Quantified intuitionistic logic over metrizable spaces

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Philip Kremer*

Abstract

In the topological semantics, quantified intuitionistic logic, QH, is known to be strongly complete not only for the class of all topological spaces, but also for some particular topological spaces — for example, for the irrational line, $\mathbb{P}$, and for the rational line, $\mathbb{Q}$, in each case with a constant countable domain for the quantifiers. Each of $\mathbb{P}$ and $\mathbb{Q}$ is a separable zero-dimensional dense-in-itself metrizable space. The main result of the current paper generalizes these known results: QH is strongly complete for any zero-dimensional dense-in-itself metrizable space with a constant domain of cardinality $\leq$ the space's weight; consequently, QH is strongly complete for any *separable* zero-dimensional dense-in-itself metrizable space with a constant *countable* domain. We also prove a result that follows from earlier work of Moerdijk: if we allow varying domains for the quantifiers then QH is strongly complete for any dense-in-itself metrizable space with countable domains.

Keywords: Quantified intuitionistic logic, topological semantics, completeness.

Assume a countable quantified intuitionistic or modal language without identity. Rasiowa and Sikorski [20] extend the the McKinsey-Tarski topological semantics ([13, 14]) for propositional intuitionistic H [propositional modal S4] to a constant-domain topological semantics for quantified QH [QS4] without

*Dept. of Philosophy, University of Toronto Scarborough, kremer@utsc.utoronto.ca.

identity.[1] They prove the completeness of QH [QS4] for the class of all topological spaces, assuming constant domains for the quantifiers, and take up the question of completeness for particular topological spaces. For example, they show how to construct a subspace of the irrational line $\mathbb{P}$, for which QH [QS4] is complete with a constant countable domain. On the negative side, they show that QS4 is not complete for any Baire space, e.g., $\mathbb{R}$, with a constant countable domain.[2]

Subsequent work has improved on the results in [20]. On the negative side, neither QS4 nor QH is complete for any locally connected space, e.g., $\mathbb{R}$, with a constant domain of any cardinality ([10, 21] and below). On the positive side, Dragalin [2] shows that QH is strongly complete for $\mathbb{N}^{\mathbb{N}}$ with a constant countable domain: since $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{P}$, QH is also strongly complete for $\mathbb{P}$, with a constant countable domain.[3] By a completely different argument, Kremer [10] shows that QS4 and consequently QH are strongly complete for the rational line, $\mathbb{Q}$, with a constant countable domain.

Each of $\mathbb{P}$ and $\mathbb{Q}$ is a separable zero-dimensional dense-in-itself metrizable space. The main result of the current paper is that QH is strongly complete for *any* zero-dimensional dense-in-itself metrizable space, with a constant domain of cardinality $\leq$ the space's weight.[4] A corollary: QH is strongly complete for any *separable* zero-dimensional dense-in-itself metrizable space, with a constant *countable* domain. This substantially generalizes the above-mentioned results for $\mathbb{P}$ and $\mathbb{Q}$: besides these, the most important separable dense-in-itself zero-dimensional metrizable space is the Cantor space, $\mathbb{C}$, but

[1]They note that, in the intuitionistic case, the extension of the topological semantics to a quantified language goes back to [18], and in the modal case to [19].

[2]Recall that a *Baire space* is a topological space where the intersection of any countable family of dense open sets is dense.

[3]$\mathbb{N}^{\mathbb{N}}$ is the set of total functions from $\mathbb{N}$ to $\mathbb{N}$, topologized as follows. Suppose that $g : \{0, ..., n-1\} \to \mathbb{N}$, for some $n \in \mathbb{N}$: let $B_g = \{f \in \mathbb{N}^{\mathbb{N}} : (\forall m < n)(f(m) = g(m))\}$. The topology on $\mathbb{N}^{\mathbb{N}}$ is determined by the base $\{B_g : (\exists n \in \mathbb{N})(g : \{0, ..., n-1\} \to \mathbb{N})\}$. It is well-known that $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{P}$ are homeomorphic.

[4]The *weight* of a topological space is the minimal cardinality of a basis for the space.

there are many others.[5,6] We leave it open whether we can improve the result to the claim that QH is strongly complete for every zero-dimensional dense-in-itself metrizable space, separable or not, with a constant *countable* domain, rather than merely a constant domain of cardinality $\leq$ the weight of the space.

The semantics in [20] is a constant-domain semantics: at each point in a topological space, the domain over which the quantifiers range is the same. This contrasts with the Kripke semantics, where completeness fails unless we allow varying domains. The semantics we present below is a topological semantics with varying domains, for which constant-domain models are a special case: this will be useful even for our constant-domain results. Our varying-domain semantics is in turn a kind of special case of the sheaf semantics originating in [6] and presented in [16] and elsewhere:[7] more precisely, our semantics is a notational variant of the sheaf semantics restricted to identity-free languages interpreted over topological spaces rather than the more general complete Heyting algebras. Thus, a straightforward consequence of the main theorem of [16] is that, in our varying-domain semantics, QH is complete for any dense-in-itself metrizable space. Here, we improve on this a bit: QH is *strongly* complete for any dense-in-itself metrizable space, with *countable* domains. Our proof is really not too different from the proof

[5] Up to homeomorphism, there are $2^{\mathfrak{c}}$ separable dense-in-themselves zero-dimensional metrizable spaces, where $\mathfrak{c}$ is the cardinality of the continuum. *Proof.* It follows from Proposition 1.3.15 in [4] that the separable dense-in-themselves zero-dimensional metrizable spaces are, up to homeomorphism, the dense-in-themselves subspaces of $\mathbb{C}$. This gives us $2^{\mathfrak{c}}$ as an upper bound on the cardinality, up to homeomorphism, of the separable dense-in-themselves zero-dimensional metrizable spaces. Now, as noted in [15], any subset of $\mathbb{N}$ can be identified with a member of $\mathbb{C}$ in a natural way, so that any subset of $\mathcal{P}(\mathbb{N})$ can be identified with a subspace of $\mathbb{C}$. In particular, any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ can be identified with a subspace of $\mathbb{C}$. Indeed, any such $\mathcal{U}$ is a dense-in-itself subspace of $\mathbb{C}$. Moreover, by Corollary 2 in [15], there are $2^{\mathfrak{c}}$ pairwise nonhomemorphic nonprincipal ultrafilters on $\mathbb{N}$. Thus, there are at least $2^{\mathfrak{c}}$ pairwise nonhomemorphic dense-in-themselves subspaces of $\mathbb{C}$. And any subspace of $\mathbb{C}$ is a separable zero-dimensional metrizable space. So there are at least $2^{\mathfrak{c}}$ pairwise nonhomemorphic separable dense-in-themselves zero-dimensional metrizable spaces. I am grateful to Henno Brandsma for pointing this out to me.

[6] From early on, Cantor space has figured prominently in QH completeness results: [20] (page 423, footnote 1) cites a annoucement by Beth, in a 1957 colloquium in Amsterdam, that QH is complete for the family of closed subspaces of $\mathbb{C}$.

[7] [1] presents a sheaf semantics, and [12] presents a closely related varying-domain topological semantics, for first-order S4 with identity.

Table 1: Summary of results for dense-in-themselves metrizable spaces

space	domains	domain size	QH		QS4	
$\mathbb{Q}$	constant	countable	✓		✓	[10]
		unrestricted	✓		✓	
	varying	countable	✓		✓	
		unrestricted	✓		✓	
$\mathbb{R}$	constant	countable	✗		✗	[20]
		unrestricted	✗	[10, 21]	✗	[10, 21]
	varying	countable	✓	*	✗	[12]
		unrestricted	✓	*	—	
$\mathbb{P}$	constant	countable	✓	[2]	✗	[20]
		unrestricted	✓		—	
	varying	countable	✓		✗	[12]
		unrestricted	✓		—	
any	constant	countable	✗		✗	
		unrestricted	✗		✗	
	varying	countable	✓	*	✗	
		unrestricted	✓	*	—	
any 0-d	constant	countable	—		✗	
		unrestricted	✓	*	—	
	varying	countable	✓	*	✗	
		unrestricted	✓	*	—	
any sep 0-d	constant	countable	✓	*	✗	
		unrestricted	✓	*	—	
	varying	countable	✓	*	✗	
		unrestricted	✓	*	—	

In this table, $\mathbb{Q}$ is the rational line, $\mathbb{R}$ the real line, and $\mathbb{P}$ the irrational line. The notation 'any' stands for any dense-in-itself metrizable space; 'any 0-d' stands for any zero-dimensional dense-in-itself metrizable space; and 'any sep 0-d' stands for any separable zero-dimensional dense-in-itself metrizable space. A checkmark, ✓, indicates that the given logic is strongly complete for the given topological space; an exmark, ✗, indicates that the logic is not complete for the given topological space (or at least one such space, in the case of 'any', 'any 0-d' and 'any sep 0-d'); and an em dash, —, indicates that the questions of completeness and strong completeness remain open. Results proved in the current paper are marked with an asterisk. Some results are cited: uncited results follow easily from cited ones.

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in [16], where the mild improvements to strong completeness and countable domains could have been achieved with minor ammendments.

The current paper proves results only for QH and not for QS4. There are some known dissimilarities: QH is strongly complete while QS4 is incomplete for $\mathbb{P}$ and for $\mathbb{C}$ with a constant countable domain.[8] It remains an open question whether QS4 is complete for $\mathbb{P}$ or for $\mathbb{C}$ with a constant uncountable domain. It remains a further open question whether QS4 is complete for every zero-dimensional dense-in-itself metrizable space with a constant domain. Table 1 summarizes some known results as well as some questions we believe to be open.

1 Preliminaries

Let $\mathcal{L}$ be a quantified intuitionistic language with a countable set Var of variables; disjoint countable sets Pred_n of n-ary predicate symbols, for each $n \geq 1$; a countable set Names of names; disjoint countable sets Func_n of n-ary function symbols, for each $n \geq 1$; connective &, $\vee$, $\to$ and $\sim$; quantifiers $\forall$ and $\exists$; and parentheses. Let $\mathsf{Pred} = \bigcup_n \mathsf{Pred}_n$ and $\mathsf{Func} = \bigcup_n \mathsf{Func}_n$; we assume that Pred is nonempty. Note that $\mathcal{L}$ has no equals sign. If A is a formula, t is a term, and x is a variable, then $[\mathsf{t}/\mathsf{x}]A$ is the result of replacing every free occurrence of x in A with t. We say that t is *substitutible for* x *in* A iff no free occurrence of x in A is in the scope of any bound variable y, where y occurs in t. Given any set D, *D-terms*, *D-formulas* and *D-sentences* are terms, formulas and sentences in the language $\mathcal{L}(D)$, which is the result of expanding the language $\mathcal{L}$ so that every member of the set D is a name of $\mathcal{L}$. (Here we assume that $D \cap S = \emptyset$, if $S = \mathsf{Var}$, Pred, Names or Func.) It will be useful to let $\mathsf{Term}(D)$ be the set of closed D-terms. Note that, given any D-formula A, any variable $\mathsf{x} \in \mathsf{Var}$ and any $d \in D$, the D-formula $[d/\mathsf{x}]A$ is the result of replacing every occurrence of x in A with d. We reserve the unprefixed expressions 'formula(s)' and 'sentence(s)' for formulas and sentences in the original language $\mathcal{L}$.

Axiomatizations quantified intuitionistic logic, QH, are easily found online or in the literature. We repeat the axiom schemes and rules as found in [17]. In the last two axiom schemes, the term t must be substitutible for x in A:

[8] The incompleteness is a special case of the incompleteness of QS4 for any Baire space with a constant countable domain ([20]).

- Axioms.
  - $A \to (B \to A)$
  - $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
  - $A \to (B \to (A \& B))$
  - $(A \& B)) \to A$
  - $(A \& B)) \to B$
  - $A \to (A \vee B)$
  - $B \to (A \vee B)$
  - $(A \to C) \to ((B \to C) \to (A \vee B) \to C)$
  - $(A \to B) \to ((A \to {\sim}B) \to {\sim}A)$
  - ${\sim}A \to (A \to B)$
  - $\forall \mathsf{x} A \to [\mathsf{t}/\mathsf{x}]A$
  - $[\mathsf{t}/\mathsf{x}]A \to \exists \mathsf{x} A$
- Rules of inference.
  - Modus Ponens: From A and $(A \to B)$, conclude B.
  - $\forall$-Introduction: From $(C \to A)$ conclude $(C \to \forall \mathsf{x} A)$, where x is any variable that does not occur free in C.
  - $\exists$-Elimination: From $(A \to C)$ conclude $(\exists \mathsf{x} A \to C)$, where x is any variable that does not occur free in C.

A nonempty finite set Γ of formulas of $\mathcal{L}$ is *consistent* iff the negation of their conjunction is not a theorem of QH. A possibly infinite nonempty set Γ is *consistent* iff every nonempty finite subset is consistent. We follow [7] in extending the notion of consistency to what they call *double theories* ([7], Section 1.1.2), i.e., *pairs* $\langle \Gamma, \Delta \rangle$ of sets of formulas. Here's why. (1) Once a semantics is on the table, completeness is usually articulated in terms of validity: if a formula is valid then it is a theorem. (2) In the classical setting, completeness is strengthened to strong completeness by first restating completeness in terms of consistency rather than validity: every consistent formula, and thus every finite consistent set of formulas, is satisfiable. (3) Strong completeness is then the stronger claim that every consistent set, finite or infinite, of formulas is satisfiable. Step (2) works in the classical setting because the non-theoremhood of a formula A is equivalent to the consistency of its classical negation, $\neg A$. But the analog fails in the intuitionistic setting: $(p \vee \sim p)$ is not a theorem, even though $\sim(p \vee \sim p)$ is not consistent. Thus, in the intuitionistic setting, the claim that every finite consistent set of formulas is satisfiable is *weaker* than completeness, i.e., the claim that every valid formula is a theorem. So this does not generalize to a suitable version of strong completeness for infinite sets of formulas.

This is fixed by considering pairs $\langle \Gamma, \Delta \rangle$ of sets of formulas as follows. A *pair* $\langle \Gamma, \Delta \rangle$ of nonempty finite sets of formulas is *consistent* iff the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is not a theorem of QH. A pair $\langle \Gamma, \Delta \rangle$ of possibly infinite nonempty sets of formulas is *consistent* iff every pair $\langle \Gamma', \Delta' \rangle$ is consistent, where Γ' is a finite subset of Γ and Δ' is a finite subset of Δ. Note that a set Γ is consistent iff the pair $\langle \Gamma, \{\bot\} \rangle$ is consistent where $\bot$ is any contradiction. More importantly, notice that the non-theoremhood of a formula A is equivalent to the consistency of the pair $\langle \emptyset, \{A\} \rangle$: thus, completeness can be restated in terms of consistency rather than theoremhood, and can be appropriately strengthened to strong completeness. See page 10, below.

2 Topological semantics

The topological semantics in [20] is a constant-domain semantics: here we present a generalization that allows varying domains. Our terminology and notation are adapted from [7] and [10]. We assume familiarity with the basics of point-set topology: [3] and [5] are standard references. We will not distinguish between a topological space and the underlying set, e.g., we will use $\mathbb{R}$ both for the set of real numbers and the topological space consisting of this set together with the standard topology on it. We will use $Int(S)$ for the interior of S and $Cl(S)$ for the closure of S.

Given a topological space X, a *system of domains* is a family, $D = \{D_x\}_{x \in X}$, of nonempty sets indexed by points in X satisfying the following condition:

the set $O_d =_{\mathrm{df}} \{x \in X : d \in D_x\}$ is open in X, for every $d \in \bigcup_{x \in X} D_x$.

For reasons that will become clear in Section 4, we refer to this condition as the *expanding-domain* condition. A *predicate topological space* is an ordered pair $\mathbf{X} = \langle X, D \rangle$, where X is a topological space and D is a system of domains. We let $D_{\mathbf{X}} =_{\mathrm{df}} \bigcup_{x \in X} D_x$. It will be useful, for any $D_{\mathbf{X}}$-sentence A

to define $O_A =_{\mathrm{df}} X$, if no members of $D_{\mathbf{X}}$ occur as names in A; and otherwise $O_A =_{\mathrm{df}} O_{d_1} \cap \ldots \cap O_{d_n}$, where $d_1, \ldots, d_n$ are the members of $D_{\mathbf{X}}$ occurring as names in A. If κ is a cardinal number, we say that $\mathbf{X}$ has a *constant domain of cardinality* $\leq \kappa$ if $\forall x \in X,\ D_x = D_{\mathbf{X}}$ and $card(D_{\mathbf{X}}) \leq \kappa$. In particular, we say that $\mathbf{X}$ has a *constant countable domain* iff $\mathbf{X}$ has a constant domain of cardinality $\leq \aleph_0$. We say that $\mathbf{X}$ has *countable domains* iff every D_x is countable. We say that $\mathbf{X}$ is *countable* iff X is countable and each D_x is countable. We say that $\mathbf{X}$ is *based on* X. Where X is a topological space and D is a single domain, i.e., a nonempty set, we write $\langle X, D\rangle$ for the predicate topological space based on X with D as a constant domain.

A *predicate topological model* is an ordered triple $\mathbf{M} = \langle X, D, V\rangle$, where $\mathbf{X} = \langle X, D\rangle$ is a predicate topological space, and

$$V : \mathsf{Pred} \cup \mathsf{Names} \cup \mathsf{Func} \to \bigcup_{n\geq 1} \mathcal{P}(X)^{{D_{\mathbf{X}}}^n} \cup D_{\mathbf{X}} \cup (\bigcup_{n\geq 1} {D_{\mathbf{X}}}^{{D_{\mathbf{X}}}^n})$$

is such that

- $V(\mathsf{P})(d_1, \ldots, d_n) \subseteq \bigcap_{i=1}^{n} O_{d_i}$ for every $\mathsf{P} \in \mathsf{Pred}_n$ and $d_1, \ldots, d_n \in D_{\mathbf{X}}$;
- $V(\mathsf{c}) \in D_x$ for every $\mathsf{c} \in \mathsf{Names}$ and $x \in X$;
- $V(\mathsf{f})(d_1, \ldots, d_n) \in D_x$ for every $x \in X$, every $\mathsf{f} \in \mathsf{Func}_n$ and every $d_1, \ldots, d_n \in D_x$; and
- $V(\mathsf{P})(d_1, \ldots, d_n)$ is an open subset of X for every $\mathsf{P} \in \mathsf{Pred}_n$ and $d_1, \ldots, d_n \in D_{\mathbf{X}}$.

We say that $\mathbf{M}$ is *based on* $\mathbf{X}$.

Suppose that $\mathbf{M} = \langle X, D, V\rangle$ is a predicate topological model. First we define $Val_{\mathbf{M}}(\mathsf{t}) \in D_{\mathbf{X}}$ for every closed $D_{\mathbf{X}}$-term t: $Val_{\mathbf{M}}(d) = d$, if $d \in D_{\mathbf{X}}$; if $\mathsf{c} \in \mathsf{Names}$ then $Val_{\mathbf{M}}(c) = V(c)$; and if $\mathsf{f} \in \mathsf{Func}_n$ and $\mathsf{t}_1, \ldots, \mathsf{t}_n$ are closed $D_{\mathbf{X}}$-terms then $Val_{\mathbf{M}}(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n) = V(\mathsf{f})(Val_{\mathbf{M}}(\mathsf{t}_1), \ldots, Val_{\mathbf{M}}(\mathsf{t}_n))$. Note that if $x \in X$ and if t is a closed $D_{\mathbf{X}}$-term such that the members of $D_{\mathbf{X}}$ occuring in t as names are all in D_x, then $Val_{\mathbf{M}}(\mathsf{t}) \in D_x$.

Next, we define $Val_{\mathbf{M}}(A) \subseteq X$, for each $D_{\mathbf{X}}$-sentence A as follows:

$$
\begin{aligned}
Val_{\mathbf{M}}(\mathsf{P}\mathsf{t}_1 \dots \mathsf{t}_n) &= V(\mathsf{P})(Val_{\mathbf{M}}(\mathsf{t}_1), \dots, Val_{\mathbf{M}}(\mathsf{t}_n)), \\
&\quad \text{where } \mathsf{P} \in \mathsf{Pred}_n \text{ and } \mathsf{t}_1, \dots, \mathsf{t}_n \text{ are } D_{\mathbf{X}}\text{-terms} \\
Val_{\mathbf{M}}(A \,\&\, B) &= Val_{\mathbf{M}}(A) \cap Val_{\mathbf{M}}(B) \\
Val_{\mathbf{M}}(A \vee B) &= O_{(A \vee B)} \cap (Val_{\mathbf{M}}(A) \cup Val_{\mathbf{M}}(B)) \\
Val_{\mathbf{M}}(\sim A) &= O_{\sim A} \cap Int(X - Val_{\mathbf{M}}(A)) \\
Val_{\mathbf{M}}(A \to B) &= O_{(A \to B)} \cap Int((X - Val_{\mathbf{M}}(A)) \cup Val_{\mathbf{M}}(B)) \\
Val_{\mathbf{M}}(\forall \mathsf{x} A) &= O_{\forall \mathsf{x} A} \cap Int(\textstyle\bigcap_{d \in D_{\mathbf{X}}}((X - O_d) \cup Val_{\mathbf{M}}([d/\mathsf{x}]A))) \\
Val_{\mathbf{M}}(\exists \mathsf{x} A) &= \textstyle\bigcup_{d \in D_{\mathbf{X}}} Val_{\mathbf{M}}([d/\mathsf{x}]A)
\end{aligned}
$$

It is routine to show that $Val_{\mathbf{M}}(A)$ is open for every $D_{\mathbf{X}}$-sentence A, and that $Val_{\mathbf{M}}(A) \subseteq O_A$.

The clause for $Val_{\mathbf{M}}(\forall \mathsf{x} A)$ looks a bit strange, but is forced on us in the presence of varying domains. With a constant domain D, the clause reduces to $Val_{\mathbf{M}}(\forall \mathsf{x} A) = Int(\bigcap_{d \in D} Val_{\mathbf{M}}([d/\mathsf{x}]A))$, which is fine.[9] In the more general varying-domain case, the clause for $Val_{\mathbf{M}}(\forall \mathsf{x} A)$ becomes more intuitive if we note that it is equivalent to the following, by design:

$$x \in Val_{\mathbf{M}}(\forall \mathsf{x} A) \text{ iff } (\exists \text{ open } O)(x \in O \text{ and } (\forall y \in O)(\forall d \in D_y)(y \in Val_{\mathbf{M}}([d/\mathsf{x}]A)))$$

We note a counterintuitive but harmless consequence of the fact that $Val_{\mathbf{M}}(A) \subseteq O_A$: if there is no $x \in X$ such that $d, d' \in D_x$ and if both d and d' occur in the $D_{\mathbf{X}}$-sentence A then $Val_{\mathbf{M}}(A) = Val_{\mathbf{M}}(\sim A) = Val(A \to A) = O_A = \emptyset$. There are workarounds for this, but they are not needed for either soundness or completeness, which will shortly be defined in terms of sentences, i.e. sentences in the original language $\mathcal{L}$, and not in terms of $D_{\mathbf{X}}$-sentences: thus, for example, for any sentence A, $Val_{\mathbf{M}}(A \to A) = X$.

If A is a sentence (i.e., not merely a $D_{\mathbf{X}}$-sentence), we say that $\mathbf{M} \Vdash A$ (A is *valid* in $\mathbf{M}$) iff $Val_{\mathbf{M}}(A) = X$. If A is an open formula, then we follow [7], Section 3.2, in saying that $\mathbf{M} \Vdash A$ iff $\mathbf{M} \Vdash \overline{\forall} A$, where $\overline{\forall} A$ is the universal

[9] We take the *interior* of $\bigcap_{d \in D} Val_{\mathbf{M}}([d/\mathsf{x}]A)$ rather than just $\bigcap_{d \in D} Val_{\mathbf{M}}([d/\mathsf{x}]A)$, to ensure that $Val_{\mathbf{M}}(\forall \mathsf{x} A)$ is open.

closure of A.[10] We say that $\mathbf{X} \Vdash A$ iff $\mathbf{M} \Vdash A$ for every $\mathbf{M}$ based on $\mathbf{X}$. We say that $X \Vdash A$ iff $\mathbf{X} \Vdash A$ for every $\mathbf{X}$ based on X. If $\mathfrak{X}$ is a class of topological spaces [of predicate topological spaces], then we say that $\mathfrak{X} \Vdash A$ iff $X \Vdash A$ [$\mathbf{X} \Vdash A$] for every $X \in \mathfrak{X}$ [$\mathbf{X} \in \mathfrak{X}$]. We say that A is *valid in* X [$\mathbf{X}$, $\mathfrak{X}$] iff $X \Vdash A$ [$\mathbf{X} \Vdash A$, $\mathfrak{X} \Vdash A$]. QH is *sound for* a class $\mathfrak{X}$ of [predicate] topological spaces iff $\mathfrak{X} \Vdash A$, for every formula $A \in$ QH; and QH is *complete for* a class $\mathfrak{X}$ of [predicate] topological spaces iff $A \in$ QH for every formula A with $\mathfrak{X} \Vdash A$. If X [$\mathbf{X}$] is a topological space [predicate topological space] We say that QH is sound or complete for X [$\mathbf{X}$] iff QH is sound or complete for $\{X\}$ [$\{\mathbf{X}\}$].

If Γ is a nonempty set of sentences, we define $Val_{\mathbf{M}}(\Gamma) = \bigcap_{A \in \Gamma} Val_{\mathbf{M}}(A)$. If Γ and Δ are nonempty sets of sentences, then $Val_{\mathbf{M}}(\langle \Gamma, \Delta \rangle) = Val_{\mathbf{M}}(\Gamma) - \bigcup_{A \in \Delta} Val_{\mathbf{M}}(A)$. The pair $\langle \Gamma, \Delta \rangle$ of nonempty sets of sentences is *satisfiable in* $\mathbf{X}$ iff $Val_{\mathbf{M}}(\langle \Gamma, \Delta \rangle) \neq \emptyset$ for some model $\mathbf{M}$ based on $\mathbf{X}$. If X is a topological space, then $\langle \Gamma, \Delta \rangle$ is *satisfiable in* X iff $\langle \Gamma, \Delta \rangle$ is satisfiable in some predicate topological space based on X. If $\mathfrak{X}$ is a class of topological spaces [predicate topological spaces], then $\langle \Gamma, \Delta \rangle$ is *satisfiable in* $\mathfrak{X}$ iff $\langle \Gamma, \Delta \rangle$ is satisfiable in some $X \in \mathfrak{X}$ [some $\mathbf{X} \in \mathfrak{X}$]. Note that QH is complete for a class $\mathfrak{X}$ of [predicate] topological spaces iff every consistent pair $\langle \Gamma, \Delta \rangle$ is satisfiable in $\mathfrak{X}$, where Γ and Δ are finite. We say that QH is *strongly complete* for $\mathfrak{X}$ iff every consistent pair of nonempty sets of sentences is satisfiable in $\mathfrak{X}$. If X [$\mathbf{X}$] is a topological space [predicate topological space] We say that QH is strongly complete for X [$\mathbf{X}$] iff QH is strongly complete for $\{X\}$ [$\{\mathbf{X}\}$].[11]

The foundational results in [20] are not for the topological semantics just presented, but rather for our semantics restricted to predicate topological spaces with a constant domain. Accordingly, we say that QH is [strongly] complete for a topological space X with *a constant domain* iff QH is [strongly] complete for $\langle X, D \rangle$ for some constant domain D. Similarly, QH is [strongly] complete for X with *a constant domain of cardinality* $\leq \kappa$ iff QH is [strongly] complete for $\langle X, D \rangle$ for some constant domain D of cardinality $\leq \kappa$.

[10] An equivalent definition: $\mathbf{M} \Vdash A$ iff $O_{A'} \subseteq Val_{\mathbf{M}}(A')$ for every $D_{\mathbf{X}}$-instance A' of A, where a $D_{\mathbf{X}}$-instance of a formula A is any $D_{\mathbf{X}}$-sentence of the form $[d_1/\mathsf{x}_1] \ldots [d_n/\mathsf{x}_n]A$.

[11] Here we follow a decision made in [7], in the context of Kripke semantics: for pairs $\langle \Gamma, \Delta \rangle$, we only define satisfiability when Γ and Δ are sets of *sentences* and not when Γ and Δ are arbitrary sets of formulas. For the latter, we would want the effect of surrounding the whole pair $\langle \Gamma, \Delta \rangle$ with an existential quantifier. There are ways of doing this and getting the desired results, but they introduce distracting complications.

3 Results

It is routine to show that QH is sound for any class of predicate topological models. The main QH-completeness result in [20] is that QH is complete for the class of all predicate topological spaces with constant domains ([20], X, 4.1), indeed with constant countable domains. For our main result, Theorem 3.1, we recall that a *metric* on a nonempty set X is a function $d : X \times X \to \mathbb{R}$ such that, for every $x, y, z \in X$, (i) $d(x, y) = d(y, x)$, (ii) $d(x, y) \geq 0$, (iii) $d(x, y) = 0$ iff $x = y$, and (iv) $d(x, z) \leq d(x, y) + d(y, z)$. An *open ball* is a subset of X of the form $\{y : d(x, y) < r\}$, where $x \in X$ and r is a positive real number: the point x is the *centre* of $\{y : d(x, y) < r\}$, and r is its *radius*. A *metric* space $\langle X, d\rangle$ is a nonempty set together with a metric. And a topological space X is *metrizable* iff there is a metric d on X such that the open balls form a basis for its topology. Given a metric d on a nonempty set X, we identify the metric space $\langle X, d\rangle$ with the topological space whose topology is given by the basis of open balls. We also note that if $x \in O \subseteq X$ where O is open, then there is an open ball B with centre x such that $x \in B \subseteq O$.

We also recall that a topological space is *zero-dimensional* iff it has a basis of *clopen* sets, i.e., sets that are both closed and open; is *dense-in-itself* iff, for no point x is the singleton $\{x\}$ open; and is *separable* iff it has a countable dense subset.

Theorem 3.1. (Main result) QH *is strongly complete for* any *zero-dimensional dense-in-itself metrizable space, with a constant domain of cardinality $\leq$ the space's weight, i.e., the minimal cardinality of a basis for the space.*

Corollary 3.2. QH *is strongly complete for any* separable *zero-dimensional dense-in-itself metrizable space, with a constant* countable *domain.*

We prove Theorem 3.1 in Section 7, below. Corollary 3.2 follows immediately since every separable metrizable space has a countable basis. As noted in our introductory remarks, Corollary 3.2 substantially generalizes the known results for $\mathbb{P}$ and $\mathbb{Q}$. We leave it open whether QH is strongly complete for any zero-dimensional dense-in-itself metrizable space, separable or not, with a constant *countable* domain.

If we allow varying domains, then we can both remove the requirement of zero-dimensionality and rely entirely on countable domains:

Theorem 3.3. *For any dense-in-itself metrizable space X, there is a system D of countable domains such that* QH *is strongly complete for the predicate topological space $\langle X, D \rangle$.*

As noted in our introductory remarks, Theorem 3.3 more or less follows from the main result of [16], where the result is stated as simple, i.e., not strong, completeness and there is no constraint on the size of the domains. See Section 7, below, for a proof.

It is worth mentioning an incompleteness result in the constant-domain semantics. A space X is *connected* if it is not the union of two nonempty disjoint open sets, and that a subset $S \subseteq X$ is *connected* (in X) if it is connected as a subspace of X. Note that an open subset of a space X is connected iff it is not the union of two nonempty disjoint open sets. A space is *locally connected* if it has a basis consisting of connected open sets. Note that $\mathbb{R}$ is locally connected, since it has as a basis the family of open intervals. Also, $\mathbb{R}$ is a dense-in-itself metrizable space. Thus, the following result from [10] shows that Theorem 3.1 cannot be generalized to all dense-in-themselves metrizable spaces:

Theorem 3.4. *For any locally connected space X,* QH *is not complete for X with constant domains; i.e.,* QH *is not complete for the class $\{\langle X, D \rangle : D$ is a constant domain$\}$.*

Proof. Let A be the formula,

$$\forall \mathsf{x}(\mathsf{Px} \vee \exists \mathsf{x} {\sim} \mathsf{Px}) \ \& \ \forall \mathsf{x}(\mathsf{Px} \vee {\sim} \mathsf{Px}) \to (\forall \mathsf{x} \mathsf{Px} \vee \exists \mathsf{x} {\sim} \mathsf{Px}),$$

where P is a unary predicate.[12] The proof of Theorem 3.4 in [10] can be adapted to show that $A \notin \mathrm{QH}$ and that $\langle X, D \rangle \Vdash A$ for any locally connected space X and any constant domain D. [21] provides two slightly simpler examples: $\forall \mathsf{x}(\mathsf{Px} \vee {\sim} \mathsf{Px}) \to (\forall \mathsf{x} \mathsf{Px} \vee \exists \mathsf{x} {\sim} \mathsf{Px})$ (Exercise 8.22) and Markov's principle $\forall \mathsf{x}(\mathsf{Px} \vee {\sim} \mathsf{Px}) \ \& \ {\sim}{\sim} \exists \mathsf{x} \mathsf{Px} \to \exists \mathsf{x} \mathsf{Px}$ (Exercise 8.23). □

4 Kripke semantics

A *Kripke frame* is an ordered pair $\langle X, R \rangle$, where X is a nonempty set and $R \subseteq X \times X$ is a *preorder*, i.e., a reflexive and transitive relation. As with

[12] If the language has no unary predicates, then choose some nonunary predicate P' of arity, say n, and replace Px in the formula with $\mathsf{P}'\mathsf{x} \dots \mathsf{x}$, with x occurring n times.

topological spaces, we will not carefully distinguish a Kripke frame $\langle X, R\rangle$ from the underlying set X, often using the notation X ambiguously for each. We say that $r \in X$ is a *root* of X iff $\forall x \in X, rRx$. We say that X is *rooted* iff X has at least one root. Given $x \in X$, $R(x) =_{\text{df}} \{x' \in x : xRx'\}$.

It is well-known that each Kripke frame can be identified with a topological space as follows: given a Kripke frame $\langle X, R\rangle$, say that a set $S \subseteq X$ is *open* iff $(\forall x \in S)(\forall x' \in X)(\text{if } xRx' \text{ then } x' \in S)$. It is easy to check that the open sets form a topology: indeed, X equipped with the resulting topology is an *Alexandrov space*, i.e., a topological space in which the arbitrary intersection of open sets is open or, equivalently, in which every point has a least open neighbourhood. Now, given any topological space X, Alexandrov or not, we can define the *specialization* preorder $R_X =_{\text{df}} \{\langle x, y\rangle : x \in Cl(\{y\})\}$. The Alexandrov spaces are exactly those spaces X whose open sets are the open sets in the Kripke frame $\langle X, R_X\rangle$. Henceforth, we will simply identify the Kripke frame $\langle X, R\rangle$ with the Alexandrov space X endowed with the above-defined topology; alternatively, we will simply identify the Alexandrov space X with the Kripke frame $\langle X, R_X\rangle$. Note that, in topological terms, r is a *root* of an Alexandrov space X iff r is in every nonempty closed subset of X.

In Section 2 we defined predicate topological spaces. If X is an Alexandrov space, then note that the expanding-domain condition on a system of domains is equivalent to the following – which explains our 'expanding-domain' terminology (see page 7):

$$\text{for every } x, x' \in X, \text{ if } xR_Xx' \text{ then } D_x \subseteq D_{x'}.$$

This is precisely the standard expanding-domain condition on Kripke frames.

At this point, the Kripke semantics just becomes a special case of the topological semantics: it is the topological semantics restricted to Alexandrov spaces. In particular, suppose that $\langle X, D\rangle$ is a predicate topological space, where X is an Alexandrov space, and that $\mathbf{M}$ is a model based on $\langle X, D\rangle$. Define $\mathbf{M}, x \vDash A$, for every $x \in X$ and every D_x-sentence A with the following clauses from standard Kripke semantics:

$$
\begin{array}{rll}
\mathbf{M}, x \Vdash \mathsf{P}\mathsf{t}_1 \ldots \mathsf{t}_n & \text{if} & x \in V(\mathsf{P})(\mathit{Val}_{\mathbf{M}}(\mathsf{t}_1), \ldots, \mathit{Val}_{\mathbf{M}}(\mathsf{t}_n)), \text{ where } \mathsf{P} \in \mathsf{Pred}_n \\
\mathbf{M}, x \Vdash (A \;\&\; B) & \text{if} & \mathbf{M}, x \Vdash A \text{ and } \mathbf{M}, x \Vdash B \\
\mathbf{M}, x \Vdash (A \vee B) & \text{if} & \mathbf{M}, x \Vdash A \text{ or } \mathbf{M}, x \Vdash B \\
\mathbf{M}, x \Vdash {\sim}A & \text{if} & \text{for every } x' \text{ with } xR_Xx', \mathbf{M}, x' \nVdash A \\
\mathbf{M}, x \Vdash (A \to B) & \text{if} & \text{for every } x' \text{ with } xR_Xx', \\
& & \text{if } \mathbf{M}, x' \Vdash A \text{ then } \mathbf{M}, x' \Vdash B \\
\mathbf{M}, x \Vdash \forall \mathsf{x} A & \text{if} & \text{for every } x' \text{ with } xR_Xx', \\
& & \text{and for every } d \in D_{x'}, \mathbf{M}, x' \Vdash [d/\mathsf{x}]A \\
\mathbf{M}, x \Vdash \exists \mathsf{x} A & \text{if} & \text{for some } d \in D_x, \mathbf{M}, x \Vdash [d/\mathsf{x}]A
\end{array}
$$

Note that $\mathit{Val}_{\mathbf{M}}(A) = \{x \in X : \mathbf{M}, x \Vdash A\}$, where $\mathit{Val}_{\mathbf{M}}(A)$ is defined as on page 8.

The following theorem, here transposed to Alexandrov spaces rather than Kripke frames, is well-known (see, e.g., [7], Theorem 6.2.25) and will be extremely useful:

Lemma 4.1. QH *is strongly complete for the class of countable rooted predicate spaces* $\langle X, D\rangle$ *where* X *is Alexandrov.*[13]

5 Satisfiability transferring maps

In propositional modal logic, p-morphisms are standard tools for transferring satisfiability and hence completeness from one Kripke frame or model to another. In the topological setting, propositional p-morphisms generalize to surjective *interior maps*. In particular, a function from one topological space to another is *continuous* iff the preimage of every open set is open, is *open* iff the image of every open set is open, and is an *interior map* iff it is continuous and open. In the propositional setting, surjective interior maps preserve [strong] completeness backwards in the following sense: if $\varphi : X \to Y$ is a surjective interior map and H is [strongly] complete for Y then H is [strongly] complete for X. The reason is that surjective interior maps preserve satisfiability backwards in the following sense: if $\varphi : X \to Y$ is a surjective interior map and the pair $\langle \Gamma, \Delta\rangle$ of sentences is satisfiable in Y then $\langle \Gamma, \Delta\rangle$ is satisfiable in X.

[7] extends the notion of a p-morphism between Kripke frames to the notion of a predicate p-morphism between *predicate* Kripke frames (i.e., Kripke

[13] [7] states this theorem for languages without function symbols, but the result extends to languages with function symbols.

frames equipped with a domain at each world). An obvious strategy is to generalize the definition in [7] to predicate topological spaces. But this won't quite do. In Section 6, below, we define a topological space $2^{\leq\omega}$, the infinite binary tree with limits, and we show that QH is strongly complete for $2^{\leq\omega}$. For every dense-in-itself space X, [9] defines a continuous function $f_X : X \to 2^{\leq\omega}$, and we would like to make use of f_X to transfer strong completeness from $2^{\leq\omega}$ back to X – *even though* f_X *sometimes fails to be an interior map*. We will define a new notion of a *quasi-interior map* between topological spaces, which will induce a notion of a *quasi-p-morphism* between predicate topological spaces and also between predicate topological models.

For any topological space X and any $S \subseteq X$, define the *openure* of S, $Op(S)$ as the intersection of all open supersets of S: $Op(S) =_{\text{df}} \bigcap\{O \subseteq X : O \text{ is open and } S \subseteq O\}$.[14] Say that S is *openish* iff $Op(S)$ is open and $Op(S) \cap O \subseteq Op(S \cap O)$ whenever $O \subseteq X$ is open.[15] Note that every open set is openish, but not vice-versa. Say that a function between topological spaces is *openish* iff the image of every open set is openish, and is a *quasi-interior map* iff it is continuous and openish.

Definition 5.1. Suppose that $\mathbf{X} = \langle X, D\rangle$ and $\mathbf{X'} = \langle X', D'\rangle$ are predicate topological spaces, and that $\mathbf{M} = \langle X, D, V\rangle$ and $\mathbf{M'} = \langle X', D', V'\rangle$ are predicate topological models based on $\mathbf{X}$ and $\mathbf{X'}$, respectively.

(i) A *predicate quasi-p-morphism from* $\mathbf{X}$ *to* $\mathbf{X'}$ is an ordered pair $\varphi = \langle \varphi_0, \varphi_1\rangle$, such that

1. $\varphi_0 : X \to X'$ is a quasi-interior map;
2. $\varphi_1 = (\varphi_{1x})_{x\in X}$ is a family of functions indexed by the members of X;
3. every $\varphi_{1x} : D_x \to {D'}_{\varphi_0(x)}$ is a surjective map; and
4. for every $x \in X$ and every $d \in D_x$, there is an open set $O^x_d \subseteq O_d$, such that $x \in O^x_d$ and for every $y \in O^x_d$, $\varphi_{1y}(d) = \varphi_{1x}(d)$. We can stipulate that O^x_d is the largest such open set. For any $x \in X$ and any D_x-sentence A, we define $O^x_A = O^x_{d_1} \cap \ldots \cap O^x_{d_n}$, where $d_1, \ldots, d_n$ are the members of D_x occurring as names in A. If there are no such members, then $O^x_A = X$.

[14] Thanks to Kenny Easwaran for suggesting the terminology 'openure' by analogy with 'closure': $Cl(S) = \bigcap\{C \subseteq X : C \text{ is closed and } S \subseteq C\}$.

[15] Thanks to Hasko von Kriegstein for suggesting 'openish' for this notion.

(ii) A *predicate quasi-p-morphism from* $\mathbf{M}$ *to* $\mathbf{M}'$ is a predicate quasi-p-morphism from $\mathbf{X}$ to $\mathbf{X}'$ such that, for every $x \in X$, for every $\mathsf{P} \in \mathsf{Pred}_n$ ($n \geq 1$), for every $\mathsf{c} \in \mathsf{Names}$, for every $\mathsf{f} \in \mathsf{Func}_n$ ($n \geq 1$), and for every $d_1, \ldots, d_n \in D_x$,

5. $x \in V(\mathsf{P})(d_1 \ldots d_n)$ iff $\varphi_0(x) \in V'(\mathsf{P})(\varphi_{1x}(d_1) \ldots \varphi_{1x}(d_n))$;
6. $\varphi_{1x}(V(\mathsf{c})) = V'(\mathsf{c})$; and
7. $\varphi_{1x}(V(\mathsf{f})(d_1, \ldots, d_n)) = V'(\mathsf{f})(\varphi_{1x}(d_1), \ldots, \varphi_{1x}(d_n))$.

(iii) We say that a predicate quasi-p-morphism $\varphi = \langle \varphi_0, \varphi_1 \rangle$ from one predicate topological space [model] to another is a *predicate p-morphism* iff φ_0 is a surjective interior map, rather than merely a quasi-interior map.

The following lemma, copied almost verbatim from [10], is standard and its proof routine.

Lemma 5.2. *If* $\varphi = \langle \varphi_0, \varphi_1 \rangle$ *is a predicate quasi-p-morphism from* $\mathbf{M} = \langle X, D, V \rangle$ *to* $\mathbf{M}' = \langle X', D', V' \rangle$*, then for every* $D_{\mathbf{X}}$*-term* t*,*

$$\text{for every } x \in X, \text{ if } \mathsf{t} \text{ is a } D_x\text{-term, then}$$
$$\varphi_{1x}(Val_{\mathbf{M}}(\mathsf{t})) = Val_{\mathbf{M}'}(\varphi_{1x} \cdot \mathsf{t}),$$

where $\varphi_{1x} \cdot \mathsf{t}$ *is the* ${D'}_{\varphi_0(x)}$*-term obtained from the* D_x*-term* t *by replacing every occurrence in* t *of every* $d \in D_x$ *with* $\varphi_{1x}(d)$*.*

The statement of the following lemma is also copied almost verbatim from [10] (transposed to the intuitionistic case). Its proof follows the proof of Lemma 5.3 in [10] closely, except that we will require more care since we are working with *quasi*-p-morphisms, rather than p-morphisms, and since both the domain and range are predicate topological spaces with varying domains.

Lemma 5.3. *If* $\varphi = \langle \varphi_0, \varphi_1 \rangle$ *is a predicate quasi-p-morphism from* $\mathbf{M} = \langle X, D, V \rangle$ *to* $\mathbf{M}' = \langle X', D', V' \rangle$*, then for every* $D_{\mathbf{X}}$*-sentence* A*,*

$$\text{for every } x \in X, \text{ if } A \text{ is a } D_x\text{-sentence then}$$
$$x \in Val_{\mathbf{M}}(A) \text{ iff } \varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot A),$$

where $\varphi_{1x} \cdot A$ *is the* $D_{\varphi_0(x)}$*-sentence obtained from the* D_x*-sentence* A *by replacing every occurrence in* A *of every* $d \in D_x$ *with* $\varphi_{1x}(d)$*.*

Proof. We prove this by strong induction on the complexity of A, i.e., the number of quantifier- or connective-occurrences in B. As an inductive hypothesis (IH), suppose that for every $D_{\mathbf{X}}$-sentence B of complexity strictly less than the complexity of A, and every $x \in X$, if B is a D_x-sentence, then $x \in Val_{\mathbf{M}}(B)$ iff $\varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)$. We will verify four cases: (1) A is atomic, (2) $A = (B \to C)$, (3) $A = \exists \mathsf{x} B$, and (4) $A = \forall \mathsf{x} B$.

Case (1): A is of the form $\mathsf{P}\mathsf{t}_1 \dots \mathsf{t}_n$, where $\mathsf{P} \in \mathsf{Pred}_n$ and $\mathsf{t}_1 \dots \mathsf{t}_n$ are $D_{\mathbf{X}}$-terms. Note: $x \in Val_{\mathbf{M}}(A)$ iff $x \in V(\mathsf{P})(Val_{\mathbf{M}}(\mathsf{t}_1), \dots, Val_{\mathbf{M}}(\mathsf{t}_n))$ iff $\varphi_0(x) \in V'(\mathsf{P})(\varphi_{1x}(Val_{\mathbf{M}}(\mathsf{t}_1)) \dots \varphi_{1x}(Val_{\mathbf{M}}(\mathsf{t}_n)))$ (by the definition of a quasi-p-morphism) iff $\varphi_0(x) \in V'(\mathsf{P})(Val_{\mathbf{M}'}(\varphi_{1x} \cdot \mathsf{t}_1) \dots Val_{\mathbf{M}'}(\varphi_{1x} \cdot \mathsf{t}_n))$ (by Lemma 5.2) iff $\varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot A)$

Case (2): $A = B \to C$. Choose any $x \in X$ and assume that A is a D_x-sentence. We want to show that $x \in Val_{\mathbf{M}}(B \to C)$ iff $\varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot (B \to C))$. It is worth recalling here that $Val_{\mathbf{M}'}(\varphi_{1x} \cdot (B \to C)) = Val_{\mathbf{M}'}(\varphi_{1x} \cdot B \to \varphi_{1x} \cdot C) = Int((X' - Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)) \cup Val_{\mathbf{M}'}(\varphi_{1x} \cdot C))$. We consider each direction of the biconditional separately.

($\Rightarrow$) Assume that $x \in Val_{\mathbf{M}}(B \to C)$. Let $O = Val_{\mathbf{M}}(B \to C) \cap O^x_{(B \to C)}$ (see Definition 5.1, (i), (4)). Note that O is open, so that $\varphi_0[O]$ is openish. Thus $Op(\varphi_0[O])$ is open. Also, $\varphi_0(x) \in \varphi_0[O] \subseteq Op(\varphi_0[O])$. To show that $\varphi_0(x) \in Int((X' - Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)) \cup Val_{\mathbf{M}'}(\varphi_{1x} \cdot C))$, it suffices to show that $Op(\varphi_0[O]) \cap Val_{\mathbf{M}'}(\varphi_{1x} \cdot B) \subseteq Val_{\mathbf{M}'}(\varphi_{1x} \cdot C)$. Given the definition of *openish* and given that $Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)$ is open in X', it suffices to show that $Op(\varphi_0[O] \cap Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)) \subseteq Val_{\mathbf{M}'}(\varphi_{1x} \cdot C)$. And given that $Val_{\mathbf{M}'}(\varphi_{1x} \cdot C)$ is open in X', it suffices to show that $\varphi_0[O] \cap Val_{\mathbf{M}'}(\varphi_{1x} \cdot B) \subseteq Val_{\mathbf{M}'}(\varphi_{1x} \cdot C)$.

So suppose that $y \in \varphi_0[O] \cap Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)$. Then $y = \varphi_0(z)$ for some $z \in O$. So $z \in Val_{\mathbf{M}}(B \to C)$ and $\varphi_{1z}(d) = \varphi_{1x}(d)$, for every d occurring in B or in C. Thus $\varphi_{1z} \cdot B = \varphi_{1x} \cdot B$ and $\varphi_{1z} \cdot C = \varphi_{1x} \cdot C$. Thus $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1z} \cdot B)$, since $y \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot B)$. Also, B is a D_z-sentence. So, by (IH), $z \in Val_{\mathbf{M}}(B)$. So $z \in Val_{\mathbf{M}}(C)$, since $z \in Val_{\mathbf{M}}(B \to C)$. So $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1z} \cdot C)$, by (IH). So $y \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot C)$, as desired.

($\Leftarrow$) Assume that $\varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot (B \to C))$. Note that ${\varphi_0}^{-1}[Val_{\mathbf{M}'}(\varphi_{1x} \cdot (B \to C))]$ is open, since φ_0 is continuous. Let $O = {\varphi_0}^{-1}[Val_{\mathbf{M}'}(\varphi_{1x} \cdot (B \to C))] \cap O^x_{(B \to C)}$. Note that $x \in O$. To show that $x \in Val_{\mathbf{M}}(B \to C) = Int((X - Val_{\mathbf{M}}(B)) - Val_{\mathbf{M}}(C))$, it suffices to show that $O \cap Val_{\mathbf{M}}(B) \subseteq Val_{\mathbf{M}}(C)$. So suppose that $z \in O \cap Val_{\mathbf{M}}(B)$. Then, since $z \in O$, we have $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot (B \to C)) = Val_{\mathbf{M}'}(\varphi_{1x} \cdot B \to \varphi_{1x} \cdot C)$, and $\varphi_{1z}(d) = \varphi_{1x}(d)$, for every d occurring in B or in C. Thus $\varphi_{1z} \cdot B = \varphi_{1x} \cdot B$

and $\varphi_{1z} \cdot C = \varphi_{1x} \cdot C$. Thus $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1z} \cdot B \to \varphi_{1z} \cdot C)$. Also, since $z \in Val_{\mathbf{M}}(B)$, by (IH) we have $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1z} \cdot B)$. So $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1z} \cdot C)$. So, again by (IH), $z \in Val_{\mathbf{M}}(C)$, as desired.

Case (3): $A = \exists \mathsf{x} B$. Choose any $x \in X$ and assume that A is a D_x-sentence. Note

$$
\begin{array}{lll}
x \in Val_{\mathbf{M}}(A) & \text{iff} & (\exists d \in D_{\mathbf{X}})(x \in Val_{\mathbf{M}}([d/\mathsf{x}]B)) \\
& \text{iff}^* & (\exists d \in D_x)(x \in Val_{\mathbf{M}}([d/\mathsf{x}]B)) \\
& \text{iff} & (\exists d \in D_x)(\varphi_0(x) \in Val_{\mathbf{M}}(\varphi_{1x} \cdot [d/\mathsf{x}]B)) \text{ (by IH)} \\
& \text{iff} & (\exists d \in D_x)(\varphi_0(x) \in Val_{\mathbf{M}}([\varphi_{1x}(d)/\mathsf{x}](\varphi_{1x} \cdot B))) \\
& \text{iff}^\dagger & (\exists d' \in D'_{\varphi_0(x)})(\varphi_0(x) \in Val_{\mathbf{M}}([d'/\mathsf{x}](\varphi_{1x} \cdot B))) \\
& \text{iff}^{**} & (\exists d' \in D'_{\mathbf{X}'})(\varphi_0(x) \in Val_{\mathbf{M}}([d'/\mathsf{x}](\varphi_{1x} \cdot B))) \\
& \text{iff} & \varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot A).
\end{array}
$$

Some remarks about the flagged iff's. The right-to-left direction of iff* is obvious, since $D_x \subseteq D_{\mathbf{X}}$. The argument for the left-to-right direction of iff* depends on whether x is free in B: supposing that x is free B, if $x \in Val_{\mathbf{M}}([d/\mathsf{x}]B)$ where $d \in D_{\mathbf{X}}$, then $d \in D_x$ since $x \in Val_{\mathbf{M}}([d/\mathsf{x}]B) \subseteq O_{[d/\mathsf{x}]B} \subseteq O_d$; supposing that x is not free in B, if $d \in D_{\mathbf{X}}$, then the $D_{\mathbf{X}}$-sentence $[d/\mathsf{x}]B$ is just the D_x-sentence B, which is the D_x-sentence $[d'/\mathsf{x}]B$ for any $d' \in D_x$. For the left-to-right direction of iff†, note that $(\forall d \in D_x)(\varphi_{1x}(d) \in D'_{\varphi_0(x)})$; for the right-to-left direction, recall that $\varphi_{1x} : D_x \to D'_{\varphi_0(x)}$ is surjective. The biconditional flagged as iff** is similar to iff*.

Case (4): $A = \forall \mathsf{x} B$. Choose any $x \in X$ and assume that A is a D_x-sentence. We want to show that $x \in Val_{\mathbf{M}}(\forall \mathsf{x} B)$ iff $\varphi_0(x) \in Val_{\mathbf{M}'}(\varphi_{1x} \cdot \forall \mathsf{x} B)$. We consider each direction of the biconditional separately.

($\Rightarrow$) Assume that $x \in Val_{\mathbf{M}}(\forall \mathsf{x} B)$. Let $O = Val_{\mathbf{M}}(\forall \mathsf{x} B) \cap O^x_{\forall \mathsf{x} B}$ (see Definition 5.1, (i), (4)). Note that O is open in X. We will first show,

$$(\forall y \in \varphi_0[O])(\forall d \in D'_y)(y \in Val_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B))) \qquad (*)$$

Note that we will not be done after showing $(*)$, since $\varphi_0[O]$ might not be open, though it is certainly openish.

To see $(*)$, choose $y \in \varphi_0[O]$ and $d \in D'_y$. Note that $y = \varphi_0(z)$, for some $z \in O$; and $d = \varphi_{1z}(d_0)$, for some $d_0 \in D_z$. Note that $z \in Val_{\mathbf{M}}(\forall \mathsf{x} B)$, since $z \in O$. So $z \in Val_{\mathbf{M}}([d_0/\mathsf{x}]B)$. So, by (IH), $\varphi_0(z) \in Val_{\mathbf{M}'}(\varphi_{1z} \cdot ([d_0/\mathsf{x}]B))$. So $y \in Val_{\mathbf{M}'}([\varphi_{1z}(d_0)/\mathsf{x}](\varphi_{1z} \cdot B))$. So $y \in Val_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1z} \cdot B))$. Also, $z \in O^x_{\forall \mathsf{x} B}$. So $\varphi_{1z}(d) = \varphi_{1x}(d)$, for every $d \in D_{\mathbf{X}}$ that occurs as a name in B. So $\varphi_{1z} \cdot B = \varphi_{1x} \cdot B$. So $y \in Val_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B))$, as desired.

For any $d \in D'_{\mathbf{X}'}$, let $O'_d = \{y \in X' : d \in D_y\}$. Recall that O'_d is open. Given $(*)$, $(\forall d \in D'_{\mathbf{X}'})(\forall y \in \varphi_0[O] \cap O'_d)(y \in \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B)))$. So, $(\forall d \in D'_{\mathbf{X}'})(\varphi_0[O] \cap O'_d \subseteq \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B)))$. So $(\forall d \in D'_{\mathbf{X}'})(Op(\varphi_0[O] \cap O'_d) \subseteq \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B)))$, since $\mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B))$ is open. Also, $Op(\varphi_0[O]) \cap O'_d \subseteq Op(\varphi_0[O] \cap O'_d)$, since $\varphi_0[O]$ is openish and O'_d is open. So $(\forall d \in D'_{\mathbf{X}'})(Op(\varphi_0[O]) \cap O'_d \subseteq \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B)))$. So $(\forall d \in D'_{\mathbf{X}'})(\forall y \in Op(\varphi_0[O]) \cap O'_d)(y \in \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B)))$. So $(\forall y \in Op(\varphi_0[O]))(\forall d \in D'_y)(y \in \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}](\varphi_{1x} \cdot B)))$. Now $Op(\varphi_0[O])$ is open since $\varphi_0[O]$ is openish. And $\varphi_0(x) \in Op(\varphi_0[O])$. So $\varphi_0(x) \in \mathit{Val}_{\mathbf{M}'}(\forall\mathsf{x}(\varphi_{1x} \cdot B))$. So $\varphi_0(x) \in \mathit{Val}_{\mathbf{M}'}(\varphi_{1x} \cdot \forall\mathsf{x}B)$, as desired.

($\Leftarrow$) Assume that $\varphi_0(x) \in \mathit{Val}_{\mathbf{M}'}(\varphi_{1x} \cdot \forall\mathsf{x}B)$. Note that ${\varphi_0}^{-1}[\mathit{Val}_{\mathbf{M}'}(\varphi_{1x} \cdot \forall\mathsf{x}B)]$ is open, since φ_0 is continuous. Let $O = {\varphi_0}^{-1}[\mathit{Val}_{\mathbf{M}'}(\varphi_{1x} \cdot \forall\mathsf{x}B)] \cap O^x_{\forall\mathsf{x}B}$. Note that $x \in O$ and O is open in X. To show that $x \in \mathit{Val}_{\mathbf{M}}(\forall\mathsf{x}B)$, it suffices to show

$$(\forall z \in O)(\forall d \in D_z)(z \in \mathit{Val}_{\mathbf{M}'}([d/\mathsf{x}]B))$$

So suppose that $z \in O$ and $d \in D_z$. Then, since $z \in O$, we have $\varphi_0(z) \in \mathit{Val}_{\mathbf{M}'}(\varphi_{1x} \cdot \forall\mathsf{x}B)$, and $\varphi_{1z}(d) = \varphi_{1x}(d)$, for every d occurring in $\forall\mathsf{x}B$. Thus $\varphi_{1z} \cdot \forall\mathsf{x}B = \varphi_{1x} \cdot \forall\mathsf{x}B$. Thus $\varphi_0(z) \in \mathit{Val}_{\mathbf{M}'}(\varphi_{1z} \cdot \forall\mathsf{x}B) = \mathit{Val}_{\mathbf{M}'}(\forall\mathsf{x}(\varphi_{1z} \cdot B))$. So $\varphi_0(z) \in \mathit{Val}_{\mathbf{M}'}([\varphi_{1z}(d)/\mathsf{x}](\varphi_{1z} \cdot B)) = \mathit{Val}_{\mathbf{M}'}(\varphi_{1z} \cdot [d/\mathsf{x}]B)$. So by (IH) we have $z \in \mathit{Val}_{\mathbf{M}}([d/\mathsf{x}]B)$, as desired. □

In the propositional case, surjective interior maps transfer strong completeness from the target topological space back to the source, by transferring satisfiability from the target back to the source. Our quasi-p-morphisms are based on quasi-interior maps, and we make no assumption of surjectivity of these maps: this endangers the nice satisfiability-transfering property. But there's a way out. Suppose that $\mathbf{X} = \langle X, D\rangle$ is a predicate topological space and that $Y \subseteq X$. Then we say that a pair $\langle \Gamma, \Delta\rangle$ of sentences is *satisfiable in* $\mathbf{X} = \langle X, D\rangle$ *by way of* Y iff there is a predicate topological model $\mathbf{M} = \langle X, D, V\rangle$ such that $\mathit{Val}_{\mathbf{M}}(\langle \Gamma, \Delta\rangle) \cap Y \neq \emptyset$.

Now we can state a satisfiability-transferring corollary to Lemma 5.3. The simplest satisfiability-transferring result is available when the language $\mathcal{L}$ has no names and no function symbols: these introduce complications that we will attend to shortly.

Corollary 5.4. *Suppose that the language $\mathcal{L}$ has no names and no function symbols. Suppose that $\varphi = \langle \varphi_0, \varphi_1\rangle$ is a predicate quasi-p-morphism from $\mathbf{X} = \langle X, D\rangle$ to $\mathbf{X}' = \langle X', D'\rangle$. And suppose that the pair $\langle \Gamma, \Delta\rangle$ of sentences* *is satisfiable in* $\mathbf{X}' = \langle X', D' \rangle$ *by way of* $\varphi_0[X]$. *Then* $\langle \Gamma, \Delta \rangle$ *is satisfiable in* $\mathbf{X}$.

Proof. We want to show that there is a predicate topological model $\mathbf{M} = \langle X, D, V \rangle$ such that $Val_{\mathbf{M}}(\langle \Gamma, \Delta \rangle) \neq \emptyset$. Since $\langle \Gamma, \Delta \rangle$ is satisfiable in $\mathbf{X}' = \langle X', D' \rangle$ by way of $\varphi_0[X]$, there is a predicate topological model $\mathbf{M}' = \langle X', D', V' \rangle$ such that $Val_{\mathbf{M}'}(\langle \Gamma, \Delta \rangle) \cap \varphi_0[X] \neq \emptyset$.

Define the valuation V for the predicate topological space $\langle X, D \rangle$ as follows: $V(\mathsf{P})(d_1 \dots d_n) = \{x \in X : \varphi_0(x) \in V'(\mathsf{P})(\varphi_{1x}(d_1) \dots \varphi_{1x}(d_n))\}$, for $d_1, \dots, d_n \in \bigcup_{x \in X} D_x$. And let $\mathbf{M} = \langle X, D, V \rangle$. Note that $\varphi = \langle \varphi_0, \varphi_1 \rangle$ is a predicate quasi-p-morphism from $\mathbf{M}$ to $\mathbf{M}'$. Choose any $y \in Val_{\mathbf{M}'}(\langle \Gamma, \Delta \rangle) \cap \varphi_0[X]$. Since $y \in \varphi_0[X]$, there is some $x \in X$ such that $\varphi_0(x) = y$. Note that no members of D_x occur in any of the sentences in $\Gamma \cup \Delta$, since these are sets of sentences in the original language $\mathcal{L}$. So $\varphi_{1x} \cdot A = A$, for every $A \in \Gamma \cup \Delta$. So, by Lemma 5.3, $x \in Val_{\mathbf{M}}(\langle \Gamma, \Delta \rangle)$. So $Val_{\mathbf{M}}(\langle \Gamma, \Delta \rangle) \neq \emptyset$, as desired. □

When the language has names or function symbols, then this simple proof of Corollary 5.4 fails: it is not so easy to proceed as in the second paragraph of the proof, where we define the valuation V for the predicate topological space $\langle X, D \rangle$ so that $\varphi = \langle \varphi_0, \varphi_1 \rangle$ is a predicate quasi-p-morphism from $\mathbf{M}$ to $\mathbf{M}'$. Indeed, we do not know whether such a valuation can, in general, be defined.

We adopt a solution to this problem from [10], updated and corrected in [11], and updated futher to handle varying domains. Suppose that $\langle X, D \rangle$ is a predicate topological space. We define new system, $D^\dagger$, of domains for X, closely related to the system D. First, let $\mathcal{L}^\dagger$ be a language just like $\mathcal{L}$, except that every name $\mathsf{c} \in \mathsf{Names}$ [function symbol $\mathsf{f} \in \mathsf{Func}$] is replaced by a name $\mathsf{c}^\dagger$ [function symbol $\mathsf{f}^\dagger$].[16] For any nonempty set S, define $\mathsf{Term}^\dagger(S)$ as the set of terms in the language $\mathcal{L}^\dagger(S)$, i.e, the language $\mathcal{L}^\dagger$ expanded with the members of S as names. And let ${D^\dagger}_x = \mathsf{Term}^\dagger(D_x)$, for every $x \in X$.

To ensure that $D^\dagger$ is indeed a system of domains, we check that it satisfies the expanding-domain condition (see page 7): we check that the set ${O^\dagger}_{\mathsf{t}} = \{x \in X : \mathsf{t} \in {D^\dagger}_x\}$ is an open subset of X, for every $\mathsf{t} \in {D^\dagger}_{\mathbf{X}^\dagger} = \bigcup_{y \in X} \mathsf{Term}^\dagger(D_y)$. We fix $y \in X$, and show by induction on $\mathsf{t} \in \mathsf{Term}^\dagger(D_y)$ that ${O^\dagger}_{\mathsf{t}}$ is open. If $\mathsf{t} = d \in D_y$, then ${O^\dagger}_{\mathsf{t}} = O_d$, which is open in X. If $\mathsf{t} = \mathsf{c}^\dagger$ where $\mathsf{c} \in \mathsf{Names}$, then ${O^\dagger}_{\mathsf{t}} = X$. For the inductive step, assume that each of

[16] A detail: none of the new names or function symbols should already occur in the syntax of $\mathcal{L}(D_{\mathbf{X}})$, where $D_{\mathbf{X}} = \bigcup_{x \in X} D_x$.

$O^{\dagger}{}_{\mathsf{t}_1}, \ldots, O^{\dagger}{}_{\mathsf{t}_n}$ is open, where $\mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}^{\dagger}(D_y)$; and consider $O^{\dagger}{}_{\mathsf{f}^{\dagger}\mathsf{t}_1\ldots\mathsf{t}_n}$, where $\mathsf{f} \in \mathsf{Func}_n$. Note that $O^{\dagger}{}_{\mathsf{f}^{\dagger}\mathsf{t}_1\ldots\mathsf{t}_n} = O^{\dagger}{}_{\mathsf{t}_1} \cap \ldots \cap O^{\dagger}{}_{\mathsf{t}_n}$, which is open in X.

Let $\mathbf{X}^{\dagger} = \langle X, D^{\dagger} \rangle$. It will be useful to note two facts about $D^{\dagger}$: (i) since $\mathcal{L}$ and therefore $\mathcal{L}^{\dagger}$ have at most countably many names and function symbols, $card(D^{\dagger}{}_x) = max(\aleph_0, card(D_x))$, for every $x \in X$. (ii) If $D_x = D_y$ for every $x, y \in X$, then $D^{\dagger}{}_x = D^{\dagger}{}_y$ for every $x, y \in X$.

Corollary 5.5. *Suppose that $\varphi = \langle \varphi_0, \varphi_1 \rangle$ is a predicate quasi-p-morphism from $\mathbf{X} = \langle X, D \rangle$ to $\mathbf{X}' = \langle X', D' \rangle$. And suppose that the pair $\langle \Gamma, \Delta \rangle$ of sentences is satisfiable in $\mathbf{X}' = \langle X', D' \rangle$ by way of $\varphi_0[X]$. Then $\langle \Gamma, \Delta \rangle$ is satisfiable in $\mathbf{X}^{\dagger} = \langle X, D^{\dagger} \rangle$.*

Proof. First (to fix notation), since $\varphi = \langle \varphi_0, \varphi_1 \rangle$ is a predicate quasi-p-morphism from $\mathbf{X} = \langle X, D \rangle$ to $\mathbf{X}' = \langle X', D' \rangle$, we have the following: for every $x \in X$ and every $d \in D_x$, there is an open set $O^x_d \subseteq O_d$, such that $x \in O^x_d$ and for every $y \in O^x_d$, $\varphi_{1y}(d) = \varphi_{1x}(d)$.

We want to show that there is a predicate topological model $\mathbf{M}^{\dagger} = \langle X, D^{\dagger}, V^{\dagger} \rangle$ such that $Val_{\mathbf{M}^{\dagger}}(\langle \Gamma, \Delta \rangle) \neq \emptyset$. Since $\langle \Gamma, \Delta \rangle$ is satisfiable in $\mathbf{X}' = \langle X', D' \rangle$ by way of $\varphi_0[X]$, there is a predicate topological model $\mathbf{M}' = \langle X', D', V' \rangle$ such that $Val_{\mathbf{M}'}(\langle \Gamma, \Delta \rangle) \cap \varphi_0[X] \neq \emptyset$.

We now define, for each $x \in X$, a function $\varphi^{\dagger}{}_{1x} : D^{\dagger}{}_x \to D'{}_{\varphi_0(x)}$. That is, we will define a function $\varphi^{\dagger}{}_{1x} : \mathsf{Term}^{\dagger}(D_x) \to D'{}_{\varphi_0(x)}$, recursively on the structure of terms in the language $\mathcal{L}^{\dagger}(D_x)$ as follows. First, $\varphi^{\dagger}{}_{1x}(d) = \varphi_{1x}(d)$, if $d \in D_x$. Second, $\varphi^{\dagger}{}_{1x}(c^{\dagger}) = V'(c)$, if $c \in \mathsf{Names}$. And third, $\varphi^{\dagger}{}_{1x}(\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n) = V'(\mathsf{f})(\varphi^{\dagger}{}_{1x}(\mathsf{t}_1), \ldots, \varphi^{\dagger}{}_{1x}(\mathsf{t}_n))$, if $\mathsf{f} \in \mathsf{Func}_n$ and $t_1, \ldots, t_n \in \mathsf{Term}^{\dagger}(D_x)$. We claim that $\varphi^{\dagger} = \langle \varphi_0, \varphi^{\dagger}{}_1 \rangle$ is a quasi-p-morphism from $\mathbf{X}^{\dagger}$ to $\mathbf{X}'$. It suffices to show that the family $\{\varphi^{\dagger}{}_{1x}\}_{x \in X}$ of functions satisfy Clauses (i)(3) and (i)(4) in Definition 5.1.

Clause (i)(3) We want to show that every $\varphi^{\dagger}{}_{1x} : D^{\dagger}{}_x \to D'{}_{\varphi_0(x)}$ is a surjective map. Note that $D_x \subseteq D^{\dagger}{}_x$, and that $\varphi^{\dagger}{}_{1x}$ agrees with φ_{1x} on D_x. Also note that $\varphi_{1x} : D_x \to D'{}_{\varphi_0(x)}$ is surjective.

Clause (i)(4) We want to show that for every $x \in X$ and every $d \in D^{\dagger}{}_x$, there is an open set $O^{\dagger x}_{d} \subseteq O_d$, such that $x \in O^{\dagger x}_{d}$ and for every $y \in O^{\dagger x}_{d}$, $\varphi^{\dagger}{}_{1y}(d) = \varphi^{\dagger}{}_{1x}(d)$. Fix $x \in X$. We will show by induction that, for every $\mathsf{t} \in \mathsf{Term}^{\dagger}(D_x)$, there is an open set $O^{\dagger x}_{\mathsf{t}} \subseteq O^{\dagger}{}_{\mathsf{t}}$, such that $x \in O^{\dagger x}_{\mathsf{t}}$ and for every $y \in O^{\dagger x}_{\mathsf{t}}$, $\varphi^{\dagger}{}_{1y}(\mathsf{t}) = \varphi^{\dagger}{}_{1x}(\mathsf{t})$. If $\mathsf{t} = d \in D_x$, then it will suffice to let $O^{\dagger x}_{\mathsf{t}} = O_d$, and if $\mathsf{t} = \mathsf{c}^{\dagger}$ where $\mathsf{c} \in \mathsf{Names}$, then it will suffice to let $O^{\dagger x}_{\mathsf{t}} = X$. For the inductive step, suppose that, for each $i = 1, \ldots, n$, we have an open set ${O^{\dagger}}^x_{\mathsf{t}_i} \subseteq {O^{\dagger}}_{\mathsf{t}_i}$, such that $x \in {O^{\dagger}}^x_{\mathsf{t}_i}$ and for every $y \in {O^{\dagger}}^x_{\mathsf{t}_i}$, ${\varphi^{\dagger}}_{1y}(\mathsf{t}_i) = {\varphi^{\dagger}}_{1x}(\mathsf{t}_i)$. And consider $\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n$, where $\mathsf{f} \in \mathsf{Func}_n$. If we let ${O^{\dagger}}^x_{\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n} = {O^{\dagger}}^x_{\mathsf{t}_1} \cap \ldots \cap {O^{\dagger}}^x_{\mathsf{t}_n}$, then note that $x \in {O^{\dagger}}^x_{\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n}$ and for every $y \in {O^{\dagger}}^x_{\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n}$, ${\varphi^{\dagger}}_{1y}(\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n) = {\varphi^{\dagger}}_{1x}(\mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n)$.

Now we define a valuation $V^{\dagger}$ for $\mathbf{X}^{\dagger} = \langle X, D^{\dagger} \rangle$:

- $V^{\dagger}(\mathsf{P})(\mathsf{t}_1 \ldots \mathsf{t}_n) = \{x \in X : \varphi_0(x) \in V'(\mathsf{P})({\varphi^{\dagger}}_{1x}(\mathsf{t}_1) \ldots {\varphi^{\dagger}}_{1x}(\mathsf{t}_n))\}$, for $\mathsf{t}_1, \ldots, \mathsf{t}_n \in {D^{\dagger}}_{\mathbf{X}^{\dagger}}$;
- $V^{\dagger}(\mathsf{c}) = \mathsf{c}^{\dagger}$; and
- $V(\mathsf{f})(\mathsf{t}_1, \ldots, \mathsf{t}_n) = \mathsf{f}^{\dagger}\mathsf{t}_1 \ldots \mathsf{t}_n$, for $\mathsf{t}_1, \ldots, \mathsf{t}_n \in {D^{\dagger}}_{\mathbf{X}^{\dagger}}$.

Note that $\varphi^{\dagger} = \langle \varphi_0, {\varphi^{\dagger}}_1 \rangle$ satisfies Clauses (ii)(5)-(ii)(7) of Definition 5.1. So $\varphi^{\dagger}$ is a quasi-p-morphism from $\mathbf{M}^{\dagger} = \langle X, D^{\dagger}, V^{\dagger} \rangle$ to $\mathbf{M}' = \langle X', D', V' \rangle$.

Recall that $Val_{\mathbf{M}'}(\langle \Gamma, \Delta \rangle) \cap \varphi_0[X] \neq \emptyset$. Choose any $y \in Val_{\mathbf{M}'}(\langle \Gamma, \Delta \rangle) \cap \varphi_0[X]$. Since $y \in \varphi_0[X]$, there is some $x \in X$ such that $\varphi_0(x) = y$. Note that no members of ${D^{\dagger}}_x$ occur in any of the sentences in $\Gamma \cup \Delta$, since these are sets of sentences in the original language $\mathcal{L}$. So $\varphi_{1x} \cdot A = A$, for every $A \in \Gamma \cup \Delta$. So, by Lemma 5.3, $x \in Val_{\mathbf{M}^{\dagger}}(\langle \Gamma, \Delta \rangle)$. So $Val_{\mathbf{M}^{\dagger}}(\langle \Gamma, \Delta \rangle) \neq \emptyset$, as desired. □

6 The infinite binary tree [with limits]

For each $n \geq 0$, let 2^n be the set of binary sequences (sequences of 0's and 1's) of length n. Let $2^{<\omega} =_{\mathrm{df}} \bigcup_{n=0}^{\infty} 2^n$, i.e., $2^{<\omega}$ is the set of finite binary sequences. Let 2^{ω} be the set of infinite binary sequences of order type ω. And let $2^{\leq\omega} =_{\mathrm{df}} 2^{<\omega} \cup 2^{\omega}$. We use Λ for the the empty binary sequence, i.e., the binary sequence of length 0. We use italic b, b', etc., to range over $2^{<\omega}$; bold $\mathbf{b}, \mathbf{b}'$, etc., to range over 2^{ω}; and bold-italic $\boldsymbol{b}, \boldsymbol{b}'$, etc., to range over $2^{\leq\omega}$. If $b \in 2^{<\omega}$ and $\boldsymbol{b} \in 2^{\leq\omega}$, then we write $b^\frown\boldsymbol{b}$ for b concatenated with $\boldsymbol{b}$. We write $b0$ and $b1$ for $b^\frown\langle 0 \rangle$ and $b^\frown\langle 1 \rangle$. For any $b \in 2^{<\omega}$, we write $|b|$ for the length of b. Given any $\mathbf{b} \in 2^{\omega}$ and any $n \in \mathbb{N}$, the finite binary sequence $\mathbf{b}|_n$ is the initial segment of length n of $\mathbf{b}$. Thus $\mathbf{b}|_0 = \Lambda$ and $|\mathbf{b}|_n| = n$. Given $b \in 2^{<\omega}$ and $\boldsymbol{b} \in 2^{\leq\omega}$, we say $b \leq \boldsymbol{b}$ iff b is an initial segment of $\boldsymbol{b}$ and $b < \boldsymbol{b}$ iff both $b \leq \boldsymbol{b}$ and $b \neq \boldsymbol{b}$. We will also use '$\leq$' for $\leq$ restricted to $2^{<\omega}$.

We now impose topologies on $2^{<\omega}$ and on $2^{\leq\omega}$:

- $2^{<\omega}$. We identify $2^{<\omega}$ with the *infinite binary tree*, i.e., the countable rooted Kripke frame $\langle 2^{<\omega}, \leq \rangle$. For each $b \in 2^{<\omega}$, let $\leq(b) =_{\mathrm{df}} \{b' \in 2^{<\omega} : b \leq b'\}$. Note that the family $\{\leq(b) : b \in 2^{<\omega}\}$ is a basis for the Alexandrov topology on $2^{<\omega}$ induced by $\leq$. We also identify $2^{<\omega}$ with the resulting Alexandrov space.

- $2^{\leq\omega}$. For any $b \in 2^{<\omega}$, define $\leq^+(b) =_{\mathrm{df}} \{\boldsymbol{b}' \in 2^{\leq\omega} : b \leq \boldsymbol{b}'\}$. (Note that $\leq^+(b) \supsetneq \leq(b)$.) And take as a basis of our topology on $2^{\leq\omega}$ the family $\{\leq^+(b) : b \in 2^{<\omega}\}$. We call $2^{\leq\omega}$, with this topology, *the infinite binary tree with limits*.[17] Unlike $2^{<\omega}$, $2^{\leq\omega}$ is not an Alexandrov space. For example, for any $\mathbf{b} \in 2^{\omega}$ the intersection of the open sets of the form $\leq^+(\mathbf{b}|_n)$, $n \in \mathbb{N}$, is not open. Indeed $\bigcap_{n\in\mathbb{N}} \leq^+(\mathbf{b}|_n) = \{\mathbf{b}\}$.

Lemma 6.1. *There is a system $D^\dagger$ of countable domains such that* QH *is strongly complete for the predicate topological space $\langle 2^{<\omega}, D^\dagger\rangle$.*

Proof. Let $S_0, S_1, S_2, \ldots$ be a sequence of countably infinite sets such that $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and such that $S_{n+1} - S_n$ is itself countably infinite for any $n \in \mathbb{N}$. Let D be the system of domains defined as follows: for any $b \in 2^{<\omega}$, $D_b = S_{|b|}$. Note that $\mathbf{X} = \langle 2^{<\omega}, D\rangle$ is a predicate topological space with countable domains, as is $\mathbf{X}^\dagger = \langle 2^{<\omega}, D^\dagger\rangle$. We will show that QH is strongly complete for $\mathbf{X}^\dagger$.

Let $\langle \Gamma, \Delta\rangle$ be a consistent pair of sentences. By Lemma 4.1, there is a countable Alexandrov space X', a system D' of countable domains and a valuation V' on $\mathbf{X}' = \langle X', D'\rangle$, such that $\langle \Gamma, \Delta\rangle$ is satisfiable in $\mathbf{X}'$. As noted in Lemma 3.3 of [9], any countable rooted Alexandrov space is the image of $2^{<\omega}$ under some interior map.[18] So there's an interior map φ_0 from $2^{<\omega}$ onto X'. And, since φ_0 is surjective, $\langle \Gamma, \Delta\rangle$ is satisfiable in $\mathbf{X}'$ by way of $\varphi_0[2^{<\omega}]$.

The fact that φ_0 is continuous means that if $b \leq b'$, then $\varphi_0(b) R_{X'} \varphi_0(b')$, where $R_{X'}$ is the specialization preorder on X'. Thus, if $b \leq b'$, then $D'_{\varphi_0(b)} \subseteq D'_{\varphi_0(b')}$. So $D'_{\varphi_0(b)} \subseteq D'_{\varphi_0(b0)}$ and $D'_{\varphi_0(b)} \subseteq D'_{\varphi_0(b1)}$. Now we will define functions φ_{1b} for each $b \in 2^{<\omega}$. Let $\varphi_{1\wedge} : D_\wedge \to D'_{\varphi_0(\wedge)}$ be any surjective function. Suppose that $\varphi_{1b} : D_b \to D'_{\varphi_0(b)}$ is defined for some $b \in 2^{<\omega}$. Recall

[17] As noted in [9], the topology defined on $2^{\leq\omega}$ is the Scott topology as defined, for example, in [8], p. 104.

[18] In [9], this is stated for Kripke frames instead of Alexandrov spaces. The proof in [9] is incorrect, but a correct proof appears in [10]: see Claim (ii.b) in the proof there of Lemma 6.2.

that $D_b = S_{|b|}$, that $D_{b0} = D_{b1} = S_{|b|+1}$. Let $\varphi_{1(b0)}$ $[\varphi_{1(b1)}]$ be any surjective function from D_{b0} to $D'_{\varphi_0(b0)}$ $[D_{b1}$ to $D'_{\varphi_0(b1)}]$ that extends φ_{1b}.

It is routine to show that $\varphi = \langle\varphi_0, \varphi_1\rangle$ is a p-morphism from $\mathbf{X}$ to $\mathbf{X}'$. So by Corollary 5.5, $\langle\Gamma, \Delta\rangle$ is satisfiable in $\mathbf{X}^\dagger = \langle 2^{<\omega}, D^\dagger\rangle$. Also note that, since each D_x is countably infinite, so is each $D^\dagger{}_x$. □

Corollary 6.2. *There is a system $D^{\leq\omega}$ of countable domains such that* QH *is strongly complete for $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$ by way of $2^{<\omega}$.*

Proof. By Lemma 6.1, there is some system D of countable domains such that QH is strongly complete for $\mathbf{X} = \langle 2^{<\omega}, D\rangle$. First we define a new system $D^{\leq\omega}$ of countable domains for the topological space $2^{\leq\omega}$: If $b \in 2^{<\omega}$, then let $D^{\leq\omega}{}_b = D_b$; and if $\mathbf{b} \in 2^\omega$, then let $D^{\leq\omega}{}_{\mathbf{b}} = \bigcup_{n\in\mathbb{N}} D_{\mathbf{b}|_n}$. Note that $D^{\leq\omega}$ satisfies the expanding domain condition. Let $\mathbf{X}^{\leq\omega} = \langle 2^{\leq\omega}, D^{\leq\omega}\rangle$. Note that $D_{\mathbf{X}^{\leq\omega}} = D_{\mathbf{X}}$. We will show that QH is strongly complete for $\mathbf{X}^{\leq\omega}$ by way of $2^{<\omega}$.

Suppose that $\langle\Gamma, \Delta\rangle$ is a consistent pair of nonempty sets of sentences. Then there is a valuation V such that $Val_{\mathbf{M}}(\langle\Gamma, \Delta\rangle) \neq \emptyset$, where $\mathbf{M} = \langle 2^{<\omega}, D, V\rangle$. We define a new valuation $V^{\leq\omega}$ for $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$, as follows – here $d_1, \ldots, d_n \in D_{\mathbf{X}^{\leq\omega}} = D_{\mathbf{X}}$:

- $V^{\leq\omega}(\mathsf{P})(d_1 \ldots d_n)$
  $= V(\mathsf{P})(d_1 \ldots d_n) \cup \{\mathbf{b} \in 2^\omega : (\exists n \in \mathbb{N})(\mathbf{b}|_n \in V(\mathsf{P})(d_1 \ldots d_n))\}$;
- $V^{\leq\omega}(\mathsf{c}) = V(\mathsf{c})$; and
- $V^{\leq\omega}(\mathsf{f})(d_1, \ldots, d_n) = V(\mathsf{f})(d_1, \ldots, d_n)$.

It is easy to check that $V^{\leq\omega}(\mathsf{P})(d_1 \ldots d_n) = \bigcup_{b\in V(\mathsf{P})(d_1\ldots d_n)} \leq^+(b)$, so that $V^{\leq\omega}(\mathsf{P})(d_1 \ldots d_n)$ is, as desired, open in $2^{\leq\omega}$. It is also routine to prove that

- $Val_{\mathbf{M}^{\leq\omega}}(\mathsf{t}) = Val_{\mathbf{M}}(\mathsf{t})$ for every $D_{\mathbf{X}^{\leq\omega}}$-term t, and
- $Val_{\mathbf{M}^{\leq\omega}}(A) = Val_{\mathbf{M}}(A) \cup \{\mathbf{b} \in 2^\omega : (\exists n \in \mathbb{N})(\mathbf{b}|_n \in Val_{\mathbf{M}}(A))\}$ for every $D_{\mathbf{X}^{\leq\omega}}$-sentence A.

Thus $Val_{\mathbf{M}^{\leq\omega}}(\langle\Gamma, \Delta\rangle) \cap 2^{<\omega} = Val_{\mathbf{M}}(\langle\Gamma, \Delta\rangle) \neq \emptyset$. So QH is strongly complete for $\mathbf{X}^{\leq\omega}$ by way of $2^{<\omega}$. □

Remark 6.3. The analogous claim fails for QS4: QS4 is not complete for any Baire space with countable domains, even with varying countable domains ([12]), and $2^{\leq\omega}$ is a Baire space. It is an open question whether QS4 is complete for $2^{\leq\omega}$ with no constraint on the size of the domains.

7 Proving the main results

For each dense-in-itself metrizable space X, [9] constructs a continuous function $f_X : X \to 2^{\leq\omega}$. We will not review the construction in the current paper. Rather, in Section 8 we will cull enough information from [9] to prove that the function f_X is openish and therefore a quasi-interior map, and that $2^{<\omega} \subseteq f_X[X]$.[19] This gives us a fairly straightforward proof of Theorem 3.3.

Proof of Theorem 3.3. Suppose that X is a dense-in-itself metrizable space. By Corollary 6.2, there is a system $D^{\leq\omega}$ of countable domains for $2^{\leq\omega}$ such that QH is strongly complete for $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$ by way of $2^{<\omega}$. Thus, QH is also strongly complete for $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$ by way of $f_X[X]$, since $2^{<\omega} \subseteq f_X[X]$. We now define a system D of countable domains for X and a predicate quasi-p-morphism from $\langle X, D\rangle$ to $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$.

The system of domains. For each $x \in X$, let $D_x = {D^{\leq\omega}}_{f_X(x)}$. Clearly each D_x is countable. We have to check that, for each $d \in \bigcup_{x\in X} D_x$, the set $O_d = \{x \in X : d \in D_x\}$ is open in X. Given that f_X is continuous, it suffices to show that $O_d = {f_X}^{-1}[{O'}_d]$, where $O_d = \{\boldsymbol{b} \in 2^{\leq\omega} : d \in D_{\boldsymbol{b}}\}$. Note $x \in O_d$ iff $d \in D_x$ iff $d \in {D^{\leq\omega}}_{f_X(x)}$ iff $f_X(x) \in {O'}_d$ iff $x \in {f_X}^{-1}[{O'}_d]$.

The quasi-p-morphism. Let $\varphi = \langle f_X, \varphi_1\rangle$, where, for each $x \in X$, the function $\varphi_{1x} : {D^{\leq\omega}}_x \to D_{f_X(x)}$ is simply the identity map. It is easy to check that φ is a predicate quasi-p-morphism from $\langle X, D\rangle$ to $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$.

Given Corollary 5.5, we are done: any consistent pair $\langle\Gamma, \Delta\rangle$ is satisfiable in $\langle 2^{\leq\omega}, D^{\leq\omega}\rangle$ by way of $f_X[X]$ and therefore satisfiable in $\langle X, D^\dagger\rangle$ – recall that each ${D^\dagger}_x$ is countable since each D_x is. □

The proof of Theorem 3.1 is somewhat more involved.

Proof of Theorem 3.1. Suppose that X is a zero-dimensional dense-in-itself metrizable space of weight κ. Note that X has a basis of cardinality κ and also a basis of clopen sets: therefore, by Theorem 1.1.15 in [5], X has a basis of cardinality κ of clopen sets.[20] We will use this fact below.

[19] The construction of f_X depends on a number of decisions made along the way, including the choice of a metric d on X, so there are actually many such f_X.

[20] I owe this reference both to Henno Brandsma and an anonymous poster, NS, on the electronic bulletin board 'Ask a Topologist'. The theorem in [5] states that, if X is a topological space of weight κ, then for any basis B for X there is a basis $B' \subseteq B$ of cardinality κ.

By Lemma 6.2, there is a system D of countable domains for $2^{\leq\omega}$ such that QH is strongly complete for $\mathbf{X} = \langle 2^{\leq\omega}, D\rangle$ by way of $2^{<\omega}$. So, by Corollary 5.5, it suffices to specify a constant domain D' of cardinality κ; and a quasi-p-morphism $\varphi = \langle\varphi_0, \varphi_1\rangle$ from $\mathbf{X}' = \langle X, D'\rangle$ to $\mathbf{X}$ such that $2^{<\omega} \subseteq \varphi_0[X]$. Before we begin, we note the following: if $b \in 2^{<\omega}$ and $\boldsymbol{b}' \in 2^{\leq\omega}$ and $b \leq \boldsymbol{b}'$, then $D_b \subseteq D_{\boldsymbol{b}'}$, since $\leq^+(b)$ is the smallest subset of $2^{\leq\omega}$ with b as a member. In particular, $D_\wedge \subseteq D_{\boldsymbol{b}}$ for every $\boldsymbol{b} \in 2^{\leq\omega}$. It will be useful to choose a $d_0 \in D_\wedge$.

Let $D' = \{\langle b, \alpha, d\rangle : b \in 2^{<\omega}, \alpha \in \kappa \text{ and } d \in D_b\}$. Note that $card(D') = \kappa$. Our predicate quasi-p-morphism is $\varphi = \langle\varphi_0, \varphi_1\rangle$, where $\varphi_0 = f_X$ and where the family $\varphi_1 = \{\varphi_{1x}\}_{x\in X}$ of functions is defined shortly, after some stage-setting.[21]

Note that, for any $b \in 2^{<\omega}$, the set $\leq^+(b)$ is open in $2^{\leq\omega}$. So, since φ_0 is continuous, the set ${\varphi_0}^{-1}[\leq^+(b)]$ is open in X. Since X has a basis of cardinality κ of clopen sets, we can express ${\varphi_0}^{-1}[\leq^+(b)]$ as a union of κ many clopen sets O^b_α as follows:

$$
{\varphi_0}^{-1}[\leq^+(b)] = \bigcup_{\alpha\in\kappa} O^b_\alpha
$$

For each $x \in X$, we define $\varphi_{1x} : D' \to D$ as follows:

$$
\varphi_{1x}(\langle b, \alpha, d\rangle) \quad = \quad \begin{cases} d \text{ if } x \in O^b_\alpha \\ d_0 \text{ if } x \notin O^b_\alpha \end{cases}
$$

We have to check that $\varphi = \langle\varphi_0, \varphi_1\rangle$ is indeed a quasi-p-morphism from $\mathbf{X}' = \langle X, D'\rangle$ to $\mathbf{X}$ such that $2^{<\omega} \subseteq \varphi_0[X]$. We already know that φ_0 is a quasi-interior map with $2^{<\omega} \subseteq \varphi_0[X]$. So we have to check that the φ_{1x} satisfy Items (3) and (4) in Definition 5.1.

Item (3). First we check that $\varphi_{1x} : D' \to D_{\varphi_0(x)}$. So suppose $\langle b, \alpha, d\rangle \in D'$. If $x \notin O^b_\alpha$, then $\varphi_{1x}(\langle b, \alpha, d\rangle) = d_0 \in D_\wedge \subseteq D_{\varphi_0(x)}$. On the other hand, suppose that $x \in O^b_\alpha$. Then $\varphi_0(x) \in \varphi_0[O^b_\alpha] \subseteq \varphi_0[\leq^+(b)]$. So $\varphi_0(x) \in \leq^+(b)$. So $b \leq \varphi_0(x)$. Also, $d \in D_b$ since $\langle b, \alpha, d\rangle \in D'$. So $\varphi_{1x}(\langle b, \alpha, d\rangle) = d \in D_b \subseteq D_{\varphi_0(x)}$, as desired. Next we check that $\varphi_{1x} : D' \to D_{\varphi_0(x)}$ is surjective. Suppose that $d \in D_{\varphi_0(x)}$. Clearly $\varphi_0(x) \in \leq^+(\varphi_0(x))$. So $x \in {\varphi_0}^{-1}[\leq^+(\varphi_0(x))]$. So, $x \in O^{\varphi_0(x)}_\alpha$ for some $\alpha \in \kappa$. So $\varphi_{1x}(\langle\varphi_0(x), \alpha, d\rangle) = d$, which suffices.

[21] From this point on, we generalize but otherwise closely follow part of the proof of Lemma 5.1 in [11].

Item (4). Choose $\langle b, \alpha, d\rangle \in D'$ and $x \in X$. We want to show that there is an open set $O^x_{\langle b,\alpha,d\rangle} \subseteq O_{\langle b,\alpha,d\rangle} = X$, such that both $x \in O^x_{\langle b,\alpha,d\rangle}$ and for every $y \in O^x_{\langle b,\alpha,d\rangle}$, $\varphi_{1y}(\langle b, \alpha, d\rangle) = \varphi_{1x}(\langle b, \alpha, d\rangle)$. If $x \in O^b_\alpha$, let $O^x_{\langle b,\alpha,d\rangle} = O^b_\alpha$. Then note that $x \in O^x_{\langle b,\alpha,d\rangle}$, and for every $y \in O^x_{\langle b,\alpha,d\rangle}$, we have $\varphi_{1y}(\langle b, \alpha, d\rangle) = d = \varphi_{1x}(\langle b, \alpha, d\rangle)$. On the other hand, if $x \notin O^b_\alpha$, then let $O^x_{\langle b,\alpha,d\rangle} = X - O^b_\alpha$. Note that $O^x_{\langle b,\alpha,d\rangle}$ is open, since O^b_α is clopen. Also note that $x \in O^x_{\langle b,\alpha,d\rangle}$, and for every $y \in O^x_{\langle b,\alpha,d\rangle}$, we have $\varphi_{1y}(\langle b, \alpha, d\rangle) = d_0 = \varphi_{1x}(\langle b, \alpha, d\rangle)$, as desired. □

8 Properties of f_X

Fix a dense-in-itself metrizable space X, and a metric d on X so that the open balls are a basis for the topology on X. For any point $x \in X$ and any set $S \subseteq X$, define $d(x, S) = inf\{d(x, y) : y \in S\}$.

At the beginning of Section 7, we promised to cull enough information from [9] to prove that f_X is openish and therefore a quasi-interior map, and that $2^{<\omega} \subseteq f_X[X]$. Section 7 of [9] shows that there are

- nonempty open sets $O_b \subseteq X$, for each $b \in 2^{<\omega}$, with $b < b' \Rightarrow O_{b'} \subsetneq O_b$,
- other nonempty sets $X_b \subsetneq O_b$, for each $b \in 2^{<\omega}$, and
- possibly empty sets $X_{\mathbf{b}} = \bigcap_n O_{\mathbf{b}|_n}$, for each $\mathbf{b} \in 2^\omega$.

such that

Lemma 8.1. *1. $d(x, X_b) \leq 1/(|b| + 1)$, for every $b \in 2^{<\omega}$ and $x \in O_b$;*

2. $(\forall b, b' \in 2^{<\omega})$(if $b \leq b'$ then $Cl(X_b) \subseteq Cl(X_{b'})$); and

3. the family $\{X_{\boldsymbol{b}} : \boldsymbol{b} \in 2^{\leq\omega}\}$ is pairwise disjoint and $X = \bigcup_{\boldsymbol{b}\in 2^{\leq\omega}} X_{\boldsymbol{b}}$.

Proof. Item (1) follows from the definition in [9] of the O_b and X_b, when $b \in 2^{<\omega}$, as well as the ε-clause in Lemma 7.1 of [9]. Item (2) is Lemma 7.7 of [9]. Item (3) is noted immediately before the statement in [9] of Lemma 7.8. □

[9] defines the function $f_X : X \to 2^{\leq\omega}$ as follows: $f_X(x)$ is the unique $\boldsymbol{b} \in 2^{\leq\omega}$ with $x \in X_{\boldsymbol{b}}$. Lemma 7.9 of [9] states that f_X is continuous. To see that $2^{<\omega} \subseteq f_X[X]$, it suffices to note that X_b is nonempty, for each $b \in 2^{<\omega}$. Our remaining task is to show that f_X is openish, i.e., that the set $f_X[O]$ is openish in $2^{\leq\omega}$ for every open $O \subseteq X$. In aid of this, we note:

Lemma 8.2. *Suppose $S \subseteq 2^{\leq\omega}$ is such that*

1. *$S \cap 2^{<\omega}$ is open in $2^{<\omega}$; and*

2. *$\forall \mathbf{b} \in S \cap 2^{\omega}$, $\exists n \in \mathbb{N}$, $\mathbf{b}|_n \in S$.*

Then S is openish in $2^{\leq\omega}$.

Proof. First, we show that, for any set O open in $2^{\leq\omega}$,

$$S \cap O \subseteq \bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b). \tag{1}$$

Proof of (1). Suppose that $\boldsymbol{b} \in S \cap O$. If $\boldsymbol{b} \in 2^{<\omega}$, then note that $\boldsymbol{b} \in \leq^+(\boldsymbol{b}) \subseteq \bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b)$. If $\boldsymbol{b} \in 2^{\omega}$, then, since S satisfies Condition (2), $\exists n \in \mathbb{N}$, $\boldsymbol{b}|_n \in S$. Also $\exists m \in \mathbb{N}$, $\boldsymbol{b}|_m \in O$, since $\boldsymbol{b} \in O$ and O is open. Let $k = max(m, n)$. Since S satisfies Condition (1) and since O is open in $2^{\leq\omega}$, $\boldsymbol{b}|_k \in S \cap O \cap 2^{<\omega}$. So $\boldsymbol{b} \in \leq^+(\boldsymbol{b}|_k) \subseteq \bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b)$.

Given (1) and the fact that $\bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b)$ is open,

$$Op(S \cap O) \subseteq \bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b). \tag{2}$$

Now we show that

$$\bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b) \subseteq Op(S \cap O) \tag{3}$$

Note that, for any $b \in S \cap O \cap 2^{<\omega}$, the smallest open set with b as a member is $\leq^+(b)$. So, for any $b \in S \cap O \cap 2^{<\omega}$, $\leq^+(b) \subseteq Op(S \cap O)$. So $\bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b) \subseteq Op(S \cap O)$, as desired. Given (2) and (3), we have

$$Op(S \cap O) = \bigcup_{b \in S \cap O \cap 2^{<\omega}} \leq^+(b). \tag{4}$$

And given (4), $Op(S)$ is open, since $Op(S \cap O)$ is open for any open set O.

It remains to show that $Op(S) \cap O \subseteq Op(S \cap O)$, for any set O open in $2^{\leq\omega}$. So choose any $\boldsymbol{b} \in Op(S) \cap O$. By (4) in the special case where $O = 2^{\leq\omega}$, we have $\boldsymbol{b} \in \leq^+(b')$ for some $b' \in S \cap 2^{<\omega}$. Also, since $\boldsymbol{b} \in O$ and O is open, we have $\boldsymbol{b} \in \leq^+(b'') \subseteq O$ for some $b'' \in O \cap 2^{<\omega}$. Since $b' \leq \boldsymbol{b}$ and $b'' \leq \boldsymbol{b}$, we have either $b'' \leq b'$ or $b' \leq b''$. Let b^* be the greater of b' and b'': so $b', b'' \leq b^* \leq \boldsymbol{b}$. Since $b'' \leq b^*$, we have $b^* \in \leq^+(b'') \subseteq O$. And since both $b' \leq b^*$ and S satisfies Condition (1) in the statement of the Lemma, we have $b^* \in S$. So $b^* \in S \cap O \cap 2^{<\omega}$ and $\boldsymbol{b} \in \leq^+(b^*)$. So $\boldsymbol{b} \in Op(S \cap O)$, by (4), as desired. $\square$

Given Lemma 8.2, it suffices to show

Lemma 8.3. *For every open $O \subseteq X$,*

1. $f_X[O] \cap 2^{<\omega}$ *is open in* $2^{<\omega}$*; and*

2. $\forall \mathbf{b} \in f_X[O] \cap 2^{\omega}$, $\exists n \in \mathbb{N}$, $\mathbf{b}|_n \in f_X[O]$.

Proof. Item (1). Choose $b \in f_X[O] \cap 2^{<\omega}$ and $b' \in 2^{<\omega}$ with $b \leq b'$. We want to show that $b' \in f_X[O]$. Since $b \in f_X[O]$, there is some $x \in O$ such that $f_X(x) = b$. Note that $x \in X_b$, since $f_X(x) = b$. By Lemma 8.1 (2), $x \in O \cap X_b \subseteq O \cap Cl(X_b) \subseteq O \cap Cl(X_{b'})$. So there is some $y \in O \cap X'_b$. So there is some $y \in O$ with $f_X(y) = b'$. So $b' \in f_X[O]$, as desired.

Item (2). Choose $\mathbf{b} \in f_X[O] \cap 2^{\omega}$. Since $\mathbf{b} \in f_X[O]$, there is some $x \in O$ such that $f_X(x) = \mathbf{b}$. Note that $x \in X_{\mathbf{b}} = \bigcap_n O_{\mathbf{b}|_n}$, since $f_X(x) = \mathbf{b}$. Choose any open ball $B = \{y : d(x, y) < r\}$ with centre x such that $B \subseteq O$. And choose any $n \in \mathbb{N}$ such that $n > 1/r$. Since $x \in \bigcap_n O_{\mathbf{b}|_n}$, we have $x \in O_{\mathbf{b}|_n}$. By Lemma 8.1 (1), $d(x, X_{\mathbf{b}|_n}) \leq 1/(n+1)$. So $d(x, X_{\mathbf{b}|_n}) < 1/n \leq r$. So there is some $y \in X_{\mathbf{b}|_n} \cap B \subseteq X_{\mathbf{b}|_n} \cap O$. So $y \in O$ and $f(y) = \mathbf{b}|_n$. So $\mathbf{b}|_n \in f_X[O]$, as desired. □

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References

[1] S. Awodey and K. Kishida, 'Topology and modality: the topological intepretation of first-order modal logic', *Review of Symbolic Logic*, 1:146–166, 2008.

[2] A. G. Dragalin, *Mathematical Intuitionism: Introduction to Proof Theory*, Translations of Mathematical Monographs, Vol. 67, translated by E. Mendelson. American Mathematical Society, Providence RI, 1988. (Russian original, 1979)

[3] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[4] R. Engelking, *Dimension Theory*, North-Holland, 1978.

[5] R. Engelking, *General Topology*, Heldermann Verlag, 1989.

[6] M. P. Fourman and D. S. Scott, 'Sheaves and logic', in *Applications of Sheaves*, Lecture Notes in Mathematics vol. 753, (M. P. Fourman, C. J. Mulvey, and D. S. Scott, eds.), Springer-Verlag, 302-401, 1979.

[7] D. M. Gabbay, V. B. Shehtman, and D. Skvortsov, *Quantification in non-classical logic, Volume 1*, Elsevier, 2009.

[8] G. Gierz, K.H. Hoffmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003.

[9] P. Kremer, 'Strong completeness of S4 for any dense-in-itself metric space', *Review of Symbolic Logic* 6:545–570, 2013.

[10] P. Kremer, 'Quantified modal logic on the rational line', *Review of Symbolic Logic* 7:439–454, 2014.

[11] P. Kremer, 'Quantified S4 in the Lebesgue measure algebra with a constant countable domain', ms 2014.

[12] T. Lando, 'First order S4 and its measure-theoretic semantics', forthcoming in the *Annals of Pure and Applied Logic*.

[13] J. C. C. McKinsey, 'A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology', *The Journal of Symbolic Logic* 6:117–134, 1941.

[14] J. C. C. McKinsey and A. Tarski, 'The algebra of topology', *Annals of Mathematics* 45:141–191, 1944.

[15] A. Medini and D. Milovich, 'The topology of ultrafilters as subspaces of 2^ω', *Topology and its Applications* 159:1318–1333, 2012.

[16] I. Moerdijk, 'Some topological spaces which are universal for intuitionistic predicate logic', *Indagationes Mathematicae (Proceedings)* 85, 227–235, 1982.

[17] J. Moschovakis, 'Intuitionistic Logic', *The Stanford Encyclopedia of Philosophy*, URL = https://plato.stanford.edu/entries/logic-intuitionistic/, accessed March 17, 2017.

[18] A. Mostowski, 'Proofs of non-deducibility in intuitionistic functional calculus', *Journal of Symbolic Logic* 13: 204–207, 1948.

[19] H. Rasiowa, 'Algebraic treatment of the functional calculi of Heyting and Lewis', *Fundamenta Mathematicae* 38:99–126, 1951.

[20] H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, Państowowe Wydawnictwo Naukowe, Warsaw, 1963.

[21] M.H. Sørensen and P. Urzyczyn, *Lectures on the Curry-Howard Isomorphism*, Volume 149 of Studies in Logic and the Foundations of Mathematics, Elsevier, Amsterdam, 2006.