
(6)	$(\Box\circ A \supset A)$	$\in$	S5Ct + $\circ A/A$	by (5)
(7)	$(\Box\Box\circ A \supset \Box A)$	$\in$	S5Ct + $\circ A/A$	by (6)
(8)	$(\Box\circ A \supset \Box\Box\circ A)$	$\in$	S5Ct + $\circ A/A$	Axiom of S5Ct
(9)	$(\Box\circ A \supset \Box A)$	$\in$	S5Ct + $\circ A/A$	by (7), (8). $\Box$

We do not know whether we can axiomatize S5H without *next removal*. We conjecture that we can:

Conjecture 2.5.3. *S5H = S5 + LTL +* $(\circ\Box A \equiv \Box\circ A)$.

3. Completeness

3.1. Common elements

The completeness proofs for our four logics have many elements in common. We recycle some of the ideas used in the literature to prove the completeness of LTL, but we do not proceed exactly as elsewhere. In particular, we have to be attentive to the topological connective $\Box$. With S5Ct and S5Ht, we are dealing with trivial spaces, so we do not have to be *especially* attentive to topological issues. The interaction, in S5Ct and S5Ht, between $\Box$ and the temporal connectives is very tractable, as is evidenced by the following theorems of these two logics: $(\Box A \supset \Box\circ A)$, $(\Box A \equiv \circ\Box A)$, and $(\Box A \equiv *\Box A)$. We will have to be more attentive when it comes to S5C and S5H, since then we will be working with nontrivial spaces.

Suppose, then that L is one of the logics S5C, S5H, S5Ct and S5Ht. A *signed formula* is an ordered pair $+C = \langle +, C\rangle$ or $-C = \langle -, C\rangle$. We identify any set of signed formulas with the corresponding formula: the formula corresponding to $\{+A, -B, -C\}$, for example, is $A \,\&\, \neg B \,\&\, \neg C$. The formula corresponding to the empty set (of signed formulas) is $(p \vee \neg p)$. We say that a formula A is *consistent* iff $\neg A \notin$ L. The notion of consistency and all notions defined in terms of consistency depend on which logic L we are working with: we will let context determine L. The points in the current subsection do not depend on L.

Suppose that Φ is a finite set of formulas. A *Φ-atom* (we often just say *atom*) is a set α of signed formulas such that,

1. α is Φ-complete, in the following sense: for each formula C, $C \in \Phi$ iff either $+C \in \alpha$ or $-C \in \alpha$; and
2. α is consistent.

A formula is *modal* iff it is of the form $\Box A$ or $\neg\Box A$. Otherwise it is *nonmodal*. Here we note that, if A is modal, then $(A \supset \Box A) \in$ L and $(\neg A \supset \Box\neg A) \in$ L: this follows from the S5 axioms used to define L. Given an atom α, we define the *modal part of* α as follows:

$$\alpha_M =_{df} \{\pm A \in \alpha : A \text{ is a modal formula}\}.$$

Note that $(\alpha_M \supset \Box\alpha_M) \in$ L.

Given a finite set Φ of formulas, we define some relations on Φ-atoms:

$$\begin{array}{rll} \alpha R\beta & \text{iff} & \alpha_M = \beta_M \\ \alpha S\beta & \text{iff} & (\alpha \,\&\, \circ\beta) \text{ is consistent} \\ \alpha S^0\beta & \text{iff} & \alpha = \beta \\ \alpha S^{n+1}\beta & \text{iff} & \alpha S\gamma \text{ and } \gamma S^n\beta, \text{ for some } \Phi\text{-complete consistent } \Phi\text{-atom } \gamma \\ \alpha S^\sharp\beta & \text{iff} & \alpha S^n\beta, \text{ for some } n \geq 0. \end{array}$$

R is clearly an equivalence relation on the Φ-atoms. We will denote the equivalence class determined by α as $|\alpha|_R$. The next few lemmas concern R and S.

Lemma 3.1.1. *Suppose that Φ is a finite set of formulas closed under subformulas, that $\Box A \in \Phi$ and that α is a Φ-atom α. Then*

$$+\Box A \in \alpha \text{ iff, for every } \beta \in |\alpha|_R, +A \in \beta.$$

Proof. Note that $A \in \Phi$, since $\Box A \in \Phi$. We consider both directions of the desired biconditional.

($\Rightarrow$) Suppose that $+\Box A \in \alpha$ and that $\beta \in |\alpha|_R$. Then $\alpha R \beta$. So $\alpha_M = \beta_M$. So $+\Box A \in \beta$. So $+A \in \beta$, since β is Φ-complete and consistent.

($\Leftarrow$) Suppose that $+\Box A \notin \alpha$. Then $-\Box A \in \alpha$. First, we claim that $(\neg A \,\&\, \alpha_M)$ is consistent. Suppose not. Then $(\alpha_M \supset A) \in$ L. So $(\Box\alpha_M \supset \Box A) \in$ L. Recall that $(\alpha_M \supset \Box\alpha_M) \in$ L. So $(\alpha_M \supset \Box A) \in$ S5Ct. But this cannot be, given that $-\Box A \in \alpha$ and that α is consistent. Given that $(\neg A \,\&\, \alpha_M)$ is consistent, there is some atom β such that $\alpha_M \cup \{-A\} \subseteq \beta$. Note that $\beta_M = \alpha_M$, so that $\beta \in |\alpha|_R$. □

Lemma 3.1.2. *Suppose that Φ is a finite set of formulas closed under subformulas with $\circ A \in \Phi$; and that α and β are Φ-atoms with $\alpha S \beta$. Then $+\circ A \in \alpha$ iff $A \in \beta$.*

Lemma 3.1.3. *Suppose that Φ is a finite set of formulas closed under subformulas and that α is a Φ-atom. Then there is some Φ-atom β such that $\alpha S \beta$.*

Proof. Let *At* be the set of Φ-atoms, and let $\bigvee At$ be the disjunction of all the (formulas corresponding to) the Φ-atoms. Note that $\bigvee At$ is an instance of a propositional tautology. So $\circ\bigvee At \in$ L. Suppose, for a reductio, that $(\alpha \,\&\, \circ\beta)$ is inconsistent, for each $\beta \in At$. Then $(\alpha \,\&\, \circ\bigvee At)$ is inconsistent. So α is inconsistent, since $\circ\bigvee At \in$ L. But this contradicts the fact that α is a Φ-atom. □

Lemma 3.1.4. *Suppose that Φ is a finite set of formulas closed under subformulas, that α and β are Φ-atoms with $\alpha S \beta$, and that $+{*}A \in \alpha$. Then $+{*}A \in \beta$.*

Proof. We need only note that $({*}A \supset \circ{*}A) \in$ L. □

Corollary 3.1.5. *Suppose that Φ is a finite set of formulas closed under subformulas, that α and β are Φ-atoms with $\alpha S^\sharp \beta$, and that $+{*}A \in \alpha$. Then $+{*}A \in \beta$.*

Lemma 3.1.6. *Suppose that Φ is a finite set of formulas closed under subformulas, that α is a Φ-atom, and that $-{*}A \in \alpha$. Then there is some Φ-atom β such that $\alpha S^\sharp \beta$ and $-A \in \beta$.*

Proof. (We adapt the third clause of the proof of Lemma 1 in [4]. The same idea is used in [5] to a slightly different end.) For any atom γ, let $\Gamma^S_\gamma = \{\delta : \gamma S \delta\}$ and let $\Gamma^\sharp_\gamma = \{\delta : \gamma S^\sharp \delta\}$. For any set Γ of atoms, let $\bigvee\Gamma$ be the disjunction of all the (formulas corresponding to) atoms in Γ. Then $(\gamma \supset \circ\bigvee\Gamma^S_\gamma) \in$ L, for any atom γ. Also, if $\gamma \in \Gamma^\sharp_\alpha$, then $\Gamma^S_\gamma \subseteq \Gamma^\sharp_\alpha$. So $(\gamma \supset \circ\bigvee\Gamma^\sharp_\alpha) \in$ L, for any atom $\gamma \in \Gamma^\sharp_\alpha$. So $(\bigvee\Gamma^\sharp_\alpha \supset \circ\bigvee\Gamma^\sharp_\alpha) \in$ L. So $(\bigvee\Gamma^\sharp_\alpha \supset {*}\bigvee\Gamma^\sharp_\alpha) \in$ L, by Fact 2.2.1. Also, $\alpha \in \Gamma^\sharp_\alpha$, so $(\alpha \supset \bigvee\Gamma^\sharp_\alpha) \in$ L. So $(\alpha \supset {*}\bigvee\Gamma^\sharp_\alpha) \in$ L.

Now suppose that $-{*}A \in \alpha$, but (for a reductio) that there is no β such that $\alpha S^\sharp \beta$ and $-A \in \beta$. Then $A \in \beta$, for every $\beta \in \Gamma^\sharp_\alpha$. So $(\bigvee\Gamma^\sharp_\alpha \supset A) \in$ L. So $({*}\bigvee\Gamma^\sharp_\alpha \supset {*}A) \in$ L. So $(\alpha \supset {*}A) \in$ L. So α is inconsistent, since $-{*}A \in \alpha$. But α is consistent. □

For our completeness proofs, we will build models out of sequences of Φ-atoms and other objects. For our purposes, a *finite* sequence is a sequence $\langle x_i\rangle^n_{i=0}$ indexed by the set $\{0, \ldots, n\}$ for some $n \in \mathbb{N}$; an *infinite* sequence is a sequence $\langle x_i\rangle_{i\geq 0}$ indexed by the natural numbers; and a *bi-infinite* sequence is a 'sequence' $\langle x_i\rangle_{i\in\mathbb{Z}}$ indexed by the integers. We can also use the following notation for infinite sequences: $\langle x_i\rangle_{i\in\mathbb{N}}$. A natural number k is a *periodic point* of an infinite sequence $\langle x_i\rangle_{i\geq 0}$ iff for some $l \geq 1$ we have $x_{i+l} = x_i$ for every $i \geq k$. Note that, if k is a periodic point, then so is any $j \geq k$. An infinite sequence is *eventually periodic* iff it has a periodic point. A bi-infinite sequence $\langle x_i\rangle_{i\in\mathbb{Z}}$ is *bi-eventually periodic* iff both infinite sequences $\langle x_i\rangle_{i\geq 0}$ and $\langle x_{-i}\rangle_{i\geq 0}$ are eventually periodic. An object x is *cofinal* in an infinite sequence $\langle x_i\rangle_{i\geq 0}$ iff for each $i \geq 0$ there is a $j \geq i$ such that $x = x_j$. A natural number k is a *cofinality point* of an infinite sequence $\langle x_i\rangle_{i\geq 0}$ iff x_i is cofinal for every $i \geq k$. Note that, if k is a cofinality point, then so is any $j \geq k$. Note also that any periodic point is also a cofinality point.

Suppose that Φ is a finite set of formulas. A finite sequence $\langle\alpha_i\rangle^n_{i=0}$ [an infinite sequence $\langle\alpha_i\rangle_{i\geq 0}$, a bi-infinite sequence $\langle\alpha_i\rangle_{i\in\mathbb{Z}}$] of Φ-atoms is an *S-sequence* iff $\alpha_i S \alpha_{i+1}$, for each $i \geq 0$ and $< n$ [for each $i \geq 0$, for each $i \in \mathbb{Z}$]. An infinite sequence $\langle\alpha_i\rangle_{i\geq 0}$ [a bi-infinite sequence $\langle\alpha_i\rangle_{i\in\mathbb{Z}}$] of Φ-atoms is $*$*-complete* iff for every $i \geq 0$ [$i \in \mathbb{Z}$] and every formula A, if $-{*}A \in \alpha_i$ then there is some $j \geq i$ such that $-A \in \alpha_j$. A finite sequence $\langle\alpha_i\rangle^n_{i=0}$ *witnesses* the signed formula $-{*}A$ iff if $-{*}A \in \alpha_0$ then $-A \in \alpha_n$.

Lemma 3.1.7. *Suppose that Φ is a finite set of formulas closed under subformulas, that α is a Φ-atom, and that ${*}A \in \Phi$. Then there is a finite S-sequence $\langle\alpha_i\rangle^n_{i=0}$ of Φ-atoms, with $\alpha_0 = \alpha$, that witnesses the signed formula $-{*}A$.*

Proof. If $-{*}A \notin \alpha$ then it is easy: just let $\alpha_0 = \alpha$ and let our sequence be $\langle\alpha_i\rangle^0_{i=0}$. If $-{*}A \in \alpha$ then, by Lemma 3.1.6, there is a Φ-atom β such that $\alpha S^\sharp \beta$ and $-A \in \beta$. Since $\alpha S^\sharp \beta$, there is a finite S-sequence $\langle\alpha_i\rangle^n_{i=0}$ with $\alpha_0 = \alpha$ and $\alpha_n = \beta$. Note that this sequence witnesses the signed formula $-{*}A$. □

Lemma 3.1.8. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. Then there is a $*$-complete infinite S-sequence $\langle\alpha_i\rangle_{i\geq 0}$, such that $\alpha_0 = \alpha$.*

Proof. If Φ contains no formulas of the form $*A$, then it is easy: just choose *any* S-sequence $\langle \alpha_i \rangle_{i \geq 0}$, such that $\alpha_0 = \alpha$. The existence of such a sequence is guaranteed by Lemma 3.1.3.

Otherwise, Φ contains some formula(s) of the form $*A$. List the set $\{-*A : *A \in \Phi\}$ as follows: $\{-*A_0, \ldots, -*A_{v-1}\}$, where $v \geq 1$. For any $j, k \geq 1$, let $rem(k, j)$ be the remainder of k divided by j; for example $rem(47, 7) = 5$. And, for each $k \geq 0$, define $-*A_k = -*A_{rem(v,k)}$. Thus, the sequence $\langle -*A_i \rangle_{i \geq 0}$ looks like this:

$$-*A_0, \ldots, -*A_{v-1}, -*A_0, \ldots, -*A_{v-1}, -*A_0, \ldots, -*A_{v-1}, \ldots.$$

For each $k \geq 0$, we will define by induction on k a finite S-sequence $\langle \alpha_i^k \rangle_{i=0}^{m_k}$, for some m_k, that witnesses the signed formula $-*A_k$. By Lemma 3.1.7, we can choose a finite S-sequence $\langle \alpha_i^0 \rangle_{i=0}^{m_0}$ that witnesses the signed formula $-*A_0$, with $\alpha_0^0 = \alpha$. Assume that we have defined a finite S-sequence $\langle \alpha_i^k \rangle_{i=0}^{m_k}$ that witnesses the signed formula $-*A_k$. Let α_0^{k+1} be any Φ-atom such that $\alpha_{m_k}^k S \alpha_0^{k+1}$. By Lemma 3.1.7, we can choose a finite S-sequence $\langle \alpha_i^{k+1} \rangle_{i=0}^{m_{k+1}}$ that witnesses the signed formula $-*A_{k+1}$.

Now define the sequence $\langle \alpha_i \rangle_{i \geq 0}$ by gluing together the sequences $\langle \alpha_i^k \rangle_{i=0}^{m_k}$ as follows:

$$\alpha_0^0, \ldots, \alpha_{m_0}^0, \alpha_0^1, \ldots, \alpha_{m_1}^1, \alpha_0^2, \ldots, \alpha_{m_2}^2, \alpha_0^3, \ldots, \alpha_{m_3}^3, \ldots.$$

To be more precise, for each $k \geq 0$, let $n_k = k + \sum_{i=0}^{k} m_k$. For each $i \geq 0$, let $k_i = min\{k : i \leq n_k\}$. Finally, let $\alpha_i = \alpha_{i+m_{k_i}-n_{k_i}}^{k_i}$. Note the following:

$$\begin{aligned} \alpha_{n_k - m_k} &= \alpha_0^k \\ \alpha_{n_k - m_k + i} &= \alpha_i^k, \quad \text{if } i \leq m_k \\ \alpha_{n_k} &= \alpha_{m_k}^k \\ \alpha_{n_k + 1} &= \alpha_0^{k+1}. \end{aligned}$$

Clearly $\langle \alpha_i \rangle_{i \geq 0}$ is an infinite S-sequence whose first member is α. We must still show that this sequence is $*$-complete. Suppose not. Then there is some $l \geq 0$ and some $-*A \in \alpha_l$ such that

$$+A \in \alpha_j \quad \text{for every } j \geq l. \qquad (\dagger)$$

We claim that

$$-*A \in \alpha_j \quad \text{for every } j \geq l. \qquad (\ddagger)$$

To see (‡), suppose not. Choose the smallest $j \geq l$ such that $+*A \in \alpha_j$. In fact, $j > l$, since $-*A \in \alpha_l$. So $-*A \in \alpha_{j-1}$. Also $+A \in \alpha_{j-1}$, since $j - 1 \geq l$. Also, $(\neg *A \,\&\, A \supset \circ\neg *A) \in \mathrm{L}$. So $(\alpha_{j-1} \supset \circ\neg *A) \in \mathrm{L}$. So α_{j-1} is not consistent with $\circ\alpha_j$, since $+*A \in \alpha_j$. But this contradicts the fact that $\langle \alpha_i \rangle_{i \geq 0}$ is an S-sequence.

Now that we have established (‡), choose $k \geq l$ so that $-*A = -*A_k$. Note that $l \leq k \leq n_k - m_k$. So $-*A_k \in \alpha_{n_k - m_k}$, by (‡). Also, as noted above, $\alpha_{n_k - m_k} = \alpha_0^k$. So $-*A_k \in \alpha_0^k$. Recall that the sequence $\langle \alpha_i^k \rangle_{i=0}^{m_k}$ witnesses the signed formula $-*A_k$. So $-A \in \alpha_{m_k}^k$. As noted above, $\alpha_{n_k} = \alpha_{m_k}^k$. So $-A \in \alpha_{n_k}$. But $l \leq k \leq n_k$, so that $+A \in \alpha_{n_k}$, by (†). A contradiction. □

We can improve on Lemma 3.1.8:

Lemma 3.1.9. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. Then there is an eventually periodic $*$-complete infinite S-sequence $\langle \alpha_i \rangle_{i \geq 0}$, such that $\alpha_0 = \alpha$.*

Proof. By Lemma 3.1.8, there is an $*$-complete infinite S-sequence $\langle \beta_i \rangle_{i \geq 0}$, such that $\beta_0 = \alpha$. Let Γ be the set of Φ-atoms cofinal in $\langle \beta_i \rangle_{i \geq 0}$. Since there are only finitely many Φ-atoms, there is a cofinality point, say k. Note that, for every $j \geq k$,

$$\{\beta_i : i \geq j\} = \{\beta_i : i \geq k\} = \Gamma.$$

So, for every $j \geq 0$,

$$\{\beta_i : i \geq j\} = \{\beta_i : j \leq i < k\} \cup \Gamma.$$

Choose the smallest $l \geq 1$ such that $\beta_k = \beta_{k+l}$. Define the new sequence $\langle \alpha_i \rangle_{i \geq 0}$ as follows:

$$\begin{aligned} \alpha_i &= \beta_i \quad \text{for } i < k \\ \alpha_{i+ml} &= \beta_i \quad \text{for } i \geq k, i < k + l, m \geq 0. \end{aligned}$$

Note first that $\alpha_0 = \alpha$. Also note that the sequence $\langle \alpha_i \rangle_{i \geq 0}$ is an S-sequence and is periodic. Finally note that, for every $j \geq k$,

$$\{\alpha_i : i \geq j\} = \{\alpha_i : i \geq k\} = \Gamma.$$

So, for every $j \geq 0$,

$$\{\alpha_i : i \geq j\} = \{\alpha_i : j \leq i < k\} \cup \Gamma = \{\beta_i : i \geq j\}.$$

So the sequence $\langle \alpha_i \rangle_{i \geq 0}$ is $*$-complete, like the sequence $\langle \beta_i \rangle_{i \geq 0}$. □

3.2. Completeness of S5Ct

The proof of completeness for S5Ct relies on the particular Lemma 3.2.1.

Lemma 3.2.1. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α and β are Φ-atoms. Then if $\alpha S\beta$ then $\alpha R\beta$.*

Proof. For a reductio, suppose that $\alpha S\beta$ but that $\alpha_M \neq \beta_M$. We consider two cases.

(Case 1) for some $\Box A \in \Phi$, we have $+\Box A \in \alpha$ and $-\Box A \in \beta$. Since $\alpha S\beta$, the following is consistent: $(\Box A \,\&\, \circ\neg\Box A)$. But this contradicts Fact 2.2.3, which says that $(\Box A \supset \circ\Box A) \in$ S5Ct.

(Case 2) for some $\Box A \in \Phi$, we have $-\Box A \in \alpha$ and $+\Box A \in \beta$. Since $\alpha S\beta$, the following is consistent: $(\neg\Box A \,\&\, \circ\Box A)$. But this contradicts Fact 2.2.4, which says that $(\circ\Box A \supset \Box A) \in$ S5Ct. □

Definition 3.2.2. Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. We will define a finite trivial topological space, X_α; a continuous function, f_α on X_α; and a valuation function $V_\alpha : PV \to \mathcal{P}(X_\alpha)$. In particular, X_α will be a finite subset of $\mathbb{N} \times \mathbb{N}$.

First, enumerate all of the atoms in $|\alpha|_R$, starting with α itself: $\alpha^0, \ldots, \alpha^n$, with $\alpha^0 = \alpha$. For each α^m, let $\langle \alpha_i^m \rangle_{i\geq 0}$ be an eventually periodic $*$-complete infinite S-sequence with $\alpha_0^m = \alpha^m$: such a sequence exists by Lemma 3.1.9. Thus we have n eventually periodic sequences,

$$\begin{array}{cccccc} \alpha_0^0 & \alpha_1^0 & \alpha_2^0 & \alpha_3^0 & \alpha_4^0 & \ldots \\ \alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_0^n & \alpha_1^n & \alpha_2^n & \alpha_3^n & \alpha_4^n & \ldots \end{array}$$

Since each of these sequence is eventually periodic, for each $m = 0, \ldots, n$ we have the following: there is a $k_m \geq 0$ and an $l_m \geq 1$ such that, for every $i \geq k_m$, we have $\alpha_{i+l_m}^m = \alpha_i^m$. We cut each sequence off at $(k_m + l_m) - 1$:

$$\begin{array}{cccccccc} \alpha_0^0 & \alpha_1^0 & \alpha_2^0 & \ldots & \alpha_{k_0}^0 & \alpha_{k_0+1}^0 & \ldots & \alpha_{(k_0+l_0)-1}^0 \\ \alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_{k_1}^1 & \alpha_{k_1+1}^1 & \ldots & \alpha_{(k_1+l_1)-1}^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_0^n & \alpha_1^n & \alpha_2^n & \ldots & \alpha_{k_n}^n & \alpha_{k_n+1}^n & \ldots & \alpha_{(k_n+l_n)-1}^n \end{array}$$

We define X_α as follows:

$$X_\alpha = \{\langle a, b\rangle \in \mathbb{N} \times \mathbb{N} : 0 \leq a \leq n \text{ and } 0 \leq b \leq (k_a + l_a) - 1\}.$$

We impose the trivial topology on X_α. We define the function $f_\alpha : X_\alpha \to X_\alpha$ as follows:

$$f_\alpha(\langle a, b\rangle) = \begin{cases} \langle a, b+1\rangle, & \text{if } b < (k_a + l_a) - 1 \\ \langle a, k_a\rangle, & \text{if } b = (k_a + l_a) - 1. \end{cases}$$

We define the valuation function V_α as follows:

$$V_\alpha(p) = \{\langle a, b\rangle \in X_\alpha : +p \in \alpha_b^a\}, \quad \text{for each propositional variable } p.$$

Finally, we define the dynamic topological model, $M_\alpha =_{df} \langle X_\alpha, f_\alpha, V_\alpha\rangle$.

Shortly we will prove the following:

Theorem 3.2.3. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. And suppose that X_α, f_α, and V_α are defined as in Definition 3.2.2. Then, for each $A \in \Phi$:*

$$\text{for each } \langle a, b\rangle \in X_\alpha, \quad \langle a, b\rangle \in V_\alpha(A) \quad \text{iff } +A \in \alpha_b^a.$$

But first we state a lemma about f_α.

Lemma 3.2.4. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. Suppose that $\langle a, b\rangle \in X_\alpha$, that $i \geq 0$ and that $\langle a, b'\rangle = f_\alpha^i(\langle a, b\rangle)$. Then $\alpha_{b'}^a = \alpha_{b+i}^a$.* (Note that the ordered pair $\langle a, b+i\rangle$ need not be in X_α.)

Proof of Theorem 3.2.3. By induction on the structure of A. We will use all the notation, terminology and so on in Definition 3.2.2.

(Case 1) $A \in PV$. The result is given by the definition of V_α.

(Case 2) A is of the form $\neg B$. Choose $\langle a, b\rangle \in X_\alpha$. Then note: $\langle a, b\rangle \in V_\alpha(A)$ iff $\langle a, b\rangle \in V_\alpha(\neg B)$ iff $\langle a, b\rangle \notin V_\alpha(B)$ iff $+B \notin \alpha_b^a$ (by the inductive hypothesis) iff $-B \in \alpha_b^a$ (since α_b^a is Φ-complete) iff $+\neg B \in \alpha_b^a$ (since α_b^a is Φ-complete and consistent) iff $+A \in \alpha_b^a$.

(Case 3) A is of the form $(B \,\&\, C)$. Choose $\langle a, b\rangle \in X_\alpha$. Then note: $\langle a, b\rangle \in V_\alpha(A)$ iff $\langle a, b\rangle \in V_\alpha(B \,\&\, C)$ iff $\langle a, b\rangle \in V_\alpha(B)$ and $\langle a, b\rangle \in V_\alpha(C)$ iff $+B \in \alpha^a_b$ or $+C \in \alpha^a_b$ (by the inductive hypothesis) iff $+(B \,\&\, C) \in \alpha^a_b$ (since α^a_b is Φ-complete and consistent) iff $+A \in \alpha^a_b$.

(Case 4) A is of the form $\Box B$. Choose $\langle a, b\rangle \in X_\alpha$. First, we note that

$$\alpha R \alpha^i_j \quad \text{for each } \langle i, j\rangle \in X_\alpha. \tag{†}$$

This follows from the following: Lemma 3.2.1, the fact that $\alpha R\alpha^i_0$, and the fact that $\alpha^i_0 S^\sharp \alpha^i_j$. Second, we note that,

$$\text{for each } \Phi\text{-atom } \beta, \quad \text{if } \alpha R\beta \text{ then, for some } \langle i, j\rangle \in X_\alpha,\ \beta = \alpha^i_j. \tag{‡}$$

Now note: $\langle a, b\rangle \in V_\alpha(A)$ iff $\langle a, b\rangle \in V_\alpha(\Box B)$ iff $\langle a, b\rangle \in Int(V_\alpha(B))$ iff $\langle i, j\rangle \in V_\alpha(B)$ for every $\langle i, j\rangle \in X_\alpha$ iff $+B \in \alpha^i_j$ for every $\langle i, j\rangle \in X_\alpha$ (by IH) iff $+B \in \beta$ for every Φ-atom β with $\alpha R\beta$ (by (†) and (‡)) iff $+\Box B \in \alpha$ (by Lemma 3.1.1) iff $+A \in \alpha^a_b$.

(Case 5) A is of the form $\circ B$. Choose $\langle a, b\rangle \in X_\alpha$. We consider two cases: (5.1) $b < (k_a + l_a) - 1$, and (5.2) $b = (k_a + l_a) - 1$. (Case 5.1): $\langle a, b\rangle \in V_\alpha(A)$ iff $\langle a, b\rangle \in V_\alpha(\circ B)$ iff $f_\alpha(\langle a, b\rangle) \in V_\alpha(B)$ iff $\langle a, b+1\rangle \in V_\alpha(B)$ iff $+B \in \alpha^a_{b+1}$ (by IH) iff $+\circ B \in \alpha^a_b$ (by Lemma 3.1.2) iff $+A \in \alpha^a_b$. (Case 5.2): $\langle a, b\rangle \in V_\alpha(A)$ iff $\langle a, (k_a + l_a) - 1\rangle \in V_\alpha(\circ B)$ iff $f_\alpha(\langle a, (k_a + l_a) - 1\rangle) \in V_\alpha(B)$ iff $\langle a, k_a\rangle \in V_\alpha(B)$ iff $+B \in \alpha^a_{k_a}$ (by IH) iff $+B \in \alpha^a_{k_a+l_a}$ (since $\alpha^a_{k_a} = \alpha^a_{k_a+l_a}$) iff $+B \in \alpha^a_{b+1}$ iff $+\circ B \in \alpha^a_b$ (by Lemma 3.1.2) iff $+A \in \alpha^a_b$.

(Case 6) A is of the form $*B$. Choose $\langle a, b\rangle \in X_\alpha$. We consider both directions of our biconditional separately.

($\Rightarrow$) We prove the contrapositive. So suppose that $+A \notin \alpha^a_b$. Then $+*B \notin \alpha^a_b$. So $-*B \in \alpha^a_b$. So, since $\langle \alpha^a_i\rangle_{i\geq 0}$ is $*$-complete, we have $-B \in \alpha^a_{b+i}$, for some $i \geq 0$. Let $\langle a, b'\rangle = f^i_\alpha(\langle a, b\rangle)$. By Lemma 3.2.4, $\alpha^a_{b'} = \alpha^a_{b+i}$. So $-B \in \alpha^a_{b'}$. So $+B \notin \alpha^a_{b'}$. So $\langle a, b'\rangle \notin V_\alpha(B)$, by IH. So $f^i_\alpha(\langle a, b\rangle) \notin V_\alpha(B)$. So $\langle a, b\rangle \notin V_\alpha(*B)$. So $\langle a, b\rangle \notin V_\alpha(A)$.

($\Leftarrow$) We prove the contrapositive. So suppose that $\langle a, b\rangle \notin V_\alpha(A)$. Then $\langle a, b\rangle \notin V_\alpha(*B)$. So $f^i_\alpha(\langle a, b\rangle) \notin V_\alpha(B)$ for some $i \geq 0$. Let $\langle a, b'\rangle = f^i_\alpha(\langle a, b\rangle)$. Then $\langle a, b'\rangle \notin V_\alpha(B)$. So $+B \notin \alpha^a_{b'}$, by IH. So $+B \notin \alpha^a_{b+i}$, by Lemma 3.2.4. So $+*B \notin \alpha^a_{b+i}$. Now note that $\alpha^a_b S^\sharp \alpha^a_{b+i}$. So $+*B \notin \alpha^a_b$, by Lemma 3.1.5. So $+A \notin \alpha^a_b$. □

Corollary 3.2.5. *Suppose that $A \notin$ S5Ct. Then there is some finite trivial topological space X such that $X \not\models A$.*

Proof. Suppose that $A \notin$ S5Ct. Let Φ be the set of subformulas of A. Choose a Φ-atom α with $-A \in \alpha$. Define the topological model $M_\alpha = \langle X_\alpha, f_\alpha, V_\alpha\rangle$ as in Definition 3.2.2. By Theorem 3.2.3 and the fact that $\alpha^0_0 = \alpha$, we have $\langle 0, 0\rangle \notin V_\alpha(A)$. So $X_\alpha \not\models A$. And X_α is a finite trivial topological space. □

The completeness of S5Ct for trivial topological spaces follows directly from Corollary 3.2.5. Indeed, this Corollary is stronger than completeness: it also entails that S5Ct has the finite model property. Thus:

Corollary 3.2.6. S5Ct *is decidable.*

3.3. *Completeness of* S5Ht

The completeness proof for S5Ht proceeds in much the same way as the completeness proof for S5Ct, with a couple of extra bells and whistles. There *is* a major glitch: S5Ht does not have the finite model property. To be more precise, the formula $(\circ *p \supset *p)$ is not a theorem of S5Ht, but is validated by every model $\langle X, f, V\rangle$ where X is a finite topological space (trivial or not) and f is a homeomorphism.

To see that $(\circ *p \supset *p)$ is not a theorem of S5Ht, it suffices to define a trivial topological space X, a bijection f on X, and a function $V : PV \to \mathcal{P}(X)$ such that $\langle X, f, V\rangle \not\models (\circ *p \supset *p)$. Here goes: $X = \mathbb{Z}$, i.e. the set of integers, with the trivial topology; $f(z) = z + 1$ for each $z \in \mathbb{Z}$; and, for each propositional variable p, we have $V(p) = \{1, 2, 3, \ldots\}$. It is easy to check that $0 \notin V(\circ *p \supset *p)$.

Now we show that $(\circ *p \supset *p)$ is validated by every model $\langle X, f, V\rangle$ where X is a finite topological space (trivial or not) and f is a homeomorphism. Consider any finite topological space X, any homeomorphism f on X, and any $V : PV \to \mathcal{P}(X)$. Suppose that $x \in X$ but $x \notin V(\circ *p \supset *p)$. Then, $x \in V(\circ *p)$ and $x \notin V(*p)$. So $f^i(x) \in V(p)$ for every $i \geq 1$; and $f^i(x) \notin V(p)$ for some $i \geq 0$. So $f^i(x) \in V(p)$ for every $i \geq 1$; and $x \notin V(p)$. But since X is finite and f is a bijection, we have $x = f^k(x)$ for some $k \geq 1$. Since $k \geq 1$, we have $f^k(x) \in V(p)$. So $x \in V(p)$, which contradicts $x \notin V(p)$.

Onto the proof of completeness for S5Ht. Suppose that Φ is a finite set of formulas. A finite sequence $\langle \alpha_i\rangle^n_{i=0}$ [an infinite sequence $\langle \alpha_i\rangle_{i\geq 0}$] of Φ-atoms is a *backward S-sequence* iff $\alpha_{i+1} S \alpha_i$, for each $i \geq 0$ and $< n$ [for each $i \geq 0$].

Lemma 3.3.1. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α and β are Φ-atoms. Then if $\alpha S\beta$ then $\alpha R\beta$.*

Proof. The same as the proof of Lemma 3.2.1. □

Lemma 3.3.2. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. Then there is some Φ-atom β such that $\beta S\alpha$.* (Compare Lemma 3.1.3.)

Proof. Since α is a Φ-atom, α is consistent. So $\neg\alpha \notin$ S5Ht. So $\circ\neg\alpha \notin$ S5Ht, by Fact 2.5.1. So $\neg\circ\alpha \notin$ S5Ht. So $\circ\alpha$ is consistent. So there is some Φ-atom β such that the following is consistent: $(\beta \,\&\, \circ\alpha)$. □

Corollary 3.3.3. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. Then there is some infinite eventually periodic backward S-sequence whose initial member is α.*

Proof. The existence of an infinite backward S-sequence whose initial member α is guaranteed by Lemma 3.3.2. The existence of an *eventually periodic* infinite backward S-sequence whose initial member α is guaranteed by the former remark and the fact that there are only finitely many Φ-atoms. □

Lemma 3.3.4. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. Then there is a bi-eventually periodic $*$-complete bi-infinite S-sequence $\langle\alpha_i\rangle_{i\in\mathbb{Z}}$, such that $\alpha_0 = \alpha$.*

Proof. By Lemma 3.1.9, there is an eventually periodic $*$-complete infinite S-sequence $\langle\alpha_i\rangle_{i\geq 0}$, such that $\alpha_0 = \alpha$. And by Lemma 3.3.3, there is an eventually periodic backward S-sequence $\langle\alpha'_i\rangle_{i\geq 0}$, such that $\alpha'_0 = \alpha$. For each $i < 0$, define $\alpha_i = \alpha'_{-i}$. Then the bi-infinite sequence $\langle\alpha_i\rangle_{i\in\mathbb{Z}}$ is bi-eventually periodic and $*$-complete, and $\alpha_0 = \alpha$. □

Definition 3.3.5. Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. We will define a trivial topological space, X_α; a homeomorphism, f_α on X_α; and a valuation function $V_\alpha : PV \to \mathcal{P}(X_\alpha)$. In particular, X_α will be an infinite subset of $\mathbb{Z}\times\mathbb{Z}$.

First, enumerate all of the atoms in $|\alpha|_R$, starting with α itself: $\alpha^0, \ldots, \alpha^n$, with $\alpha^0 = \alpha$. For each α^m, let $\langle\alpha^m_i\rangle_{i\in\mathbb{Z}}$ be a bi-eventually periodic $*$-complete infinite S-sequence with $\alpha^m_0 = \alpha^m$. Thus we have n bi-eventually periodic bi-infinite sequences,

$$\begin{array}{ccccccccccc}
\ldots & \alpha^0_{-4} & \alpha^0_{-3} & \alpha^0_{-2} & \alpha^0_{-1} & \alpha^0_{0} & \alpha^0_{1} & \alpha^0_{2} & \alpha^0_{3} & \alpha^0_{4} & \ldots \\
\ldots & \alpha^1_{-4} & \alpha^1_{-3} & \alpha^1_{-2} & \alpha^1_{-1} & \alpha^1_{0} & \alpha^1_{1} & \alpha^1_{2} & \alpha^1_{3} & \alpha^1_{4} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ldots & \alpha^n_{-4} & \alpha^n_{-3} & \alpha^n_{-2} & \alpha^n_{-1} & \alpha^n_{0} & \alpha^n_{1} & \alpha^n_{2} & \alpha^n_{3} & \alpha^n_{4} & \ldots
\end{array}$$

We define X_α as follows:

$$X_\alpha = \{0, \ldots, n\} \times \mathbb{Z}.$$

We impose the trivial topology on X_α. We define the function $f_\alpha : X_\alpha \to X_\alpha$ as follows:

$$f_\alpha(\langle a, b\rangle) = \langle a, b+1\rangle.$$

We define the valuation function V_α as follows:

$$V_\alpha(p) = \{\langle a, b\rangle : 0 \leq a \leq n \text{ and } +p \in \alpha^a_b\}, \quad \text{for each propositional variable } p.$$

Finally, we define the dynamic topological model, $M_\alpha =_{df} \langle X_\alpha, f_\alpha, V_\alpha\rangle$.

The proof of the following theorem is similar to the proof of Theorem 3.2.3:

Theorem 3.3.6. *Suppose that Φ is a finite set of formulas closed under subformulas, and that α is a Φ-atom. And suppose that X_α, f_α, and V_α are defined as in Definition 3.3.5. Then, for each $A \in \Phi$:*

for each $\langle a, b\rangle \in X_\alpha$, $\langle a, b\rangle \in V_\alpha(A)$ iff $+A \in \alpha^a_b$.

And we thus get an analogue (without the finiteness condition) of Corollary 3.2.5:

Corollary 3.3.7. *Suppose that $A \notin$ S5Ht. Then there is some trivial topological space X and some homeomorphism $f : X \to X$ such that $\langle X, f\rangle \not\models A$.*

The completeness of S5Ht for homeomorphisms on trivial topological spaces follows from Corollary 3.3.7.

What about the decidability of S5Ht? We do not get it through any finite model property. But decidability does follow from the fact that each model X_α is of a kind that can be finitely represented. To be more precise.

Definition 3.3.8. A *premodel* is an ordered quartuple $M = \langle X, g, h, V\rangle$ satisfying the following:

1. For some $n \geq 0$ and some $m_0, \ldots, m_n, m'_0, \ldots, m'_n \in \mathbb{Z}$,

   $$X = \{\langle a, b\rangle : 0 \leq a \leq n \text{ and } m_a \leq b \leq m'_a\}.$$

2. $g, h : \{0, \ldots, n\} \to \mathbb{Z}$.
3. $m_a \leq g(a) \leq 0 \leq h(a) \leq m'_a$.
4. $V : PV \to \mathcal{P}(X)$.

Note that $\langle a, g(a)\rangle \in X$ and $\langle a, h(a)\rangle \in X$. Given a premodel $M = \langle X, g, h, V\rangle$, we define $n_X = max\{a : \exists b, \langle a, b\rangle \in X\}$. And for each $a \in \{0, \ldots, n_X\}$, we define $m_a = min\{b : \langle a, b\rangle \in X\}$ and $m'_a = max\{b : \langle a, b\rangle \in X\}$. Note that $m_a \leq g(a) \leq 0 \leq h(a) \leq m'_a$ for each $a \leq n_X$. Also note that every premodel is finite.

Definition 3.3.9. Given a premodel $M = \langle X, g, h, V\rangle$, we define the dynamic topological model $M' = \langle X', f', V'\rangle$ *generated by M* as follows:

$$\begin{aligned} X' &= \{0, \dots, n_X\} \times \mathbb{Z}, \quad \text{with the trivial topology} \\ f'(\langle a, b\rangle) &= \langle a, b+1\rangle, \quad \text{for } a = \{0, \dots, n_X\} \text{ and } b \in \mathbb{Z} \\ V'(p) &= V(p) \cup \\ &\quad \{\langle a, b - k((g(a) - m_a) + 1)\rangle : \langle a, b\rangle \in V(p) \text{ and } b \le g(a) \text{ and } k \ge 1\} \cup \\ &\quad \{\langle a, b + k((m'_a - h(a)) + 1)\rangle : \langle a, b\rangle \in V(p) \text{ and } b \ge h(a) \text{ and } k \ge 1\}. \end{aligned}$$

Definition 3.3.10. If M is a premodel and A is a formula, we say that $M \vDash A$ iff $M' \vDash A$ where M' is the dynamic topological model generated by M.

Theorem 3.3.11. *$A \in$ S5Ht iff $M \vDash A$, for every premodel M.*

Proof. The ($\Rightarrow$) direction of the biconditional is soundness of S5Ht for premodels. This follows from soundness of S5Ht for dynamic topological models $\langle X, f, V\rangle$, where X is trivial and f is a homeomorphism, because every dynamic topological model generated by a premodel is of this kind. The ($\Leftarrow$) direction of the biconditional follows from Theorem 3.3.6 and the fact that the model M_α, defined in Definition 3.3.5, is generated by some premodel. □

Corollary 3.3.12. S5Ht *is decidable.*

3.4. *Completeness of* S5C *and* S5H*: Common elements*

The completeness proofs for S5C and S5H have some elements in common. They also require modifying the approach we have taken so far in important ways. For starters, for some of our results it will not suffice that the set Φ be closed under subformulas: we will add an additional closure condition, which we explain presently. For this subsection, we assume that the logic L is either S5C or S5H.

Suppose that Φ is a finite set of formulas (closed under subformulas or not). We define the *modal part of* Φ as follows:

$$\Phi_M =_{df} \{A \in \Phi : A \text{ is a modal formula}\}.$$

And we define the *nonmodal part of* Φ as follows:

$$\Phi_{NM} =_{df} \Phi - \Phi_M.$$

Suppose that α is a Φ-atom. We have already defined the modal part of α as follows:

$$\alpha_M =_{df} \{\pm A \in \alpha : A \text{ is a modal formula}\}.$$

We define the *nonmodal part of* α as follows:

$$\alpha_{NM} =_{df} \alpha - \alpha_M.$$

Notice that if α is a Φ-atom then α_{NM} is a Φ_{NM}-atom.

Suppose that Φ is a finite set of formulas (closed under subformulas or not). Suppose that $A \in \Phi_{NM}$ and that α is a Φ-atom. We say that *A nonmodally dominates* α iff A is consistent and $(A \supset \alpha_{NM}) \in$ L. Finally, we say that Φ is *closed* iff both

1. Φ is closed under subformulas; and
2. for every Φ-atom α, there is some $A \in \Phi_{NM}$ that nonmodally dominates α, and such that $\Diamond A \in \Phi$ (i.e. such that $\neg\Box\neg A \in \Phi$).

Lemma 3.4.1. *For every formula A there is a finite closed set Φ of formulas such that $A \in \Phi$.*

Proof. Suppose that A is a formula. First, let Ψ be the set of subformulas of A. And let $\Psi_{NM} = \{B_1, \dots, B_n\}$. We will now define a large number of conjunctions of the members of Ψ_{NM}. To define these, let $\mathcal{S}$ be the set of sequences of 0s and 1s of length between 1 and n. For $s \in \mathcal{S}$, let $ln(s)$ be the length of s, and if $ln(s) < n$, then let $s0$ [$s1$] be s concatenated with 0 [1]. We define A_s for each sequence $s \in \mathcal{S}$:

$$\begin{aligned} A_0 &= \neg B_1 \\ A_1 &= B_1 \\ A_{s0} &= (A_s \,\&\, \neg B_{ln(s)+1}) \\ A_{s1} &= (A_s \,\&\, B_{ln(s)+1}). \end{aligned}$$

Thus, for example $A_{11001} = ((((B_1 \,\&\, B_2) \,\&\, \neg B_3) \,\&\, \neg B_4) \,\&\, B_5)$. Now we define the set Φ as follows:

$$\begin{aligned} \Phi &= \Psi_M \cup \{B_1, \dots, B_n, \neg B_1, \dots, \neg B_n\} \cup \{A_s : s \in \mathcal{S}\} \\ &\quad \cup \{\neg A_s : s \in \mathcal{S} \text{ and } ln(s) = n\} \\ &\quad \cup \{\Box\neg A_s : s \in \mathcal{S} \text{ and } ln(s) = n\} \\ &\quad \cup \{\neg\Box\neg A_s : s \in \mathcal{S} \text{ and } ln(s) = n\}. \end{aligned}$$

Our set Φ is clearly both finite and closed under subformulas. We still have to show that for every Φ-atom α, there is some $A \in \Phi_{NM}$ that nonmodally dominates α, and such that $\Diamond A \in \Phi$. So suppose that α is a Φ-atom. Let α' be the following subset of α: $\alpha' = \{\pm_i B_i : B_i \in \Psi_{NM} \text{ and } \pm_i B_i \in \alpha\}$. Here $\pm_i B_i$ is either $+B_i$ or $-B_i$ depending on which of these two is in α (exactly one is). Note that α' is a Ψ_{NM}-atom. Let s_α be the member of $\mathcal{S}$ determined as follows: the ith member of s_α is 1 if $+B_i \in \alpha$ and is 0 if $-B_i \in \alpha$. Note that $\Diamond A_{s_\alpha} \in \Phi$, by the definition of Φ. It now suffices to show that A_{s_α} nonmodally dominates α. First, notice that A_{s_α} is consistent, since α is consistent. So now it suffices to show that for every signed formula $\pm B \in \alpha_{NM}$ we have $(A_{s_\alpha} \supset \pm B) \in \mathrm{L}$. Here $\pm B$ is $+B$ if $+B \in \alpha_{NM}$ and $-B$ if $-B \in \alpha_{NM}$.

So choose $\pm B \in \alpha_{NM}$. Then $B \in \Phi_{NM}$. So,

$$B \in \{B_1, \ldots, B_n, \neg B_1, \ldots, \neg B_n\} \cup \{A_s : s \in \mathcal{S}\} \\ \cup \{\neg A_s : s \in \mathcal{S} \text{ and } ln(s) = n\}.$$

We consider four cases.

(Case 1) $B = B_i$, for some i. If $+B_i \in \alpha$, then B_i is one of the conjuncts of A_{s_α}. And if $-B_i \in \alpha$, then $\neg B_i$ is one of the conjuncts of A_{s_α}. In either case, the formula corresponding to the signed formula $\pm B$ is a conjunct of A_{s_α}. So $(A_{s_\alpha} \supset \pm B) \in \mathrm{L}$.

(Case 2) $B = \neg B_i$, for some i. If $+\neg B_i \in \alpha$, then $-B_i \in \alpha$, so that B is one of the conjuncts of A_{s_α}. And if $-\neg B_i \in \alpha$, then $+B_i \in \alpha$, so that $\neg B$ is the double negation of one of the conjuncts of A_{s_α}. In either case, the formula corresponding to the signed formula $\pm B$ is a conjunct, or the double negation of a conjunct, of A_{s_α}. So $(A_{s_\alpha} \supset \pm B) \in \mathrm{L}$.

(Case 3) $B = A_s$ for some $s \in \mathcal{S}$. If s is an initial segment of s_α, then A_s is a conjunct of A_{s_α} and $+A_s \in \alpha$. Thus $+B \in \alpha$ and $(A_{s_\alpha} \supset +B) \in \mathrm{L}$. If s is not an initial segment of α, then $-A_s \in \alpha$ and $(A_{s_\alpha} \supset \neg A_s) \in \mathrm{L}$. So $-B \in \alpha$ and $(A_{s_\alpha} \supset \neg B) \in \mathrm{L}$.

(Case 4) $B = \neg A_s$ for some $s \in \mathcal{S}$ with $ln(s) = n$. If $s = s_\alpha$, then $+A_s = +A_{s_\alpha} \in \alpha$ so that $-B \in \alpha$ and $(A_{s_\alpha} \supset \neg B) \in \mathrm{L}$. If $s \neq s_\alpha$, then $-A_s \in \alpha$ so that $+B \in \alpha$ and $(A_{s_\alpha} \supset B) \in \mathrm{L}$. □

Lemma 3.4.2. *Suppose that Φ is a closed finite set of formulas, that α is a Φ-atom, that $A \in \Phi_{NM}$ and that A nonmodally dominates α. Then $A \in \alpha$.*

Proof. Suppose not. Then $-A \in \alpha$. So $(A \supset \neg A) \in \mathrm{L}$. So $\neg A \in \mathrm{L}$. But this contradicts the consistency of A. □

Suppose that Φ is a closed finite set of formulas. Recall the relation S and the equivalence relation R defined on Φ-atoms, and recall that we are using the notation $|\alpha|_R$ for the equivalence class determined by the Φ-atom α.

Lemma 3.4.3. *Suppose that Φ is a closed finite set of formulas. Suppose that α, β and γ are Φ-atoms such that $\alpha S \beta$ and $\alpha R \gamma$. Then there is a Φ-atom δ such that $\beta R \delta$ and $\gamma S \delta$. Thus the bottom right corner of the square on the left can be filled in as indicated:*

$$\begin{array}{ccc} \alpha & S & \beta \\ R & & R \\ \gamma & S & ?? \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \alpha & S & \beta \\ R & & R \\ \gamma & S & \delta \end{array}$$

Proof. Suppose that α, β and γ are Φ-atoms such that $\alpha S \beta$ and $\alpha R \gamma$. Since $\alpha S \beta$, the formula $(\alpha \,\&\, \circ\beta)$ is consistent. So the formula $(\alpha_M \,\&\, \circ\beta_M)$ is consistent. Since $\alpha R \gamma$, we have $\alpha_M = \gamma_M$. So the formula $(\gamma_M \,\&\, \circ\beta_M)$ is consistent. We claim that

$$(\gamma \,\&\, \circ\beta_M) \text{ is consistent.} \qquad (\dagger)$$

To see this, suppose not. Let $\gamma_{NM} = \gamma - \gamma_M$. So $(\gamma_{NM} \,\&\, \gamma_M \,\&\, \circ\beta_M)$ is inconsistent. By the closure of Φ, we can choose a formula A such that A nonmodally dominates γ. In other words, A is consistent, $\Diamond A \in \Phi$ and $(A \supset \gamma_{NM}) \in \mathrm{L}$. So $(A \,\&\, \gamma_M \,\&\, \circ\beta_M)$ is inconsistent. So $((\gamma_M \,\&\, \circ\beta_M) \supset \neg A) \in \mathrm{L}$. So $((\Box\gamma_M \,\&\, \Box\circ\beta_M) \supset \Box\neg A) \in \mathrm{L}$. So $((\Box\gamma_M \,\&\, \circ\Box\beta_M) \supset \neg\Diamond A) \in \mathrm{L}$. Recall that $(\gamma_M \equiv \Box\gamma_M) \in \mathrm{L}$ and $(\beta_M \equiv \Box\beta_M) \in \mathrm{L}$. So $((\gamma_M \,\&\, \circ\beta_M) \supset \neg\Diamond A) \in \mathrm{L}$. So $(\Diamond A \,\&\, \gamma_M \,\&\, \circ\beta_M)$ is inconsistent. But since A nonmodally dominates γ, we have $+A \in \gamma$. So $+\Diamond A \in \gamma$, since $\Diamond A \in \Phi$. So $+\Diamond A \in \gamma_M$. So, $(\gamma_M \,\&\, \circ\beta_M)$ is inconsistent. But we have already noted that $(\gamma_M \,\&\, \circ\beta_M)$ is consistent. This proves (†).

Given the consistency of $(\gamma \,\&\, \circ\beta_M)$, we can add signed nonmodal formulas to the set β_M until we get a Φ-atom δ with $\delta_M = \beta_M$ and with $(\gamma \,\&\, \circ\delta)$ consistent. □

Suppose that Φ is a finite set of formulas. A *Φ-cluster* is a function $\mathfrak{f} : |\alpha|_R \to |\beta|_R$, for some α and some β such that $\gamma\ S\ \mathfrak{f}(\gamma)$ for each $\gamma \in |\alpha|_R$. Given a Φ-cluster $\mathfrak{f}$, we use the notation $dom(\mathfrak{f})$ and $range(\mathfrak{f})$ for the domain and range of $\mathfrak{f}$. The Φ-cluster $\mathfrak{f}$ *coheres with* the Φ-cluster $\mathfrak{g}$ iff $range(\mathfrak{f}) \subseteq dom(\mathfrak{g})$.

A finite sequence $\langle \mathfrak{f}_i \rangle_{i=0}^n$ [an infinite sequence $\langle \mathfrak{f}_i \rangle_{i \geq 0}$, a bi-infinite sequence $\langle \mathfrak{f}_i \rangle_{i \in \mathbb{Z}}$] of Φ-clusters is *coherent* iff $\mathfrak{f}_i$ coheres with $\mathfrak{f}_{i+1}$, for each $i \geq 0$ and $< n$ [for each $i \geq 0$, for each $i \in \mathbb{Z}$]. Suppose that $\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i=0}^n$ [$\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i \geq 0}$, $\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i \in \mathbb{Z}}$] is a coherent finite [infinite, bi-infinite] sequence of Φ-clusters, that $i \geq 0$ and $\leq n$ [$i \geq 0$, $i \in \mathbb{Z}$], that $j \geq i$ and $\leq n$ [$j \geq i$] and that $\alpha \in dom(\mathfrak{f}_i)$. We define $\mathfrak{F}^{i \to j}(\alpha)$ as follows:

$$\mathfrak{F}^{i \to i}(\alpha) = \alpha$$
$$\mathfrak{F}^{i \to j+1}(\alpha) = \mathfrak{f}_j(\mathfrak{F}^{i \to j}(\alpha)).$$

Note that $\mathfrak{F}^{i \to j} : dom(\mathfrak{f}_i) \to dom(\mathfrak{f}_j)$. An infinite sequence $\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i \geq 0}$ [a bi-infinite sequence $\langle \mathfrak{f}_i \rangle_{i \in \mathbb{Z}}$] of Φ-clusters is *$*$-complete* iff for every $i \geq 0$ [$i \in \mathbb{Z}$], for every $\alpha \in dom(\mathfrak{f}_i)$ and for every formula A, if $-*A \in \alpha$ then there is some $j \geq i$ such that $-A \in \mathfrak{F}^{i \to j}(\alpha)$. A finite sequence $\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i=0}^n$ *witnesses* the signed formula $-*A$ iff for every $\alpha \in dom(\mathfrak{f}_0)$ if $-*A \in \alpha$ then there is some $m \leq n$ such that $-A \in \mathfrak{F}^{0 \to m}(\alpha)$.

We have not shown that there *are* any Φ-clusters, let alone any eventually periodic $*$-complete coherent infinite sequences of Φ-clusters. But we will.

Lemma 3.4.4. *Suppose that Φ is a closed finite set of formulas, and that α and β are Φ-atoms with $\alpha S\beta$. Then there is a Φ-cluster $\mathfrak{f} : |\alpha|_R \to |\beta|_R$ with $\mathfrak{f}(\alpha) = \beta$.*

Proof. This is a direct consequence of Lemma 3.4.3. □

Lemma 3.4.5. *Suppose that Φ is a closed finite set of formulas, and that α and β are Φ-atoms with $\alpha S^\sharp\beta$. Then there is a coherent finite sequence $\mathfrak{F} = \langle \mathfrak{f}_i\rangle_{i=0}^n$ of Φ-clusters such that $dom(\mathfrak{f}_0) = |\alpha|_R$ and $dom(\mathfrak{f}_n) = |\beta|_R$ and $\mathfrak{F}^{0\to n}(\alpha) = \beta$.*

Proof. Since $\alpha S^\sharp\beta$, there is some S-sequence $\alpha_0, \ldots, \alpha_n$ of atoms with $\alpha_0 = \alpha$ and $\alpha_n = \beta$. Choose some α_{n+1} so that $\alpha_n S\alpha_{n+1}$. For each $k \le n$, choose a Φ-cluster $\mathfrak{f}_k : |\alpha_k|_R \to |\alpha_{k+1}|_R$ with $\mathfrak{f}(\alpha_k) = \alpha_{k+1}$. Then the finite sequence $\mathfrak{F} = \langle \mathfrak{f}_i\rangle_{i=0}^n$ of Φ–clusters is as desired. □

Lemma 3.4.6. *Suppose that Φ is a finite set of formulas closed under subformulas, that α is a Φ-atom, and that $*A \in \Phi$. Then there is a coherent finite sequence $\mathfrak{F} = \langle \mathfrak{f}_i\rangle_{i=0}^n$ of Φ-clusters, with $dom(\mathfrak{f}_0) = |\alpha|_R$, that witnesses the signed formula $-*A$.*

Proof. List $|\alpha|_R$ as follows: $\{\alpha_1, \ldots, \alpha_n\}$. We will define n increasingly long coherent finite sequences of Φ-clusters, $\mathfrak{F}_1 = \langle \mathfrak{f}_i\rangle_{i=0}^{m_1}, \ldots, \mathfrak{F}_n = \langle \mathfrak{f}_i\rangle_{i=0}^{m_n}$. For each $k \in \{1, \ldots, n\}$, we will ensure the following:

$$\text{if } 1 \le j \le k \text{ and } -*A \in \alpha_j \text{ then there is an } i \le m_k \text{ such that } -A \in \mathfrak{F}_k^{0\to i}(\alpha_j). \qquad (\dagger)$$

$\mathfrak{F}$ will then be the last of these sequence, i.e. $\mathfrak{F}_n$.

Define $\mathfrak{F}_1$ as follows. Find a Φ-atom β_1 such that $\alpha_1 S^\sharp\beta_1$ and if $-*A \in \alpha_1$ then $-A \in \beta_1$. By Lemma 3.4.5, there is a coherent finite sequence $\mathfrak{F}_1 = \langle \mathfrak{f}_i\rangle_{i=0}^{m_1}$ of Φ-clusters such that $dom(\mathfrak{f}_0) = |\alpha_1|_R = |\alpha|_R$ and $dom(\mathfrak{f}_{m_1}) = |\beta_1|_R$ and $\mathfrak{F}_1^{0\to m_1}(\alpha_1) = \beta_1$.

Suppose that the coherent finite sequence $\mathfrak{F}_k = \langle \mathfrak{f}_i\rangle_{i=0}^{m_k}$ has been defined so that (†) holds, and that $k < n$. Define $\mathfrak{F}_{k+1}$ as follows, consider two cases.

(Case 1) Suppose that there is an $i \le m_k$ such that if $-*A \in \alpha_{k+1}$ then $-A \in \mathfrak{F}_k^{0\to i}(\alpha_{k+1})$. Then let $m_{k+1} = m_k$ and let $\mathfrak{F}_{k+1} = \mathfrak{F}_k$.

(Case 2) Suppose that there is no $i \le m_k$ such that if $-*A \in \alpha_{k+1}$ then $-A \in \mathfrak{F}_k^{0\to i}(\alpha_{k+1})$. Then $-*A \in \alpha_{k+1}$ and for every $i \le m_k$, we have $+A \in \mathfrak{F}_k^{0\to i}(\alpha_{k+1})$. We claim that

$$-*A \in \mathfrak{F}_k^{0\to i}(\alpha_{k+1}), \quad \text{for every } i \le m_k. \qquad (\ddagger)$$

The argument for (‡) is pretty much the same as the argument, in the proof of Lemma 3.1.8, for the claim labelled (‡) there: we do not repeat that argument here. Given (‡), we have $-*A \in \mathfrak{F}_k^{0\to m_k}(\alpha_{k+1}) \in dom(\mathfrak{f}_{m_k})$. Indeed, we have $-*A \in \mathfrak{f}_k(\mathfrak{F}_k^{0\to m_k}(\alpha_{k+1})) \in range(\mathfrak{f}_{m_k})$. It will simplify things if we let $\alpha' = \mathfrak{f}_k(\mathfrak{F}_k^{0\to m_k}(\alpha_{k+1}))$. So $-*A \in \alpha' \in range(\mathfrak{f}_{m_k})$.

Find a Φ-atom β' such that $\alpha' S^\sharp\beta'$ and $-A \in \beta'$. By Lemma 3.4.5, there is a coherent finite sequence $\mathfrak{G} = \langle \mathfrak{g}_i\rangle_{i=0}^u$ of Φ-clusters such that $dom(\mathfrak{g}_0) = |\alpha'|_R$ and $dom(\mathfrak{g}_u) = |\beta'|_R$ and $\mathfrak{G}^{0\to u}(\alpha') = \beta'$. We define the sequence $\mathfrak{F}_{k+1} = \langle \mathfrak{f}_i\rangle_{i=0}^{m_{k+1}}$ by gluing $\mathfrak{G}$ at the end of $\mathfrak{F}_k$. More precisely, let $m_{k+1} = m_k + u + 1$ and for $i \in \{m_k + 1, \ldots, m_{k+1}\}$, let $\mathfrak{f}_i = \mathfrak{g}_{i-(m_k+1)}$.

In either Case 1 or Case 2, note that $\mathfrak{F}_{k+1} = \langle \mathfrak{f}_i\rangle_{i=0}^{m_{k+1}}$ is a coherent finite sequence and that

$$\text{if } 1 \le j \le k+1 \text{ and } -*A \in \alpha_j \text{ then there is an } i \le m_{k+1} \text{ such that } -A \in \mathfrak{F}_{k+1}^{0\to i}(\alpha_j). \qquad (\dagger\dagger)$$

Given that $\mathfrak{F}_{k+1}$ was built from $\mathfrak{F}_k$ so as to ensure (††), we conclude that we have successfully ensured (†) for each k. Now let $\mathfrak{F} = \mathfrak{F}_n$. Note that $\mathfrak{F}$ is a coherent finite sequence of Φ-clusters and that $dom(\mathfrak{f}_0) = |\alpha|_R$. Also, since (†) holds for $k = n$, the sequence $\mathfrak{F}$ witnesses $-*A$. □

Lemma 3.4.7. *Suppose that Φ is a closed finite set of formulas and that α is a Φ-atom. Then there is a $*$-complete coherent infinite sequence $\langle \mathfrak{f}_i\rangle_{i\ge 0}$ of Φ-clusters, such that $\alpha \in dom(\mathfrak{f}_0)$.*

Proof (*This Proof is Very Similar to the Proof of the Analogous Lemma* 3.1.8). If Φ contains no formulas of the form $*A$, then it is easy. First, by Lemma 3.4.4, we can choose a Φ-cluster $\mathfrak{f}_0$ with $\alpha \in dom(\mathfrak{f}_0)$. For each $n \ge 0$, if we have a Φ-cluster $\mathfrak{f}_n$, then, by Lemma 3.4.4, we can choose a Φ-cluster $\mathfrak{f}_{n+1}$ with $range(\mathfrak{f}_n) \subseteq dom(\mathfrak{f}_{n+1})$. The sequence $\langle \mathfrak{f}_i\rangle_{i\ge 0}$ of Φ-clusters will be infinite, $*$-complete, and coherent.

Otherwise, Φ contains some formula(s) of the form $*A$. List the set $\{-*A : *A \in \Phi\}$ as follows: $\{-*A_0, \ldots, -*A_{v-1}\}$, where $v \ge 1$. For any $j, k \ge 1$, let $rem(k, j)$ be the remainder of k divided by j; for example $rem(47, 7) = 5$. And, for each $k \ge 0$, define $-*A_k = -*A_{rem(v,k)}$. Thus, the sequence $\langle -*A_i\rangle_{i\ge 0}$ looks like this:

$$-*A_0, \ldots, -*A_{v-1}, -*A_0, \ldots, -*A_{v-1}, -*A_0, \ldots, -*A_{v-1}, \ldots.$$

For each $k \ge 0$, we will define by induction on k a coherent finite sequence $\mathfrak{F}_k = \langle \mathfrak{f}_i^k\rangle_{i=0}^{m_k}$, for some m_k, that witnesses the signed formula $-*A_k$. By Lemma 3.4.6, we can choose a sequence $\mathfrak{F}_0 = \langle \mathfrak{f}_i^0\rangle_{i=0}^{m_0}$ that witnesses the signed formula $-*A_0$, with $dom(\mathfrak{f}_0^0) = |\alpha|_R$. Assume that we have defined a sequence $\mathfrak{F}_k = \langle \mathfrak{f}_i^k\rangle_{i=0}^{m_k}$ that witnesses the signed formula $-*A_k$. Choose any

$\beta \in range(\mathfrak{f}^k_{m_k})$. By Lemma 3.4.6, we can choose a sequence $\mathfrak{F}_{k+1} = \langle \mathfrak{f}^{k+1}_i \rangle^{m_{k+1}}_{i=0}$ that witnesses the signed formula $-*A_{k+1}$, with $dom(\mathfrak{f}^{k+1}_0) = |\beta|_R$. Notice that $range(\mathfrak{f}^k_{m_k}) \subseteq dom(\mathfrak{f}^{k+1}_0)$.

Now define the infinite sequence $\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i \geq 0}$ by gluing together the sequences $\mathfrak{F}_k$ as follows:

$$\mathfrak{f}^0_0, \ldots, \mathfrak{f}^0_{m_0}, \mathfrak{f}^1_0, \ldots, \mathfrak{f}^1_{m_1}, \mathfrak{f}^2_0, \ldots, \mathfrak{f}^2_{m_2}, \mathfrak{f}^3_0, \ldots, \mathfrak{f}^3_{m_3}, \ldots.$$

To be more precise, for each $k \geq 0$, let $n_k = k + \sum^k_{i=0} m_k$. For each $i \geq 0$, let $k_i = min\{k : i \leq n_k\}$. Finally, let $\mathfrak{f}_i = \mathfrak{f}^{k_i}_{i+m_{k_i}-n_{k_i}}$. Note the following:

$$\begin{aligned} \mathfrak{f}_{n_k - m_k} &= \mathfrak{f}^k_0 \\ \mathfrak{f}_{n_k - m_k + i} &= \mathfrak{f}^k_i, \quad \text{if } i \leq m_k \\ \mathfrak{f}_{n_k} &= \mathfrak{f}^k_{m_k} \\ \mathfrak{f}_{n_k + 1} &= \mathfrak{f}^{k+1}_0. \end{aligned}$$

Also notice that,

$$\text{if } l \leq (n_k - m_k) \text{ and } m \leq m_k \text{ and } \gamma \in dom(\mathfrak{f}_l), \text{ then } \mathfrak{F}^{0 \to m}_k(\mathfrak{F}^{l \to (n_k - m_k)}(\gamma)) = \mathfrak{F}^{l \to (n_k - m_k)+m}(\gamma). \qquad (\star)$$

Clearly $\mathfrak{F} = \langle \mathfrak{f}_i \rangle_{i \geq 0}$ is a coherent infinite sequence whose first member is α. We must still show that $\mathfrak{F}$ is $*$-complete. Suppose not. Then there is some $l \geq 0$ and some $\gamma \in dom(\mathfrak{f}_l)$ and some $-*A \in \gamma$ such that

$$+A \in \mathfrak{F}^{l \to j}(\gamma) \quad textrmforeveryj \geq l. \qquad (\dagger)$$

We claim that

$$-*A \in \mathfrak{F}^{l \to j}(\gamma) \quad \text{for every } j \geq l. \qquad (\ddagger)$$

The argument for ($\ddagger$) is pretty much the same as the argument, in the proof of Lemma 3.1.8, for the claim labelled ($\ddagger$) there: we do not repeat that argument here.

Choose some $k \geq l$ for which $-*A = -*A_k$. Note that $l \leq k \leq n_k - m_k$. So $-*A_k \in \mathfrak{F}^{l \to (n_k - m_k)}(\gamma)$, by ($\ddagger$). Let $\delta = \mathfrak{F}^{l \to (n_k - m_k)}(\gamma)$. So $-*A_k \in \delta \in dom(\mathfrak{f}_{n_k - m_k})$. Also, as noted above, $\mathfrak{f}_{n_k - m_k} = \mathfrak{f}^k_0$. So $-*A_k \in \delta \in dom(\mathfrak{f}^k_0)$. Recall that the sequence $\mathfrak{F}_k = \langle \mathfrak{f}^k_i \rangle^{m_k}_{i=0}$ witnesses the signed formula $-*A_k$. So there is some $m \leq m_k$ such that $-A \in \mathfrak{F}^{0 \to m}_k(\delta)$. Sos $-A \in \mathfrak{F}^{0 \to m}_k(\mathfrak{F}^{l \to (n_k - m_k)}(\gamma))$. So $-A \in \mathfrak{F}^{l \to (n_k - m_k)+m}(\gamma)$, by ($\star$), above. But this contradicts ($\dagger$). □

We can improve on Lemma 3.4.7:

Lemma 3.4.8. *Suppose that Φ is a closed finite set of formulas and that α is a Φ-atom. Then there is an eventually periodic $*$-complete coherent infinite sequence $\langle \mathfrak{f}_i \rangle_{i \geq 0}$ of Φ-clusters, such that $\alpha \in dom(\mathfrak{f}_0)$.*

Proof. By Lemma 3.4.7, there is a $*$-complete coherent infinite sequence $\mathfrak{G} = \langle \mathfrak{g}_i \rangle_{i \geq 0}$ of Φ-clusters, such that $\alpha \in dom(\mathfrak{g}_0)$. We will now define five natural numbers $a \leq b \leq c \leq d \leq e$.

For each $k \geq 0$, Let $\Gamma_k = \{\beta : \beta \in dom(\mathfrak{g}_i)$ for some $i \geq 0$ such that $i \leq k\}$. And let $\Gamma = \{\beta : \beta \in dom(\mathfrak{g}_i)$ for some $i \geq 0\} = \cup_k \Gamma_k$. Note that Γ is finite, since there are finitely many Φ-atoms. So we can let a be the smallest natural number such that $\Gamma_a = \Gamma$.

$$\underbrace{\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_a}_{\substack{\text{every member of } \Gamma \\ \text{is in the domain of} \\ \text{one of these clusters.}}}, \mathfrak{g}_{a+1}, \mathfrak{g}_{a+2}, \ldots$$

Since there are finitely Φ-clusters, the sequence $\langle \mathfrak{g}_i \rangle_{i \geq 0}$ has a cofinality point. Let b be the smallest cofinality point greater than a. So for each $i \geq b$, the Φ-cluster $\mathfrak{g}_i$ is cofinal in the sequence $\langle \mathfrak{g}_i \rangle_{i \geq 0}$.

$$\underbrace{\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_{b-1}}_{\substack{\text{every member of } \Gamma \\ \text{is in the domain of} \\ \text{one of these clusters.}}}, \quad \underbrace{\mathfrak{g}_b, \mathfrak{g}_{b+1}, \mathfrak{g}_{b+2}, \ldots}_{\substack{\text{every cofinal } \Phi\text{-cluster} \\ \text{is among these clusters}}}$$

For each $k \geq b$, et $\Sigma_k = \{\beta : \beta \in dom(\mathfrak{g}_i)$ for some $i \geq b$ such that $i \leq k\}$. And let $\Sigma = \{\beta : \beta \in dom(\mathfrak{g}_i)$ for some $i \geq b\} = \cup_k \Gamma_k$. Note that Σ is the set of all Φ-clusters cofinal in the sequence $\langle \mathfrak{g}_i \rangle_{i \geq 0}$. So we can let c be the smallest natural number greater than b such that $\Sigma_c = \Sigma$.

$$\underbrace{\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_{b-1}}_{\substack{\text{every member of } \Gamma \\ \text{is in the domain of} \\ \text{one of these clusters.}}}, \quad \underbrace{\mathfrak{g}_b, \mathfrak{g}_{b+1}, \mathfrak{g}_{b+2}, \ldots, \mathfrak{g}_c}_{\substack{\text{every cofinal } \Phi\text{-cluster} \\ \text{is among these clusters}}}, \mathfrak{g}_{c+1}, \ldots$$

Suppose that $-*A \in \beta \in dom(\mathfrak{g}_i)$ where $i \leq c$. Since $\mathfrak{G}$ is $*$-complete, there is a $j \geq i$ such that $-A \in \mathfrak{G}^{i\to j}(\beta)$. Since there are only finitely formulas in Φ and since there are only finitely many Φ-atoms, there is a number $d > c$ with the following property: For each formula A and each Φ-atom β and each $i \leq c$, if $-*A \in \beta \in dom(\mathfrak{g}_i)$ then there is a $j \geq i$ such that both $j < d$ and $-A \in \mathfrak{G}^{i\to j}(\beta)$.

$$\underbrace{\mathfrak{g}_0,\mathfrak{g}_1,\dots,\mathfrak{g}_{b-1}}_{\substack{\text{every member of }\Gamma\\ \text{is in the domain of}\\ \text{one of these clusters.}}},\quad \underbrace{\mathfrak{g}_b,\mathfrak{g}_{b+1},\mathfrak{g}_{b+2},\dots,\mathfrak{g}_c}_{\substack{\text{every cofinal }\Phi\text{-cluster}\\ \text{is among these clusters}}},\quad \mathfrak{g}_{c+1},\dots,\mathfrak{g}_{d-1},\quad \underbrace{\mathfrak{g}_d}_{\substack{(\forall A)(\forall\beta)(\forall i\leq c)\\ (\text{if } -*A\in\beta\in dom(\mathfrak{g}_i)\\ \text{then } \exists j(j\geq i\ \&\ j<d\\ \&\ -A\in\mathfrak{G}^{i\to j}(\beta)))}},\mathfrak{g}_{d+1},\dots$$

Finally, since the Φ-cluster $\mathfrak{g}_b$ is cofinal in the sequence $\mathfrak{G}$, there is an $e \geq d$ such that $\mathfrak{g}_{e+1} = \mathfrak{g}_b$. Note: for each formula A and each Φ-atom β and each $i \leq c$, if $-*A \in \beta \in dom(\mathfrak{g}_i)$ then there is a $j \geq i$ such that both $j < e+1$ and $-A \in \mathfrak{G}^{i\to j}(\beta)$. Also note: for every $i > e$ there is a j such that $j \geq b$ and $j \leq c$ and $\mathfrak{g}_j = \mathfrak{g}_i$.

$$\underbrace{\mathfrak{g}_0,\mathfrak{g}_1,\dots,\mathfrak{g}_{b-1}}_{\substack{\text{every member of }\Gamma\\ \text{is in the domain of}\\ \text{one of these clusters.}}},\quad \underbrace{\mathfrak{g}_b,\mathfrak{g}_{b+1},\mathfrak{g}_{b+2},\dots,\mathfrak{g}_c}_{\substack{\text{every cofinal }\Phi\text{-cluster}\\ \text{is among these clusters}}},\quad \mathfrak{g}_{c+1},\dots,\mathfrak{g}_{e-1},\mathfrak{g}_e,\quad \underbrace{\mathfrak{g}_{e+1}=\mathfrak{g}_b}_{\substack{(\forall A)(\forall\beta)(\forall i\leq c)\\ (\text{if } -*A\in\beta\in dom(\mathfrak{g}_i)\\ \text{then } \exists j(j\geq i\ \&\ j<e+1\\ \&\ -A\in\mathfrak{G}^{i\to j}(\beta)))}},\dots$$

We define our new infinite sequence $\mathfrak{F}$ of Φ-clusters as follows:

$$\underbrace{\mathfrak{g}_0,\mathfrak{g}_1,\dots,\mathfrak{g}_{b-1}}_{\substack{\text{initial}\\ \text{segment}}},\quad \underbrace{\mathfrak{g}_b,\dots,\mathfrak{g}_c,\mathfrak{g}_{c+1},\dots,\mathfrak{g}_e}_{\substack{\text{repeating}\\ \text{segment}}},\quad \underbrace{\mathfrak{g}_b,\dots,\mathfrak{g}_c,\mathfrak{g}_{c+1},\dots,\mathfrak{g}_e}_{\substack{\text{repeating}\\ \text{segment}}},\quad \dots$$

More precisely, let $\mathfrak{F} = \langle \mathfrak{f}_i\rangle_{i\geq 0}$, where for $i \geq 0$,

$$\mathfrak{f}_i = \mathfrak{g}_i, \quad \text{if } i < b; \text{ and}$$
$$\mathfrak{f}_{i+m(1+e-b)} = \mathfrak{g}_i, \quad \text{if } i \geq b \text{ and } i \leq e \text{ and } m \geq 0.$$

Note that $\mathfrak{F}$ is an eventually periodic coherent infinite sequence. $\mathfrak{F}$ is also $*$-complete. To see this, suppose that $-*A \in \beta \in \mathfrak{f}_i$ for some A and some β and some i. We want to show that

$$-A \in \mathfrak{F}^{i\to j}(\beta) \quad \text{for some } j \geq i. \tag{$\star$}$$

Suppose that $(\star)$ is false. Then

$$+A \in \mathfrak{F}^{i\to j}(\beta) \quad \text{for every } j \geq i. \tag{$\dagger$}$$

We claim that

$$-*A \in \mathfrak{F}^{i\to j}(\beta) \quad \text{for every } j \geq i. \tag{$\ddagger$}$$

The argument for $(\ddagger)$ is pretty much the same as the argument, in the proof of Lemma 3.1.8, for the claim labelled $(\ddagger)$ there: we do not repeat that argument here.

Let m be the smallest natural number such that $i \leq b + m(1+e-b)$. And let $b' = b + m(1+e-b)$ and let $\gamma = \mathfrak{F}^{i\to b'}(\beta)$. By $(\ddagger)$, we have $-*A \in \gamma \in dom(\mathfrak{f}_{b'})$. Note also that $\mathfrak{f}_{b'} = \mathfrak{g}_b$. So $-*A \in \gamma \in dom(\mathfrak{g}_b)$. So, since $b \leq c$, for some j we have $j \geq b$ and $j \leq e$ and $-A \in \mathfrak{G}^{b\to j}(\gamma)$. Let $j' = j + m(1 + (e-b))$.

Now, for any $k \in \{b, \dots, j\}$ we have $\mathfrak{f}_{k+m(1+e-b)} = \mathfrak{g}_k$. So

$$\mathfrak{F}^{(b+m(1+e-b))\to(j+m(1+e-b))}(\gamma) = \mathfrak{G}^{b\to j}(\gamma).$$

That is, $\mathfrak{F}^{b'\to j'}(\gamma) = \mathfrak{G}^{b\to j}(\gamma)$. Therefore $-A \in \mathfrak{F}^{b'\to j'}(\gamma) = \mathfrak{F}^{b'\to j'}(\mathfrak{F}^{i\to b'}(\beta)) = \mathfrak{F}^{i\to j'}(\beta)$. But this contradicts $(\dagger)$. □

3.5. *Completeness of* S5C

Definition 3.5.1. Suppose that Φ is a closed finite set of formulas, and that α is a Φ-atom. We will define a finite almost discrete topological space, X_α; a continuous function, f_α on X_α; and a valuation function $V_\alpha : PV \to \mathcal{P}(X_\alpha)$.

First, choose an eventually periodic $*$-complete coherent infinite sequence $\langle \mathfrak{f}_i\rangle_{i\geq 0}$ of Φ-clusters, such that $\alpha \in dom(\mathfrak{f}_0)$. Choose $k \geq 1$ and $l \geq 1$ so that for every $i \geq k$, we have $\mathfrak{f}_{i+l} = \mathfrak{f}_i$. We cut the sequence $\mathfrak{F}$ off at $(k+l)-1$:

$$\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_k, \mathfrak{f}_{k+1}, \dots, \mathfrak{f}_{k+l-1}.$$

We define X_α as follows:

$$X_\alpha = \{\langle i, \beta\rangle : 0 \leq i \leq (k+l-1) \text{ and } \beta \in dom(\mathfrak{f}_i)\}.$$

For each $i \leq (k+l-1)$, define the set O_i as follows: $O_i = \{\langle i, \beta\rangle : \beta \in dom(\mathfrak{f}_i)\}$. The topology we impose on X_α is as follows: a set is *open* iff it is either empty or a union of some of the O_i's. In other words, the O_i's form a basis for our topology. Since our topology has a basis of pairwise disjoint open sets, the space X_α is almost discrete.

We define a function $f_\alpha : X_\alpha \to X_\alpha$ as follows:

$$f_\alpha(\langle i, \beta\rangle) = \begin{cases} \langle i+1, \mathfrak{f}_i(\beta)\rangle, & \text{if } i < (k+l)-1 \\ \langle k, \ \mathfrak{f}_i(\beta)\rangle, & \text{if } i = (k+l)-1. \end{cases}$$

The function f_α is continuous, since the inverse image of every basis set O_i is open. In particular, $f_\alpha^{-1}(O_0) = \emptyset$; $f_\alpha^{-1}(O_k) = O_{k-1} \cup O_{(k+l)-1}$; and if $i \neq 0$ and $i \neq k$ then $f_\alpha^{-1}(O_i) = O_{i-1}$. We define the valuation function V_α as follows:

$$V_\alpha(p) = \{\langle i, \beta\rangle \in X_\alpha : +p \in \beta\}, \quad \text{for each propositional variable } p.$$

Finally, we define the dynamic topological model, $M_\alpha =_{df} \langle X_\alpha, f_\alpha, V_\alpha\rangle$.

The following lemma is analogous to Lemma 3.2.4.

Lemma 3.5.2. *Suppose that Φ is a closed finite set of formulas and that α is a Φ-atom. Suppose that $\langle i, \beta\rangle \in X_\alpha$, that $j \geq 0$ and that $\langle i', \gamma\rangle = f_\alpha^j(\langle i, \beta\rangle)$. Then $\mathfrak{f}_{i'} = \mathfrak{f}_{i+j}$ and $\gamma = \mathfrak{F}^{i\to i+j}(\beta)$.* (Note that the ordered pair $\langle i+j, \gamma\rangle$ need not be in X_α.)

Theorem 3.5.3. *Suppose that Φ is a closed finite set of formulas, and that α is a Φ-atom. And suppose that X_α, f_α, and V_α are defined as in Definition 3.5.1. Then, for each $A \in \Phi$:*

for each $\langle i, \beta\rangle \in X_\alpha$, $\langle i, \beta\rangle \in V_\alpha(A)$ iff $+A \in \beta$.

Proof. By induction on the structure of A. We will use all the notation, terminology and so on in Definition 3.5.1.

(Case 1) $A \in PV$. The result is given by the definition of V_α.

(Case 2) A is of the form $\neg B$. Choose $\langle i, \beta\rangle \in X_\alpha$. Then note: $\langle i, \beta\rangle \in V_\alpha(A)$ iff $\langle i, \beta\rangle \in V_\alpha(\neg B)$ iff $\langle i, \beta\rangle \notin V_\alpha(B)$ iff $+B \notin \beta$ (by the inductive hypothesis) iff $-B \in \beta$ (since β is Φ-complete) iff $+\neg B \in \beta$ (since β is Φ-complete and consistent) iff $+A \in \beta$.

(Case 3) A is of the form $(B \,\&\, C)$. Choose $\langle i, \beta\rangle \in X_\alpha$. Then note: $\langle i, \beta\rangle \in V_\alpha(A)$ iff $\langle i, \beta\rangle \in V_\alpha(B \,\&\, C)$ iff $\langle i, \beta\rangle \in V_\alpha(B)$ and $\langle i, \beta\rangle \in V_\alpha(C)$ iff $+B \in \beta$ or $+C \in \beta$ (by the inductive hypothesis) iff $+(B \,\&\, C) \in \beta$ (since β is Φ-complete and consistent) iff $+A \in \beta$.

(Case 4) A is of the form $\Box B$. Choose $\langle i, \beta\rangle \in X_\alpha$. So $\beta \in dom(f_i) = |\beta|_R$. By Lemma 3.1.1, we have

$+\Box B \in \beta$ iff, for every $\gamma \in |\beta|_R$, $+B \in \gamma$.

Thus,

(1)	$+\Box B \in \beta$ iff, for every $\gamma \in dom(\mathfrak{f}_i)$, $+B \in \gamma$.	
(2)	$+\Box B \in \beta$ iff, for every $\gamma \in dom(\mathfrak{f}_i)$, $\langle, i, \gamma\rangle \in V(B)$	by IH.
(3)	$+\Box B \in \beta$ iff $O_i \subseteq V_\alpha(B)$	by the def'n of O_i.
(4)	$+\Box B \in \beta$ iff $O_i \subseteq Int(V_\alpha(B))$	since O_i is open.

Now note that O_i is the *smallest* open set containing $\langle i, \beta\rangle$. Thus, for any $Y \subseteq X_\alpha$, we have $O_i \subseteq O$ iff $\langle i, \beta\rangle \in O$. In particular, $O_i \subseteq Int(V_\alpha(B))$ iff $\langle i, \beta\rangle \in Int(V_\alpha(B)) = V_\alpha(\Box B)$. So $+\Box B \in \beta$ iff $\langle i, \beta\rangle \in V_\alpha(\Box B)$.

(Case 5) A is of the form $\circ B$. Choose $\langle i, \beta\rangle \in X_\alpha$. We consider two cases: (5.1) $i < (k+l)-1$, and (5.2) $i = (k+l)-1$. (Case 5.1) $\langle i, \beta\rangle \in V_\alpha(A)$ iff $\langle i, \beta\rangle \in V_\alpha(\circ B)$ iff $f_\alpha(\langle i, \beta\rangle) \in V_\alpha(B)$ iff $\langle i+1, \mathfrak{f}_i(\beta)\rangle \in V_\alpha(B)$ iff $+B \in \mathfrak{f}_i(\beta)$ (by IH) iff $+\circ B \in \beta$ (by Lemma 3.1.2, since $\beta\, S\, \mathfrak{f}_i(\beta)$) iff $+A \in \beta$. (Case 5.2) $\langle i, \beta\rangle \in V_\alpha(A)$ iff $\langle k+l-1, \beta\rangle \in V_\alpha(\circ B)$ iff $f_\alpha(\langle k+l-1, \beta\rangle) \in V_\alpha(B)$ iff $\langle k, \mathfrak{f}_i(\beta)\rangle$ iff $+B \in \mathfrak{f}_i(\beta)$ iff $\circ B \in \beta$ (by Lemma 3.1.2, since $\beta\, S\, \mathfrak{f}_i(\beta)$) iff $+A \in \beta$.

(Case 6) A is of the form $*B$. Choose $\langle i, \beta\rangle \in X_\alpha$. We consider both directions of our biconditional separately.

($\Rightarrow$) We prove the contrapositive. So suppose that $+A \notin \beta$. Then $+*B \notin \beta$. So $-*B \in \beta \in dom(\mathfrak{f}_i)$. So, since the sequence $\mathfrak{F} = \langle \mathfrak{f}_i\rangle_{i\geq 0}$ is $*$-complete, we have $-B \in \mathfrak{F}^{i\to j}(\beta)$, for some $j \geq i$. Let $\langle i', \gamma\rangle = f_\alpha^{j-i}(\langle i, \beta\rangle)$. Then $\mathfrak{f}_{i'} = \mathfrak{f}_j$ and $\gamma = \mathfrak{F}^{i\to j}(\beta)$, by Lemma 3.5.2. So $+B \notin \gamma$. So $\langle i', \gamma\rangle \notin V_\alpha(B)$, by IH. So $f_\alpha^{j-i}(\langle i, \beta\rangle) \notin V_\alpha(B)$. So $\langle i, \beta\rangle \notin V_\alpha(*B)$. So $\langle i, \beta\rangle \notin V_\alpha(A)$.

($\Leftarrow$) We prove the contrapositive. So suppose that $\langle i, \beta\rangle \notin V_\alpha(A)$. Then $\langle i, \beta\rangle \notin V_\alpha(*B)$. So $f_\alpha^j(\langle i, \beta\rangle) \notin V_\alpha(B)$ for some $j \geq 0$. Let $\langle i', \gamma\rangle = f_\alpha^j(\langle i, \beta\rangle)$. Then $\langle i', \gamma\rangle \notin V_\alpha(B)$. So $+B \notin \gamma$, by IH. So $+B \notin \mathfrak{F}^{i\to i+j}(\beta)$, by Lemma 3.5.2. So $+*B \notin \beta$. Now note that $\beta\, S^\sharp\, \mathfrak{F}^{i\to i+j}(\beta)$. So $+*B \notin \beta$, by Lemma 3.1.5. So $+A \notin \beta$. □

Corollary 3.5.4. *Suppose that $A \notin$ S5C. Then there is some finite almost discrete topological space X such that $X \not\models A$.*

Proof. Suppose that $A \notin$ S5Ct. By Lemma 3.4.1, there is a finite closed set Φ of formulas such that $A \in \Phi$. Choose a Φ-atom α with $-A \in \alpha$. Define the topological model $M_\alpha = \langle X_\alpha, f_\alpha, V_\alpha\rangle$ as in Definition 3.5.1. By Theorem 3.5.3 and the fact that $\alpha \in dom(\mathfrak{f}_0)$, we have $\langle 0, \alpha\rangle \notin V_\alpha(A)$. So $X_\alpha \not\models A$. And X_α is a finite almost discrete topological space. □

The completeness of S5C for almost discrete topological spaces follows directly from Corollary 3.5.4. Indeed, this Corollary is stronger than completeness: it also entails that S5C has the finite model property. Thus:

Corollary 3.5.5. S5C *is decidable.*

In Section 2.5, we promised a proof of the following:

Theorem 3.5.6. Next removal *is admissible in* S5C.

Proof. Suppose, for a reductio, that $\circ A \in$ S5C, but $A \notin$ S5C. Since $A \notin$ S5C, there is an almost discrete topological space X, a continuous function $f : X \to X$, and a valuation function $V : PV \to \mathcal{P}(X)$ such that $V(A) \neq X$. Choose some $b \in X - V(A)$. Define a new topological space X', a new continuous function f', and a new valuation function V' as follows. Choose any object $a \notin X$, and let $X' = X \cup \{a\}$, where the following subsets of X are open: the sets $O \subseteq X$ that are open in X, and the sets of the form $O \cup \{a\}$ where $O \subseteq X$ is open in X. Note that X' is an almost discrete topological space. Define f' by extending f to X' as follows: $f'(a) = b$. Note that f' is continuous. And define V' as follows: $V'(p) = V(p)$. It is easy to prove that $V'(B) \cap X = V(B)$, for every formula B. Thus $b \notin V'(A)$. Thus $a \notin V'(\circ A)$. Thus $X' \not\models \circ A$. Thus, by the soundness of S5C for almost discrete spaces, $\circ A \notin$ S5C. But this contradicts our original assumption. □

3.6. *Completeness of* S5H

The completeness proof for S5H borrows ideas from both the completeness proof for S5Ht and the completeness proof for S5C. Two things must be noted right away. The first thing is that S5H fails to satisfy the finite model property in the same sense that S5Ht fails: the formula $(\circ * p \supset * p)$ is not a theorem of S5H, even though it is validated by every model $\langle X, f, V\rangle$ where X is a finite topological space (almost discrete or not) and f is a homeomorphism. The second thing is that it will suffice to prove that S5H is complete for *open onto continuous functions* on almost discrete spaces.[10]

Lemma 3.6.1. *Suppose that $M \not\models A$ where $M = \langle X, f, V\rangle$, where X is an almost discrete topological space, and where f is an open continuous function from X* onto *X. And suppose that $M \not\models A$. Then there is some almost discrete topological space X', some homeomorphism $f' : X' \to X'$ and some valuation function $V' : PV \to \mathcal{P}(X')$ such that $M' \not\models A$ where $M' = \langle X', f', V'\rangle$.*

Proof. Say that an infinite sequence $\langle x_i\rangle_{i\geq 0}$ is a *backwards f-sequence* iff $x_i = f(x_{i+1})$ for each $i \geq 0$. Since f is onto, every $x \in X$ is the initial member of some backwards f-sequence, perhaps many. Let X' be the set of backwards f-sequences. Let the open subsets of X' be the sets of the following form: $\{\langle x_i\rangle_{i\geq 0} \in X' : x_0 \in O\}$, where O is open in X. Note that these open sets form an almost discrete topology on X'. And define f' as follows: $f'(\langle x_i\rangle_{i\geq 0}) = \langle f(x_i)\rangle_{i\geq 0}$. Note that f' is a homeomorphism on X'. Define $V'(p) = \{\langle x_i\rangle_{i\geq 0} \in X' : x_0 \in V(p)\}$. It is a straightforward matter to show that, for each formula A, we have $V'(A) = \{\langle x_i\rangle_{i\geq 0} \in X' : x_0 \in V(A)\}$. Thus $M' \not\models A$ since $M \not\models A$. □

Our canonical model (see Definition 3.6.9) will use an open onto continuous function, which will not necessarily be one-one.

Recall that a Φ-cluster is a function $\mathfrak{f} : |\alpha|_R \to |\beta|_R$ for some Φ-atoms α and β. We will say that a Φ-cluster $\mathfrak{f}$ is an *onto* Φ-cluster iff $\mathfrak{f}$ is a function from $|\alpha|_R$ *onto* $|\beta|_R$ for some Φ-atoms α and β. We will want our sequences of Φ-clusters to be sequences of *onto* Φ-clusters. The following Lemma, similar to Lemma 3.4.3, helps with this:

Lemma 3.6.2. *Suppose that Φ is a closed finite set of formulas. Suppose that α, β and δ are Φ-atoms such that $\alpha S\beta$ and $\beta R\delta$. Then there is a Φ-atom γ such that $\alpha R\gamma$ and $\gamma S\delta$. Thus the bottom left corner of the square on the left can be filled in as indicated:*

$$\begin{array}{ccc} \alpha & S & \beta \\ R & & R \\ ?? & S & \delta \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \alpha & S & \beta \\ R & & R \\ \gamma & S & \delta \end{array}$$

Proof. Suppose that α, β and δ are Φ-atoms such that $\alpha S\beta$ and $\beta R\delta$. Since $\alpha S\beta$, the formula $(\alpha \,\&\, \circ\beta)$ is consistent. So the formula $(\alpha_M \,\&\, \circ\beta_M)$ is consistent. Since $\beta R\delta$, we have $\beta_M = \delta_M$. So the formula $(\alpha_M \,\&\, \circ\delta_M)$ is consistent. We claim that

$$(\alpha_M \,\&\, \circ\delta) \text{ is consistent.} \qquad (\dagger)$$

To see this, suppose not. Let $\delta_{NM} = \delta - \delta_M$. So $(\alpha_M \,\&\, \circ(\delta_{NM} \,\&\, \delta_M))$ is inconsistent. By the closure of Φ, we can choose a formula A such that A nonmodally dominates δ. In other words, A is consistent, $\Diamond A \in \Phi$ and $(A \supset \delta_{NM}) \in$ S5H. So $(\alpha_M \,\&\, \circ(A \,\&\, \delta_M))$ is inconsistent. So $(\alpha_M \,\&\, \circ A \,\&\, \circ\delta_M)$ is inconsistent. So $((\alpha_M \,\&\, \circ\delta_M) \supset \neg\circ A) \in$ S5H. So $((\Box\alpha_M \,\&\, \Box\circ\delta_M) \supset \Box\neg\circ A) \in$ S5H. So $((\Box\alpha_M \,\&\, \Box\circ\delta_M) \supset \Box\circ\neg A) \in$ S5H. Recall that $(\circ\Box\neg A \equiv \Box\circ\neg A)$ and $(\circ\Box\delta \equiv \Box\circ\delta)$ are axioms of S5H. So $((\Box\alpha_M \,\&\, \circ\Box\delta_M) \supset \circ\Box\neg A) \in$ S5H. So $((\Box\alpha_M \,\&\, \circ\Box\delta_M) \supset \neg\circ\Diamond A) \in$ S5H. Recall that $(\gamma_M \equiv \Box\gamma_M) \in$ S5H and $(\delta_M \equiv \Box\delta_M) \in$ S5H. So $((\alpha_M \,\&\, \circ\delta_M) \supset \neg\circ\Diamond A) \in$ S5H. So $(\alpha_M \,\&\, \circ\delta_M \,\&\, \circ\Diamond A)$ is inconsistent. But since A nonmodally dominates δ, we have $+A \in \delta$. So $+\Diamond A \in \delta$, since $\Diamond A \in \Phi$. So $+\Diamond A \in \delta_M$. So $(\alpha_M \,\&\, \circ\delta_M)$ is inconsistent. But we have already noted that $(\alpha_M \,\&\, \circ\delta_M)$ is consistent. This proves (†).

Given the consistency of $(\alpha_M \,\&\, \circ\delta)$, we can add signed nonmodal formulas to the set α_M until we get a Φ-atom γ with $\gamma_M = \alpha_M$ and with $(\gamma \,\&\, \circ\delta)$ consistent. □

The following Lemma is a strengthening of Lemma 3.4.4:

Lemma 3.6.3. *Suppose that Φ is a closed finite set of formulas, and that α and β are Φ-atoms with $\alpha S\beta$. Then there is an onto Φ-cluster $\mathfrak{f} : |\alpha|_R \to |\beta|_R$ with $\mathfrak{f}(\alpha) = \beta$.*

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10 We say that a function on a topological space is *open* iff the image of every open set is open.

Proof. List the members of $|\beta|_R$ as follows: $\beta_1, \ldots, \beta_n$, with $\beta_1 = \beta$. By Lemma 3.6.2, there are $\alpha_1, \ldots, \alpha_n \in |\alpha|_R$, with $\alpha_1 = \alpha$, such that $\alpha_i S \beta_i$ for each $i \in \{1, \ldots, n\}$. Define $\mathfrak{f}(\alpha_i)$ inductively as follows:

$$\mathfrak{f}(\alpha_1) = \beta_1$$
$$\mathfrak{f}(\alpha_{i+1}) = \begin{cases} \beta_{i+1}, & \text{if } \alpha_{i+1} \neq \alpha_j, \quad \text{for any } j \leq i \\ \mathfrak{f}(\alpha_j), & \text{if } \alpha_{i+1} = \alpha_j \text{ and } j \leq i. \end{cases}$$

If $|\alpha|_R = \{\alpha_1, \ldots, \alpha_n\}$, then we are done: $\mathfrak{f}$ is a Φ-cluster from $|\alpha|_R$ *onto* $|\beta|_R$ with $\mathfrak{f}(\alpha) = \beta$.

Otherwise, list the members of $|\alpha|_R - \{\alpha_1, \ldots, \alpha_n\}$ as follows: $\gamma_1, \ldots, \gamma_m$. By Lemma 3.4.3, there are $\delta_1, \ldots, \delta_m \in |\beta|_R$ such that $\gamma_i S \delta_i$ for each $i \in \{1, \ldots, m\}$. Define $\mathfrak{f}(\gamma_i) = \delta_i$. Now $\mathfrak{f}$ is a Φ-cluster from $|\alpha|_R$ *onto* $|\beta|_R$ with $\mathfrak{f}(\alpha) = \beta$. □

Given Lemma 3.6.3, we get stronger analogues of Lemmas 3.4.5–3.4.7, with *onto* Φ-clusters and sequences of *onto* Φ-clusters: the proofs are virtually the same. In particular, we get the following:

Lemma 3.6.4. *Suppose that Φ is a closed finite set of formulas and that α is a Φ-atom. Then there is an eventually periodic $*$-complete coherent infinite sequence $\langle \mathfrak{f}_i \rangle_{i \geq 0}$ of onto Φ-clusters, such that $\alpha \in dom(\mathfrak{f}_0)$.*

Recall that the proof of completeness for S5Ht relied on *backwards S*-sequences of Φ-atoms, so that we could build *bi*-infinite S-sequences. For S5H, we need *backwards coherent* sequences: a finite sequence $\langle \mathfrak{f}_i \rangle_{i=0}^{n}$ [an infinite sequence $\langle \mathfrak{f}_i \rangle_{i \geq 0}$] of Φ-clusters is *backwards-coherent* iff $\mathfrak{f}_{i+1}$ coheres with $\mathfrak{f}_i$, for each $i \geq 0$ and $< n$ [for each $i \geq 0$]. We start with the following analogue of Lemma 3.3.2:

Lemma 3.6.5. *Suppose that Φ is a closed finite set of formulas, and that α is a Φ-atom. Then there is a Φ-atom β such that $\beta S \alpha$.*

Proof. Since α is a Φ-atom, α is consistent. So $\neg\alpha \notin$ S5H. So $\circ\neg\alpha \notin$ S5H, since S5H is closed under the rule of *next removal*. So $\neg\circ\alpha \notin$ S5Ht. So $\circ\alpha$ is consistent. So there is some Φ-atom β such that the following is consistent: $(\beta \,\&\, \circ\alpha)$. □

Corollary 3.6.6. *Suppose that Φ is a closed finite set of formulas, and that α is a Φ-atom. Then there is an onto cluster $\mathfrak{f}$ such that $\alpha \in range(\mathfrak{f})$.*

Proof. This follows from Lemma 3.6.5 and Lemma 3.6.3. □

Corollary 3.6.7. *Suppose that Φ is a closed finite set of formulas, and that $\mathfrak{f}$ is an onto Φ-cluster. Then there is an infinite eventually periodic backwards-coherent sequence $\langle \mathfrak{f}_i \rangle_{i \geq 0}$ of onto Φ-clusters with $\mathfrak{f}_0 = \mathfrak{f}$.*

Proof. The existence of an infinite backwards-coherent sequence $\langle \mathfrak{f}_0 \rangle_{i \geq 0}$ of onto Φ-clusters with $\mathfrak{f}_0 = \mathfrak{f}$ is guaranteed by Corollary 3.6.6. The existence of an *eventually periodic* infinite eventually periodic backwards-coherent sequence $\langle \mathfrak{f}_0 \rangle_{i \geq 0}$ of onto Φ-clusters with $\mathfrak{f}_0 = \mathfrak{f}$ is guaranteed by the former remark and the fact that there are only finitely many Φ-clusters. □

Lemma 3.6.8. *Suppose that Φ is a closed finite set of formulas and that α is a Φ-atom. Then there is a bi-eventually periodic $*$-complete coherent bi-infinite sequence $\langle \mathfrak{f}_i \rangle_{i \in \mathbb{Z}}$ of onto clusters such that $\alpha_0 \in \mathfrak{f}_0$.*

Proof. By Lemma 3.6.4, there is an eventually periodic $*$-complete coherent infinite sequence $\langle \mathfrak{f}_i \rangle_{i \geq 0}$ of onto Φ-clusters, such that $\alpha \in dom(\mathfrak{f}_0)$. And by Lemma 3.6.7, there is an infinite eventually periodic backwards-coherent sequence $\langle \mathfrak{f}'_i \rangle_{i \geq 0}$ of onto Φ-clusters with $\mathfrak{f}'_0 = \mathfrak{f}_0$. For each $i < 0$, define $\mathfrak{f}_i = \mathfrak{f}'_{-i}$. Then the bi-infinite sequence $\langle \mathfrak{f}_i \rangle_{i \in \mathbb{Z}}$ of onto clusters is bi-eventually periodic, coherent and $*$-complete; and $\alpha \in \mathfrak{f}_0$. □

Definition 3.6.9. Suppose that Φ is a closed finite set of formulas, and that α is a Φ-atom. We will define an almost discrete topological space, X_α; a continuous open onto function, f_α on X_α; and a valuation function $V_\alpha : PV \to \mathcal{P}(X_\alpha)$.

First, choose a bi-eventually periodic $*$-complete coherent bi-infinite sequence $\langle \mathfrak{f}_i \rangle_{i \in \mathbb{Z}}$ of onto Φ-clusters, such that $\alpha \in dom(\mathfrak{f}_0)$. We define X_α as follows:

$$X_\alpha = \{\langle i, \beta \rangle : i \in \mathbb{Z} \text{ and } \beta \in dom(\mathfrak{f}_i)\}.$$

For each $i \in \mathbb{Z}$, define the set O_i as follows: $O_i = \{\langle i, \beta \rangle : \beta \in dom(\mathfrak{f}_i)\}$. The topology we impose on X_α is as follows: a set is *open* iff it is either empty or a union of some of the O_i's. In other words, the O_i's form a basis for our topology. Since our topology has a basis of pairwise disjoint open sets, the space X_α is almost discrete.

We define a function $f_\alpha : X_\alpha \to X_\alpha$ as follows:

$$f_\alpha(\langle i, \beta \rangle) = \langle i + 1, \mathfrak{f}_i(\beta) \rangle.$$

Note that both the image under f_α and the inverse image under f_α of any basis set O_i is open: the image is O_{i+1} and the inverse image is O_{i-1}. So the function f_α is continuous and open. The function f_α also maps X_α *onto* X_α: suppose that $\langle i, \beta \rangle \in X_\alpha$; then $\beta \in dom(\mathfrak{f}_i)$; and since $\mathfrak{f}_{i-1} : dom(\mathfrak{f}_{i-1}) \to dom(\mathfrak{f}_i)$ is onto, there is a $\gamma \in dom(\mathfrak{f}_{i-1})$ such that $\mathfrak{f}_{i-1}(\gamma) = \beta$; thus $f(\langle i - 1, \gamma \rangle) = \langle i, \beta \rangle$.

We define the valuation function V_α as follows:

$$V_\alpha(p) = \{\langle i, \beta \rangle \in X_\alpha : +p \in \beta\}, \quad \text{for each propositional variable } p.$$

Finally, we define the dynamic topological model, $M_\alpha =_{df} \langle X_\alpha, f_\alpha, V_\alpha \rangle$.

The proof of the following theorem is similar to the proof of Theorem 3.5.3:

Theorem 3.6.10. *Suppose that Φ is a closed finite set of formulas, and that α is a Φ-atom. And suppose that X_α, f_α, and V_α are defined as in Definition 3.6.9. Then, for each $A \in \Phi$:*

for each $\langle i, \beta\rangle \in X_\alpha$, $\langle i, \beta\rangle \in V_\alpha(A)$ iff $+A \in \beta$.

And we thus get an analogue (without the finiteness condition) of Corollary 3.5.4:

Corollary 3.6.11. *Suppose that $A \notin$ S5H. Then there is almost discrete topological space X and some continuous open onto function $f : X \to X$ such that $\langle X, f\rangle \not\models A$.*

The completeness of S5H for continuous open onto functions on almost discrete spaces follows from Corollary 3.3.7. The completeness of S5H for homeomorphisms on almost discrete spaces follows from Corollary 3.3.7 and Lemma 3.6.1.

What about the decidability of S5H? We do not get it through any finite model property. But, as in the case of S5Ht, decidability does follow from the fact that each model X_α is of a kind that can be finitely represented: using a method of finite premodels similar to that used at the end of Section 3.3, we can prove the following:

Theorem 3.6.12. S5H *is decidable.*

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