The Topological Product of S4 and S5

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Abstract

The most obvious bimodal logic generated from unimodal logics L_1 and L_2 is their *fusion*, $L_1 \otimes L_2$, axiomatized by the theorems of L_1 for $\Box_1$ and of L_2 for $\Box_2$, and by the rules of modus ponens, substitution and necessitation for $\Box_1$ and for $\Box_2$. Shehtman introduced the *frame product* $L_1 \times L_2$, as the logic of the products of certain Kripke frames. Typically, $L_1 \otimes L_2 \subsetneq L_1 \times L_2$, e.g. $S4 \otimes S4 \subsetneq S4 \times S4$. Van Benthem, Bezhanishvili, ten Cate and Sarenac generalized Shehtman's idea and introduced the *topological product* $L_1 \times_t L_2$, as the logic of the products of certain topological spaces: they showed, in particular, that $S4 \times_t S4 = S4 \otimes S4$. In this paper, we axiomatize $S4 \times_t S5$, which is strictly between $S4 \otimes S5$ and $S4 \times S5$. We also apply our techniques to proving a conjecture of van Benthem *et al* concerning the logic of products of Alexandrov spaces with arbitrary topological spaces.

Keywords: Bimodal logic, multimodal logic, topological semantics, topological product, product space.

Let $\mathcal{L}$ be a propositional language with a set PV of propositional variables; standard Boolean connectives &, $\vee$ and $\neg$; and one modal operator, $\Box$. And let $\mathcal{L}_{12}$ be like $\mathcal{L}$, except with *two* modal operators, $\Box_1$ and $\Box_2$. We use standard definitions of the Boolean connectives $\supset$ and $\equiv$ and the modal operators $\Diamond$ in $\mathcal{L}$ and $\Diamond_1$ and $\Diamond_2$ in $\mathcal{L}_{12}$.

There are several ways to combine two normal modal logics[1] L_1 and L_2 formulated in the language $\mathcal{L}$ to get a bimodal logic formulated in the language $\mathcal{L}_{12}$. The simplest is to define $L_1 \otimes L_2$, the *fusion* of L_1 and L_2, as

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[1]A set L of formulas of $\mathcal{L}$ is a *normal modal logic* iff every propositional tautology is in L, $(\Box(p \supset q) \supset (\Box p \supset q)) \in$ L, and L is closed under modus ponens, necessitation for $\Box$, and substitution.

follows: let L_1' $[L_2']$ be the set of formulas of $\mathcal{L}_{12}$ got by replacing each occurrence of $\Box$ in each formula in L_1 $[L_2]$ by $\Box_1$ $[\Box_2]$; and let $L_1 \otimes L_2$ be the smallest set of formulas of $\mathcal{L}_{12}$ that contains $L_1' \cup L_2'$ and that is closed under modus ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution.[2] Thus, S4 $\otimes$ S4 [S4 $\otimes$ S5, S5 $\otimes$ S5] is the bimodal logic axiomatized by S4-axioms [S4-axioms, S5-axioms] for $\Box_1$ and S4-axioms [S5-axioms, S5-axioms] for $\Box_2$, as well as the rules of modus ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution.

Shehtman [19] uses Kripke semantics to produce combinations, stronger than fusions, of modal logics. He defines the *product* of two Kripke frames, as a particular *bi*relational Kripke frame. The *frame product* of logics L_1 and L_2, denoted $L_1 \times L_2$, is then the set of formulas in the language $\mathcal{L}_{12}$ validated by every product of a Kripke frame validating L_1 with a Kripke frame validating L_2.[3] We always have $L_1 \otimes L_2 \subseteq L_1 \times L_2$ and almost always $L_1 \otimes L_2 \subsetneq L_1 \times L_2$;[4] and for many popular modal logics, $L_1 \times L_2$ is the result of adding the following three axioms to $L_1 \otimes L_2$ (see Theorem 7.12 of [5] and Theorem 1.2, below):

$com_{\supset}$	(left commutativity)	$\Box_1\Box_2 p \supset \Box_2\Box_1 p$
$com_{\subset}$	(right commutativity)	$\Box_2\Box_1 p \supset \Box_1\Box_2 p$
chr	(Church-Rosser)	$\Diamond_1\Box_2 p \supset \Box_2\Diamond_1 p$.

For modal logics stronger than S4, the McKinsey-Tarski topological semantics ([15, 16, 18]) for the unimodal language $\mathcal{L}$ generalizes the Kripke semantics. In the topological semantics, interpretations of $\mathcal{L}$ are based on topological spaces rather than Kripke frames. Van Benthem, Bezhanishvili, ten Cate and Sarenac [21] generalize Shehtman's products of frames to products of topological spaces: given topological spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, they define a *bi*topological space, $\mathcal{X}_1 \times \mathcal{X}_2$, equipped with two topologies. They define the *topological product* of logics L_1 and L_2, denoted $L_1 \times_t L_2$, as the set of formulas in the language $\mathcal{L}_{12}$ validated by all bitopological spaces of the form

[2] We are following the notation in [4] and [13] for $\otimes$, though other notation is used: [21] uses '$\oplus$' and [5] uses '$*$'.

[3] As noted in [21], a systematic study of multi-dimensional modal logics of products of Kripke frames can be found in [5], and an up-to-date account of the most important results in the field can be found in [4].

[4] But not always. Suppose that one of L_1 or L_2 is either the modal logic Triv $=$ S4$+(p \supset \Box p)$ or the modal logic Verum $=$ K $+ \Box p$, and that the other is Kripke complete, i.e., is determined by some class of Kripke frames. Then $L_1 \otimes L_2 = L_1 \times L_2$.

$\mathcal{X}_1 \times \mathcal{X}_2$, where $\mathcal{X}_1$ is a space validating L_1 and $\mathcal{X}_2$ is a space validating L_2. In general,

$$L_1 \otimes L_2 \quad \subseteq \quad L_1 \times_t L_2 \quad \subseteq \quad L_1 \times L_2.$$

It is a nontrivial theorem of [21] that

$$S4 \otimes S4 \quad = \quad S4 \times_t S4 \quad \subsetneq \quad S4 \times S4.$$

Once the definitions are on the table, we will see (Theorem 2.1) that,

$$S5 \otimes S5 \quad \subsetneq \quad S5 \times_t S5 \quad = \quad S5 \times S5,$$

and that

$$S4 \otimes S5 \quad \subsetneq \quad S4 \times_t S5 \quad \subsetneq \quad S4 \times S5.$$

Our main result is an axiomatization of $S4 \times_t S5$. In [14], Kurucz and Zakharyaschev define the logic $[L_1, L_2]^{\mathsf{EX}}$ for any normal modal logics L_1 and L_2, by adding two axioms to $L_1 \otimes L_2$, namely $com_\supset$ and *chr*. Shehtman [20] suggests the term *semiproducts* for such logics. Note the absence of $com_\subset$, the converse of $com_\supset$. We show that $S4 \times_t S5 = [S4, S5]^{\mathsf{EX}}$, i.e., that $S4 \times_t S5$ can be axiomatized by adding $com_\supset$ and *chr* to $S4 \otimes S5$. We appeal to a special case of Theorem 6 in [14] (Theorem 49 in [13]): $[S4, S5]^{\mathsf{EX}}$ is identical to the *expanding relativized product* of S4 and S5, denoted $(S4 \times S5)^{\mathsf{EX}}$: this is defined as the logic of all *expanding relativized product frames* (see subsection 3.3, below), i.e., special subframes of product frames. We note here that Kurucz and Zakharyaschev reduce the decidability of a large class of expanding relativized product logics to the the decidability of the corresponding product logics ([14], Theorem 7). In particular since $S4 \times S5$ is decidable ([5], Theorem 12.12), so is $S4 \times_t S5 = (S4 \times S5)^{\mathsf{EX}}$.

Our proof that $[S4, S5]^{\mathsf{EX}}$ axiomatizes $S4 \times_t S5$ proceeds by showing that $[S4, S5]^{\mathsf{EX}}$, and hence $S4 \times_t S5$, is sound and complete wrt a particular bitopological space: $\mathbb{Q} \times \mathbb{N}$, where $\mathbb{Q}$ is the rational line equipped with the standard topology and $\mathbb{N}$ is the set of natural numbers equipped with the trivial topology.[5] Thus we get a kind of bitopological analogue to the classic result of [16] that unimodal S4 is sound and complete wrt the rational line, $\mathbb{Q}$.

Our proof technique can be generalized. Van Benthem *et al* ask what the bimodal logic is of the products of Alexandrov spaces[6] with arbitrary

[5] A *trivial* topology allows exactly two open sets: see subsection 1.3, below.

[6] A topological space is *Alexandrov* iff arbitrary intersections of open sets are open. See subsection 1.3.

topological spaces, conjecturing that it is $(\mathrm{S4} \otimes \mathrm{S4}) + com_{\subset} + chr$. This is equivalent to the mirror image claim that the bimodal logic of the products of arbitrary topological spaces with Alexandrov spaces is $[\mathrm{S4}, \mathrm{S4}]^{\mathsf{EX}} = (\mathrm{S4} \otimes \mathrm{S4}) + com_{\supset} + chr$. We prove not only this claim, but also that $[\mathrm{S4}, \mathrm{S4}]^{\mathsf{EX}}$ is sound and complete for a particular product of a topological space with an Alexandrov space: $\mathbb{Q} \times \mathbb{N}^*$, where $\mathbb{Q}$ is the rational line equipped with the standard topology and where $\mathbb{N}^*$ is the set of finite sequences of natural numbers, equipped with the following Alexandrov topology: a set $O \subseteq \mathbb{N}^*$ is open iff, for each $\overline{a}, \overline{b} \in \mathbb{N}^*$, if $\overline{a} \in O$ and $\overline{a}$ is an initial segment of $\overline{b}$, then $\overline{b} \in O$.

We take the current paper to be part of a larger project, initiated by van Benthem, Bezhanishvili, ten Cate and Sarenac in [21], of getting clear on topological products of modal logics in general. It often helps to start with relatively basic cases. [21] investigates the most basic topological product, $\mathrm{S4} \times_t \mathrm{S4}$, showing it to be identical to $\mathrm{S4} \otimes \mathrm{S4}$. Arguably the second most basic topological product is $\mathrm{S5} \times_t \mathrm{S5}$: it is easy to show this is identical to the frame product $\mathrm{S5} \times \mathrm{S5}$ (see Theorem 2.1, below). The topic of the current paper, $\mathrm{S4} \times_t \mathrm{S5}$, can be seen as the third most basic topological product.

1 Background definitions

1.1 Unirelational and birelational semantics

A *unirelational* (*Kripke*) *frame* (uniframe) is a pair $\mathcal{U} = \langle W, R \rangle$, where W is a nonempty set and R is a relation on W. We say that $\mathcal{U}$ is *reflexive* [*transitive*, *symmetric*] iff R is, and that $r \in W$ is a *root* of $\mathcal{U}$ iff $\forall w \in W, rRw$. We say that $\mathcal{U}$ is *rooted* iff $\mathcal{U}$ has at least one root. A *unirelational model* is a triple $\mathcal{M} = \langle W, R, V \rangle$, where $\langle W, R \rangle$ is a uniframe and $V : PV \to \mathcal{P}(W)$. We extend V to all formulas of $\mathcal{L}$ as follows:

$$\begin{array}{rcl} V(\neg A) & = & W - V(A) \\ V(A \,\&\, B) & = & V(A) \cap V(B) \\ V(A \vee B) & = & V(A) \cup V(B) \\ V(\Box A) & = & \{w \in W : \forall v \in W (wRv \Rightarrow v \in V(A))\}. \end{array}$$

(We will call the first three clauses *standard Boolean clauses*.) We say that $\mathcal{M} \vDash A$ iff $V(A) = W$. Given a uniframe $\mathcal{U} = \langle W, R \rangle$, we say that $\mathcal{U} \vDash A$ iff $\mathcal{M} \vDash A$ for every unirelational model $\mathcal{M} = \langle W, R, V \rangle$. And if Γ is a set of formulas of $\mathcal{L}$, we say that $\mathcal{U} \vDash \Gamma$ iff $\mathcal{U} \vDash A$ for each $A \in \Gamma$. We read $\vDash$ as *validates.*

A *birelational frame* (biframe) is a triple $\mathcal{B} = \langle W, R_1, R_2 \rangle$, where W is a nonempty set and R_1 and R_2 are relations on W. $\mathcal{B}$ is 1-*reflexive* iff R_1 is reflexive. Similarly for 1-transitive, 2-symmetric, etc. $\mathcal{B}$ is *bireflexive* iff $\mathcal{B}$ is both 1-reflexive and 2-reflexive, and similarly for bitransitive and bisymmetric. A *birelational model* is a quartuple $\mathcal{M} = \langle W, R_1, R_2, V \rangle$, where $\langle W, R_1, R_2 \rangle$ is a biframe and $V : PV \to \mathcal{P}(W)$. We extend V to all formulas of $\mathcal{L}_{12}$ with standard Boolean clauses for $\neg$, &, and $\vee$; and with the following clauses for $\Box_1$ and $\Box_2$:

$$\begin{aligned} V(\Box_1 A) &= \{w \in W : \forall v \in W(wR_1 v \Rightarrow v \in V(A))\} \\ V(\Box_2 A) &= \{w \in W : \forall v \in W(wR_2 v \Rightarrow v \in V(A))\}. \end{aligned}$$

The definitions of $\mathcal{M} \vDash A$, of $\mathcal{B} \vDash A$, and of $\mathcal{B} \vDash \Gamma$ are analogous to the unirelational case. The proof of the following theorem is a straightforward generalization of the unimodal case for S4 and for S5.

Theorem 1.1. $A \in \mathrm{S4} \otimes \mathrm{S4}$ $[\mathrm{S4} \otimes \mathrm{S5}, \mathrm{S5} \otimes \mathrm{S5}]$ *iff* $\mathcal{B} \vDash A$ *for every bireflexive bitransitive* [*bireflexive bitransitive 2-symmetric, bireflexive bitransitive bisymmetric*] *biframe* $\mathcal{B}$.

1.2 Products of uniframes, and frame products of unimodal logics

Shehthman [19] initiates the study of a particular class of biframes, those that are the *products* of uniframes: he uses these to introduce products of modal logics. Given two uniframes $\mathcal{U}_1 = \langle W_1, R_1 \rangle$ and $\mathcal{U}_2 = \langle W_2, R_2 \rangle$, define the biframe $\mathcal{U}_1 \times \mathcal{U}_2 =_{\mathrm{df}} \langle W_1 \times W_2, R_1', R_2' \rangle$ where

$$\begin{aligned} \langle w, v \rangle R_1' \langle x, y \rangle \quad &\text{iff} \quad wR_1 x \text{ and } v = y\text{; and} \\ \langle w, v \rangle R_2' \langle x, y \rangle \quad &\text{iff} \quad w = x \text{ and } vR_2 y. \end{aligned}$$

A biframe of the form $\mathcal{U}_1 \times \mathcal{U}_2$ is a *product frame.*

Given two normal modal logics L_1 and L_2, define the *frame product* of L_1 and L_2 as follows:

$$\begin{aligned} \mathrm{L}_1 \times \mathrm{L}_2 \quad =_{\mathrm{df}} \quad &\{A : \mathcal{U}_1 \times \mathcal{U}_2 \vDash A \text{ for any two uniframes} \\ &\mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ such that } \mathcal{U}_1 \vDash \mathrm{L}_1 \text{ and } \mathcal{U}_2 \vDash \mathrm{L}_2\}. \end{aligned}$$

Thus,

$$\begin{array}{lcl} \mathrm{S4}\times\mathrm{S4} & = & \{A : \mathcal{U}_1\times\mathcal{U}_2 \models A,\text{where } \mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ are reflexive and transitive}\}, \\ \mathrm{S4}\times\mathrm{S5} & = & \{A : \mathcal{U}_1\times\mathcal{U}_2 \models A,\text{where } \mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ are reflexive and transitive} \\ & & \text{and } \mathcal{U}_2 \text{ is symmetric}\}, \text{ and} \\ \mathrm{S5}\times\mathrm{S5} & = & \{A : \mathcal{U}_1\times\mathcal{U}_2 \models A,\text{where } \mathcal{U}_1 \text{ and } \mathcal{U}_2 \text{ are reflexive,} \\ & & \text{transitive and symmetric}\}. \end{array}$$

For any bimodal logic L and any formulas $A_1, \ldots, A_n$, define the new bimodal logic $\mathrm{L} + A_1 + \ldots + A_n$ as the smallest bimodal logic L' such that $\mathrm{L} \cup \{A_1, \ldots, A_n\} \subseteq \mathrm{L}'$ and such that L' is closed under modus ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution. For any normal unimodal logics L_1 and L_2, define the the *commutator* of L_1 and L_2 as follows:

$$[\mathrm{L}_1, \mathrm{L}_2] \quad = \quad \mathrm{L}_1 \otimes \mathrm{L}_2 + com_{\supset} + com_{\subset} + chr.$$

The following is an immediate corollary to Theorem 7.12 in [5]:

Theorem 1.2. $\mathrm{S4} \times \mathrm{S4} = [\mathrm{S4}, \mathrm{S4}]$. $\mathrm{S4} \times \mathrm{S5} = [\mathrm{S4}, \mathrm{S5}]$. $\mathrm{S5} \times \mathrm{S5} = [\mathrm{S5}, \mathrm{S5}]$.

1.3 Topological and bitopological semantics

For modal logics stronger than S4, the McKinsey-Tarski topological semantics ([15, 16, 18]) for the unimodal language $\mathcal{L}$ generalizes the unirelational Kripke semantics. A *topological space* (or simply *space*) is an ordered pair $\mathcal{X} = \langle X, \tau \rangle$, where X is a nonempty set and τ is a *topology* on X: i.e., $\tau \subseteq \mathcal{P}(X)$; $\emptyset, X \in \tau$; and τ is closed under arbitrary unions and finite intersections. The sets in τ are the *open* sets, and their complements are the *closed* sets. Given any $S \subseteq X$, $Int(S)$ is the *interior* of S, i.e., the largest open subset of S. We say that a space $\mathcal{X} = \langle X, \tau \rangle$ is *Alexandrov* iff τ is closed under arbitrary intersections, is *almost discrete* (AD) iff X is the union of disjoint open sets, and is *trivial* iff $\tau = \{\emptyset, X\}$. Note that every trivial space is AD, and that every AD space is Alexandrov.

A *topological model* is a triple $\mathcal{M} = \langle X, \tau, V \rangle$, where $\langle X, \tau \rangle$ is a space and $V : PV \to \mathcal{P}(X)$. We extend V to all formulas of $\mathcal{L}$ with standard Boolean clauses for $\neg$, &, and $\vee$; and with the following clause for $\Box$:

$$V(\Box A) \quad = \quad Int(V(A))$$

We sometimes write $x \Vdash A$ instead of $x \in V(A)$. We say that $\mathcal{M} \models A$ iff $V(A) = X$. Given a space $\mathcal{X} = \langle X, \tau \rangle$, we say that $\mathcal{X} \models A$ iff $\mathcal{M} \models A$ for every topological model $\mathcal{M} = \langle X, \tau, V \rangle$. And if Γ is a set of formulas of $\mathcal{L}$, we say that $\mathcal{X} \models \Gamma$ iff $\mathcal{X} \models A$ for each $A \in \Gamma$.

Here's why, for logics stronger than S4, the topological semantics generalizes the unirelational Kripke semantics. Given any reflexive transitive uniframe $\mathcal{U} = \langle W, R\rangle$, let $\mathcal{X}_{\mathcal{U}}$ be the following space: $\mathcal{X}_{\mathcal{U}} =_{\mathrm{df}} \langle W, \tau_{\mathcal{U}}\rangle$ where $O \in \tau_{\mathcal{U}}$ iff $\forall w \in O \forall w' \in W(wRw' \Rightarrow w' \in O)$. Note that a space is Alexandrov iff it is of the form $\mathcal{X}_{\mathcal{U}}$ for some uniframe $\mathcal{U}$. Note also that the clauses for $V(\Box A)$ for the uniframe $\mathcal{U}$ and for the space $\mathcal{X}_{\mathcal{U}}$ coincide, since $Int(S)$ (in $\mathcal{X}_{\mathcal{U}}) = \{w \in W : \forall v \in W(wRv \Rightarrow v \in S)\}$. So we can identify any reflexive transitive frame $\mathcal{U}$ with the Alexandrov space $\mathcal{X}_{\mathcal{U}}$. So, if L is a logic stronger than S4, every uniframe $\mathcal{U}$ that validates L (i.e., $\mathcal{U} \models$ L) can be identified with an Alexandrov space $\mathcal{X}_{\mathcal{U}}$ that also validates L. But since every space validates S4 (which is easily checked), and since there are non-Alexandrov spaces (such as $\mathbb{Q}$ and $\mathbb{R}$), there are topological models of S4 (for example) that cannot be identified with any uniframes.

The first four clauses of the following theorem are classic results of [16], and the last three are easily proved:

Theorem 1.3.

1. *$A \in$ S4 iff for every space $\mathcal{X}$, $\mathcal{X} \models A$;*
2. *$A \in$ S4 iff, for every Alexandrov space $\mathcal{X}$, $\mathcal{X} \models A$;*
3. *$A \in$ S4 iff $\mathbb{Q} \models A$;*
4. *$A \in$ S4 iff $\mathbb{R} \models A$;*
5. *If $\mathcal{X}$ is a space and $\mathcal{X} \models$ S5, then $\mathcal{X}$ is AD (almost discrete);*
6. *$A \in$ S5 iff, for every AD space $\mathcal{X}$, $\mathcal{X} \models A$; and*
7. *$A \in$ S5 iff, for every trivial space $\mathcal{X}$, $\mathcal{X} \models A$.*

Remark 1.4. Theorem 1.3 (2) follows immediately from the analogous claim about unimodal frames, together with the identification of unimodal frames and Alexandrov spaces. (1) follows from (2) and from the soundness of S4 in the topological semantics (which can easily be checked). (3) is much easier to prove than (4): although a proof of (4) was published in 1944 ([16]), a fair amount of recent work has gone into giving (4) more perspicuous proofs ([1, 3, 12, 17]). Given (5) and the fact that every AD space is Alexandrov, the topological semantics for logics stronger than S5 is a notational variant, rather than a true generalization, of the Kripke semantics.

The topological semantics for the language $\mathcal{L}$ generalizes naturally to a bitopological semantics for the language $\mathcal{L}_{12}$. A *bitopological space* (bispace) is a triple $\mathcal{X} = \langle X, \tau_1, \tau_2 \rangle$, where X is a nonempty set and each of τ_1 and τ_2 is a topology on X. Given any $S \subseteq X$, we can consider two interiors of S, $Int_1(S)$ and $Int_2(S)$, associated with the topologies τ_1 and τ_2 respectively. The sets in τ_1 are the *1-open* sets, and the sets in τ_2 are the *2-open* sets. We say that a bispace $\mathcal{X} = \langle X, \tau_1, \tau_2 \rangle$ is 1-*Alexandrov* [2-*Alexandrov*] iff the space $\langle X, \tau_1 \rangle$ [$\langle X, \tau_2 \rangle$] is Alexandrov. Similarly for 1-AD and 2-AD. We say that a bispace is *biAlexandrov* [*biAD*], iff it is 1-Alexandrov and 2-Alexandrov [1-AD and 2-AD].

A *bitopological model* is a quartuple $\mathcal{M} = \langle X, \tau_1, \tau_2, V \rangle$, where $\langle X, \tau_1, \tau_2 \rangle$ is a bispace and $V : PV \to \mathcal{P}(X)$. We extend V to all formulas of $\mathcal{L}_{12}$ with standard Boolean clauses for $\neg$, &, and $\vee$; and with the following clauses for $\Box_1$ and $\Box_2$:

$$\begin{array}{rcl} V(\Box_1 A) & = & Int_1(V(A)) \\ V(\Box_2 A) & = & Int_2(V(A)) \end{array}$$

We sometimes write $x \Vdash A$ instead of $x \in V(A)$. We say that $\mathcal{M} \models A$ iff $V(A) = X$. Given a bispace $\mathcal{X} = \langle X, \tau_1, \tau_2 \rangle$, we say that $\mathcal{X} \models A$ iff $\mathcal{M} \models A$ for every bitopological model $\mathcal{M} = \langle X, \tau_1, \tau_2, V \rangle$. And if Γ is a set of formulas of $\mathcal{L}$, we say that $\mathcal{X} \models \Gamma$ iff $\mathcal{X} \models A$ for each $A \in \Gamma$.

The following theorem generalizes Theorem 1.1 and Theorem 1.3, above:

Theorem 1.5. *1. $A \in \mathrm{S4} \otimes \mathrm{S4}$ iff $\mathcal{X} \models A$, for every bispace $\mathcal{X}$;*

2. If $\mathcal{X}$ is a bispace and $\mathcal{X} \models \mathrm{S4} \otimes \mathrm{S5}$, then $\mathcal{X}$ is 2-AD;

3. $A \in \mathrm{S4} \otimes \mathrm{S5}$ iff $\mathcal{X} \models A$, for every 2-AD bispace $\mathcal{X}$;

4. If $\mathcal{X}$ is a bispace and $\mathcal{X} \models \mathrm{S5} \otimes \mathrm{S5}$, then $\mathcal{X}$ is biAD; and

5. $A \in \mathrm{S5} \otimes \mathrm{S5}$ iff $\mathcal{X} \models A$, for every biAD bispace $\mathcal{X}$.

1.4 Products of topological spaces, and topological products of unimodal logics

Van Benthem, Bezhanishvili, ten Cate and Sarenac [21] define *product spaces*: these are generalizations of Shehtman's product frames ([19]; and subsection 1.2, above). Given two spaces $\mathcal{X}_1 = \langle X_1, \tau_1 \rangle$ and $\mathcal{X}_2 = \langle X_2, \tau_2 \rangle$, define the bispace $\mathcal{X}_1 \times \mathcal{X}_2 =_{\rm df} \langle X_1 \times X_2, \tau_1', \tau_2' \rangle$ where the following two families of subsets of $X_1 \times X_2$ form bases for the topologies τ_1' and τ_2', respectively:

$$\begin{array}{ll} \text{Basis for } \tau_1': & \{O \times \{x\} : O \in \tau_1 \ \&\ x \in X_2\} \\ \text{Basis for } \tau_2': & \{\{x\} \times O : x \in X_1 \ \&\ O \in \tau_2\} \end{array}$$

A bispace of the form $\mathcal{X}_1 \times \mathcal{X}_2$ is a *product space.*[7] [21] refers to the induced topologies τ_1' and τ_2' as the *horizontal* and *vertical* topologies, respectively.

Given two normal modal logics L_1 and L_2 stronger than S4, define the *topological product* of L_1 and L_2 as follows:

$$\begin{array}{lll} \mathrm{L}_1 \times_t \mathrm{L}_2 & =_{\rm df} & \{A : \mathcal{X}_1 \times \mathcal{X}_2 \models A \text{ for any two topological spaces} \\ & & \mathcal{X}_1 \text{ and } \mathcal{X}_2 \text{ such that } \mathcal{X}_1 \models \mathrm{L}_1 \text{ and } \mathcal{X}_2 \models \mathrm{L}_2\}. \end{array}$$

Thus,

$$\begin{array}{lll} \mathrm{S4} \times_t \mathrm{S4} & = & \{A : \mathcal{X}_1 \times \mathcal{X}_2 \models A, \text{ where } \mathcal{X}_1 \text{ and } \mathcal{X}_2 \text{ are topological spaces}\}, \\ \mathrm{S4} \times_t \mathrm{S5} & = & \{A : \mathcal{X}_1 \times \mathcal{X}_2 \models A, \text{ where } \mathcal{X}_2 \text{ is AD}\}, \text{ and} \\ \mathrm{S5} \times_t \mathrm{S5} & = & \{A : \mathcal{X}_1 \times \mathcal{X}_2 \models A, \text{ where } \mathcal{X}_1 \text{ and } \mathcal{X}_2 \text{ are AD}\}. \end{array}$$

2 Results

Not much is known about topological products of modal logics. As noted on page 3, we have the following:

Theorem 2.1.

1. *For any normal logics* L_1 *and* L_2 *stronger than* S4*,* $\mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times_t \mathrm{L}_2 \subseteq \mathrm{L}_1 \times \mathrm{L}_2$.
2. $\mathrm{S4} \otimes \mathrm{S4} = \mathrm{S4} \times_t \mathrm{S4} \subsetneq \mathrm{S4} \times \mathrm{S4}$.
3. $\mathrm{S5} \otimes \mathrm{S5} \subsetneq \mathrm{S5} \times_t \mathrm{S5} = \mathrm{S5} \times \mathrm{S5}$.
4. $\mathrm{S4} \otimes \mathrm{S5} \subsetneq \mathrm{S4} \times_t \mathrm{S5} \subsetneq \mathrm{S4} \times \mathrm{S5}$.

[7] This terminology is at odds with the standard terminology in topology, where the *product space* $\mathcal{X}_1 \times \mathcal{X}_2$ is a topological space with a single topology defined in terms of τ_1 and τ_2. The current notion of a product space as a *bi*topological space is the analog of the notion of a product frame, as defined above, as a *bi*relational frame. We consider the standard product topology in subsection 2.1, below.

Proof. 1. To see that $L_1 \otimes L_2 \subseteq L_1 \times_t L_2$, suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are spaces such that $\mathcal{X}_1 \models L_1$ and $\mathcal{X}_2 \models L_2$. To see that $\mathcal{X}_1 \times \mathcal{X}_2 \models L_1 \otimes L_2$, it suffices to note that $\mathcal{X}_1 \times \mathcal{X}_2 \models L'_1[L'_2]$, where L'_1 $[L'_2]$ is the set of formulas of $\mathcal{L}_{12}$ got by replacing each occurrence of $\Box$ in each formula in L_1 $[L_2]$ by $\Box_1$ $[\Box_2]$.

To see that $L_1 \times_t L_2 \subseteq L_1 \times L_2$, it suffices to recall that every uniframe can be identified with an Alexandrov space. Thus, if $A \notin L_1 \times L_2$, there are Alexandrov spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}_1 \models L_1$ and $\mathcal{X}_2 \models L_2$ and $\mathcal{X}_1 \times \mathcal{X}_2 \not\models A$. Thus $A \notin L_1 \times_t L_2$.

2. A good deal of [21] is devoted to showing that $S4 \otimes S4 = S4 \times_t S4$. To see that $S4 \otimes S4 \subsetneq S4 \times S4$, it suffices to construct a bireflexive bitransitive biframe that does not validate $com_{\supset}$. We leave this to the reader.

3. Given (1), $S5 \times_t S5 \subseteq S5 \times S5$. To see that $S5 \times S5 \subseteq S5 \times_t S5$, suppose that $A \notin S5 \times_t S5$. Then, by Theorem 1.3 (5) and (6), there are AD spaces $\mathcal{X}_1 = \langle X_1, \tau_1 \rangle$ and $\mathcal{X}_2 = \langle X_2, \tau_2 \rangle$ such that $\mathcal{X}_1 \times \mathcal{X}_2 \not\models A$. Since $\mathcal{X}_1$ and $\mathcal{X}_2$ are AD, they are Alexandrov. So $\mathcal{X}_1 = \mathcal{X}_{\mathcal{U}_1}$ and $\mathcal{X}_2 = \mathcal{X}_{\mathcal{U}_2}$ for some reflexive transitive uniframes $\mathcal{U}_1 = \langle X_1, R_1 \rangle$ and $\mathcal{U}_2 = \langle X_2, R_2 \rangle$. Note that $\mathcal{U}_1$ and $\mathcal{U}_2$ will be symmetric, since $\mathcal{X}_1$ and $\mathcal{X}_2$ are AD. Indeed $\mathcal{U}_1 \times \mathcal{U}_2$ validates the same formulas as $\mathcal{X}_1 \times \mathcal{X}_2$. So $\mathcal{U}_1 \times \mathcal{U}_2 \not\models A$. So $A \notin S5 \times S5$.

To see that $S5 \otimes S5 \subsetneq S5 \times S5$, it suffices to construct a bireflexive bitransitive bisymmetric biframe that does not validate $com_{\supset}$. We leave this to the reader.

4. To show that $S4 \otimes S5 \subsetneq S4 \times_t S5$, we note two things: (4.1) $com_{\supset} \notin S4 \otimes S5$; and (4.2) $com_{\supset} \in S4 \times_t S5$. For (4.1), it suffices to construct a bireflexive bitransitive 2-symmetric biframe that does not validate $com_{\supset}$. We leave this to the reader. For (4.2), recall that every space $\mathcal{X}$ with $\mathcal{X} \models S5$ is AD (Theorem 1.3 (3)) and that every AD space is Alexandrov. So (4.2) follows from Proposition 4.15 in [21]. ((4.2) could also be checked directly.)

To show that $S4 \times_t S5 \subsetneq S4 \times S5$, it suffices, given Theorem 1.2, to show that $com_{\subset} \notin S4 \times_t S5$. We will identify the set, $\mathbb{Q}$, of rational numbers with the topological space $\langle \mathbb{Q}, \tau_{\mathbb{Q}} \rangle$, where $\tau_{\mathbb{Q}}$ is the standard topology on $\mathbb{Q}$; and the set, $\mathbb{N}$, of natural numbers with the topological space $\langle \mathbb{N}, \tau_{\mathbb{N}} \rangle$, where $\tau_{\mathbb{N}}$ is the trivial topology on $\mathbb{N}$.[8] Thus $\mathbb{Q} \models S4$ and $\mathbb{N} \models S5$. For each $n \in \mathbb{N}$, let

$$O_n = \{x \in \mathbb{Q} : \frac{-\pi}{n+1} < x < \frac{\pi}{n+1}\}.$$

Note that each O_n is an open subset of $\mathbb{Q}$ (and, incidentally, a closed subset of $\mathbb{Q}$). And let $\mathcal{M}$ be the bitopological model $\langle \mathbb{Q} \times \mathbb{N}, \tau'_{\mathbb{Q}}, \tau'_{\mathbb{N}}, V \rangle$, where $\tau'_{\mathbb{Q}}$

[8] This is a little careless, but will not get us into trouble.

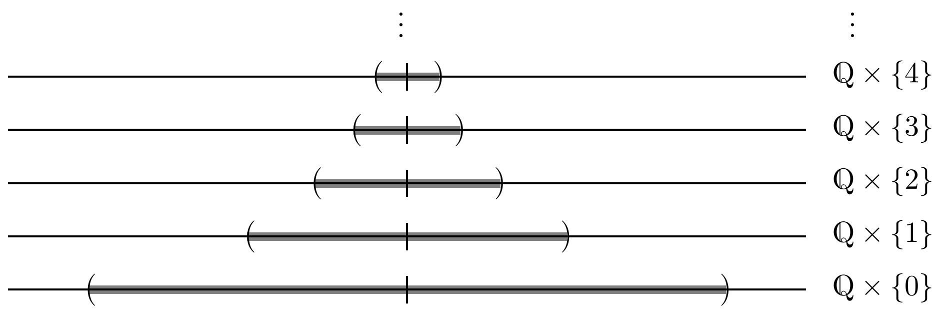

Figure 1: A countermodel for $com_{\subset}$: $V(p)$ is in grey.

and $\tau'_{\mathbb{N}}$ are the horizontal and vertical topologies induced on $\mathbb{Q} \times \mathbb{N}$, and where

$$V(p) = \bigcup_{n \in \mathbb{N}} O_n \times \{n\}.$$

Thus $V(p)$ is a 1-open but not 2-open subset of $\mathbb{Q} \times \mathbb{N}$. See Figure 1 for a picture. Note the following:

$$\begin{array}{rcl} V(\Box_1 p) & = & V(p) = \bigcup_{n \in \mathbb{N}} O_n \times \{n\}. \\ V(\Box_2 p) & = & \{0\} \times \mathbb{N}. \\ V(\Box_2 \Box_1 p) & = & \{0\} \times \mathbb{N}. \\ V(\Box_1 \Box_2 p) & = & \emptyset. \end{array}$$

So $\langle 0, 0 \rangle \not\Vdash (\Box_2 \Box_1 p \supset \Box_1 \Box_2 p)$. So $\mathbb{Q} \times \mathbb{N} \not\models com_{\subset}$. So $com_{\subset} \notin \mathrm{S4} \times_t \mathrm{S5}$. ■

For any normal unimodal logics L_1 and L_2, define the the *e-commutator* of L_1 and L_2 as in [14] and [13]:

$$[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} \quad = \quad \mathrm{L}_1 \otimes \mathrm{L}_2 + com_{\supset} + chr.$$

Our main result is as follows:

Theorem 2.2. $\mathrm{S4} \times_t \mathrm{S5} = [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$.

Theorem 2.2 is, in effect, a soundness and completeness theorem for the logic $[\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ wrt the class of product spaces of the form $\mathcal{X}_1 \times \mathcal{X}_2$, where $\mathcal{X}_2$ is AD: we will call these the vertically AD product spaces. Establishing soundness, i.e., that $[\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}} \subseteq \mathrm{S4} \times_t \mathrm{S5}$, requires checking that $com_{\supset}$ and *chr* are validated by every vertically AD product space. We leave this to the reader. To establish completeness, i.e., that $\mathrm{S4} \times_t \mathrm{S5} \subseteq [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$, we specify a particular vertically AD, indeed vertically trivial, product space wrt which $[\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ is sound and complete: $\mathbb{Q} \times \mathbb{N}$. This suffices:

Theorem 2.3. $\mathbb{Q} \times \mathbb{N} \vDash A$ *iff* $A \in [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$.

Of course, the ($\Leftarrow$) direction is simply soundness (again). Section 3, below, is devoted to proving the ($\Rightarrow$) direction, i.e., completeness.

Theorem 2.3 is of interest independently of its utility in establishing Theorem 2.2. In the unimodal topological semantics, not only is S4 sound and complete wrt the class of all topological spaces, but wrt the particular spaces $\mathbb{Q}$ and $\mathbb{R}$. By definition, $\mathrm{S4} \times_t \mathrm{S5}$ is complete wrt the class of all vertically AD spaces. The fact that $\mathrm{S4} \times_t \mathrm{S5}$ is complete wrt the particular space $\mathbb{Q} \times \mathbb{N}$ is a kind of generalization of the completeness of S4 wrt $\mathbb{Q}$.

In the unimodal case, the result for $\mathbb{Q}$ transfers to $\mathbb{R}$: Does anything like Theorem 2.3 transfer to $\mathbb{R}$? It seems not:

Theorem 2.4. *There is a formula A of $\mathcal{L}_{12}$ such that $A \notin \mathrm{S4} \times_t \mathrm{S5}$, and such that, for each Alexandrov space $\mathcal{X}$, $\mathbb{R} \times \mathcal{X} \vDash A$.*

Proof. Let A be the following formula, a cousin of $com_{\subset}$:

$$\Box_2\Box_1 p \supset (\Box_1\Box_2 p \vee \Diamond_1\Diamond_2(\Diamond_1 p \mathbin{\&} \Diamond_1 \neg p)).$$

We will show (1) that $\mathbb{Q} \times \mathbb{N} \nvDash A$, from which it follows that $A \notin \mathrm{S4} \times_t \mathrm{S5}$; and (2) that $\mathbb{R} \times \mathcal{X} \vDash A$, for each Alexandrov space $\mathcal{X}$. For (1), let $\mathcal{M} = \langle \mathbb{Q} \times \mathbb{N}, \tau'_{\mathbb{Q}}, \tau'_{\mathbb{N}}, V \rangle$ be the model specified in the proof of Theorem 2.1 (4) and pictured in Figure 1. We noted in that proof that

$$\begin{array}{rcl} V(\Box_1 p) &=& V(p) = \bigcup_{n \in \mathbb{N}} O_n \times \{n\}, \\ V(\Box_2 p) &=& \{0\} \times \mathbb{N}, \\ V(\Box_2\Box_1 p) &=& \{0\} \times \mathbb{N}, \text{ and} \\ V(\Box_1\Box_2 p) &=& \emptyset. \end{array}$$

Recall the sets $O_n = \{x \in \mathbb{Q} : \frac{-\pi}{n+1} < x < \frac{\pi}{n+1}\}$ used to specify $V(p)$: each O_n is of the form $(a, b) \cap \mathbb{Q}$, where (a, b) is an open interval on the real line *with irrational endpoints*. Thus O_n is not only an open subset of $\mathbb{Q}$, but is also a closed subset of $\mathbb{Q}$. Thus,

$$
\begin{array}{rcl}
V(\Diamond_1 p) & = & V(p) = \bigcup_{n\in\mathbb{N}} O_n \times \{n\}. \\
V(\Diamond_1 \neg p) & = & V(\neg p) = (\mathbb{Q}\times\mathbb{N}) - V(p). \\
V(\Diamond_1 p \ \&\ \Diamond_1\neg p) & = & \emptyset. \\
V(\Diamond_1\Diamond_2(\Diamond_1 p \ \&\ \Diamond_1\neg p)) & = & \emptyset.
\end{array}
$$

So $\langle 0,0\rangle \not\Vdash \Box_1\Box_2 p \vee \Diamond_1\Diamond_2(\Diamond_1 p \ \&\ \Diamond_1\neg p)$. But $\langle 0,0\rangle \Vdash \Box_2\Box_1 p$. So $\langle 0,0\rangle \not\Vdash A$. So $\mathbb{Q}\times\mathbb{N} \not\models A$, as desired.

To show (2), suppose that $\mathcal{X} = \langle X, \tau\rangle$ is an Alexandrov space and that for some bitopological model $\mathcal{M} = \langle \mathbb{R}\times X, \tau'_{\mathbb{R}}, \tau', V\rangle$, where $\tau'_{\mathbb{R}}$ and τ' are the horizontal and vertical topologies induced on $\mathbb{R}\times X$, we have $\mathcal{M} \not\models A$. Then we have $\langle a,x\rangle \not\Vdash A$, for some $a\in\mathbb{R}$ and $x\in X$. Since $\mathcal{X}$ is Alexandrov, there is a smallest open set O such that $x\in O$.

Since $\langle a,x\rangle \not\Vdash A$, we have the following:

$$\langle a,x\rangle \Vdash \Box_2\Box_1 p. \tag{2.1}$$

$$\langle a,x\rangle \Vdash \Diamond_1\Diamond_2\neg p. \tag{2.2}$$

$$\langle a,x\rangle \Vdash \Box_1\Box_2(\Box_1\neg p \vee \Box_1 p). \tag{2.3}$$

Given (2.3), there is an open interval $I\subseteq\mathbb{R}$ such that $a\in I$ and

$$\forall b\in I,\ \langle b,x\rangle \Vdash \Box_2(\Box_1\neg p \vee \Box_1 p). \tag{2.4}$$

Given (2.2), there is a $c\in I$ and a $y\in O$ such that

$$\langle c,y\rangle \Vdash \neg p. \tag{2.5}$$

Given (2.3), $\langle c,x\rangle \Vdash \Box_2(\Box_1\neg p\vee\Box_1 p)$. So $\langle c,y\rangle \Vdash (\Box_1\neg p\vee\Box_1 p)$. So, given (2.5), we get

$$\langle c,y\rangle \Vdash \Box_1\neg p. \tag{2.6}$$

Also, given (2.1),

$$\langle a,y\rangle \Vdash \Box_1 p. \tag{2.7}$$

Finally, by (2.4),

$$\forall b\in I,\ \langle b,y\rangle \Vdash \Box_1\neg p \text{ or } \langle b,y\rangle \Vdash \Box_1 p. \tag{2.8}$$

Define two subsets of $\mathbb{R}$ as follows:

$$
\begin{array}{rcl}
O^- & =_{\mathrm{df}} & \{d\in I : \langle d,y\rangle \models \Box_1\neg p\}. \\
O^+ & =_{\mathrm{df}} & \{d\in I : \langle d,y\rangle \models \Box_1 p\}.
\end{array}
$$

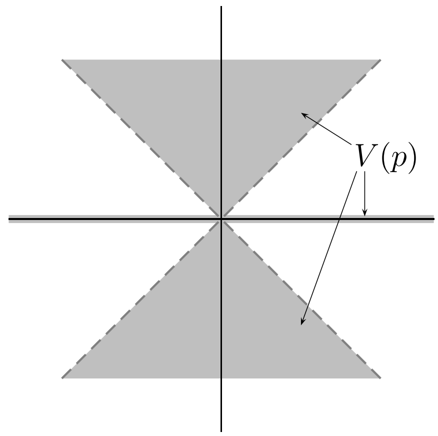

Figure 2: The model from Remark 2.5: $V(p)$ is indicated in grey, in the triangles above and below the horizontal axis, and along the horizontal axis.

Note that $O^- \times \{y\}$ and $O^+ \times \{y\}$ are 1-open subsets of $\mathbb{R} \times X$, so that O^- and O^+ are open subsets of $\mathbb{R}$. By (2.6) and (2.7), O^- and O^+ are nonempty. Clearly O^- and O^+ are disjoint. Finally, by (2.8), $I = O^- \cup O^+$. So the open interval $I \subseteq \mathbb{R}$ is the union of two disjoint nonempty open sets. But this cannot be. ■

Remark 2.5. In proving that $\mathbb{R} \times \mathcal{X} \vDash \Box_2\Box_1 p \supset (\Box_1\Box_2 p \vee \Diamond_1\Diamond_2(\Diamond_1 p \,\&\, \Diamond_1\neg p))$ for every Alexandrov space $\mathcal{X}$, we needed the assumption that $\mathcal{X}$ is Alexandrov. For there are non-Alexandrov spaces $\mathcal{X}$ such that $\mathbb{R} \times \mathcal{X} \nvDash \Box_2\Box_1 p \supset (\Box_1\Box_2 p \vee \Diamond_1\Diamond_2(\Diamond_1 p \,\&\, \Diamond_1\neg p))$. For example, $\mathbb{R} \times \mathbb{R} \nvDash \Box_2\Box_1 p \supset (\Box_1\Box_2 p \vee \Diamond_1\Diamond_2(\Diamond_1 p \,\&\, \Diamond_1\neg p))$. To see this, let $\mathcal{M}$ be the model $\langle \mathbb{R} \times \mathbb{R}, \tau_1, \tau_2, V\rangle$, where τ_1 and τ_2 are the horizontal and vertical topologies induced by the standard topology on $\mathbb{R}$ and where

$$V(p) = \{\langle x, y\rangle : x, y \in \mathbb{R} \,\&\, |x| < |y|\} \cup \{\langle x, 0\rangle : x \in \mathbb{R}\}.$$

See Figure 2 for a picture of this model. Note: $\langle 0, 0\rangle \Vdash \Box_2\Box_1 p$, but $\langle 0, 0\rangle \nVdash \Box_1\Box_2 p$ and $\langle 0, 0\rangle \nVdash \Diamond_1\Diamond_2(\Diamond_1 p \,\&\, \Diamond_1\neg p)$.

2.1 The standard product topology

Van Benthem *et al* [21] also consider a language $\mathcal{L}_{\Box 12}$ with three modalities, $\Box$, $\Box_1$ and $\Box_2$: they interpret $\Box$ with the standard product topology. More precisely, given any two topological space $\mathcal{X}_1 = \langle X_1, \tau_1 \rangle$ and $\mathcal{X}_2 = \langle X_1, \tau_1 \rangle$, they consider the *tri*topological space $(\mathcal{X}_1 \times \mathcal{X}_2)^+ =_{\rm df} \langle X_1 \times X_2, \tau_{12}, \tau_1', \tau_2' \rangle$, where τ_1' and τ_2' are the horizontal and vertical topologies already defined, and τ_{12} is the standard product topology with the basis,

$$\{O_1 \times O_2 : O_1 \in \tau_1 \ \&\ O_2 \in \tau_2\}.$$

The modality $\Box$ is then interpreted via τ_{12}:

$$V(\Box A) = Int_{12}(V(A)),$$

where Int_{12} is the interior operator associated with the topology τ_{12}. Given any two unimodal logics L_1 and L_2, define the trimodal logic

$$\begin{aligned} (\mathrm{L}_1 \times_t \mathrm{L}_2)^+ \quad =_{\rm df} \quad & \{A \text{ is a formula of } \mathcal{L}_{\Box 12} : (\mathcal{X}_1 \times \mathcal{X}_2)^+ \vDash A, \\ & \text{for any two topological spaces } \mathcal{X}_1 \vDash \mathrm{L}_1 \ \&\ \mathcal{X}_2 \vDash \mathrm{L}_2\} \end{aligned}$$

Van Benthem *et al* show that

$$(\mathrm{S4} \times_t \mathrm{S4})^+ = (\mathrm{S4} \otimes \mathrm{S4} \otimes \mathrm{S4}) + (\Box p \supset (\Box_1 p \ \&\ \Box_2 p)),^9$$

and that $\Box$ cannot be defined, in $(\mathrm{S4} \times_t \mathrm{S4})^+$, in terms of $\Box_1$ and $\Box_2$. Indeed, they show that the axiomatization $(\mathrm{S4} \otimes \mathrm{S4} \otimes \mathrm{S4}) + (\Box p \supset \Box_1 p \ \&\ \Box_2 p)$ is complete for the tritopological space $(\mathbb{Q} \times \mathbb{Q})^+$.

Life is easier in the case of $(\mathrm{S4} \times_t \mathrm{S5})^+$. If $\mathcal{X}_1 = \langle X_1, \tau_1 \rangle$ is any topological space and if $\mathcal{X}_2 = \langle X_2, \tau_2 \rangle$ is any almost disjoint space (i.e., any space that validates S5), then in the tritopological space $(\mathcal{X}_1 \times \mathcal{X}_2)^+ = \langle X_1 \times X_2, \tau_{12}, \tau_1', \tau_2' \rangle$ we have $Int_{12}(S) = Int_1(Int_2(S))$ for any $S \subseteq X_1 \times X_2$. So $(\Box p \equiv \Box_1 \Box_2 p) \in (\mathrm{S4} \times_t \mathrm{S5})^+$. Indeed, given Theorem 2.2,

$$\begin{aligned} (\mathrm{S4} \times_t \mathrm{S5})^+ \quad &= \quad (\mathrm{S4} \otimes [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}) + (\Box p \equiv \Box_1 \Box_2 p)^{10} \\ &= \quad (\mathrm{S4} \otimes \mathrm{S4} \otimes \mathrm{S5}) + com_{\supset} + chr + (\Box p \equiv \Box_1 \Box_2 p). \end{aligned}$$

[9] Given any three logics L and L_1 and L_2 formulated in the language $\mathcal{L}$, define the trimodal logic $\mathrm{L} \otimes \mathrm{L}_1 \otimes \mathrm{L}_2$, formulated in $\mathcal{L}_{\Box 12}$, as follows: let L_1' $[\mathrm{L}_2']$ be the set of formulas of $\mathcal{L}_{\Box 12}$ got by replacing each occurrence of $\Box$ in each formula in L_1 $[\mathrm{L}_2]$ by $\Box_1$ $[\Box_2]$; and let $\mathrm{L} \otimes \mathrm{L}_1 \otimes \mathrm{L}_2$ be the smallest set of formulas, of $\mathcal{L}_{\Box 12}$, that contains $\mathrm{L} \cup \mathrm{L}_1' \cup \mathrm{L}_2'$ and that is closed under modus ponens; necessitation for $\Box$, for $\Box_1$, and for $\Box_2$; and substitution.

We can do better, given Theorem 2.3: the trimodal logic $(\mathrm{S4} \otimes [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}) + (\Box p \equiv \Box_1 \Box_2 p)$ is complete for the tritopological space $(\mathbb{Q} \times \mathbb{N})^+$. To see this, suppose that $(\mathbb{Q} \times \mathbb{N})^+ \vDash A$. Let A' be the result of replacing, in A, every occurrence of $\Box$ with $\Box_1 \Box_2$. Then $(\mathbb{Q} \times \mathbb{N})^+ \vDash A'$, since $Int_{12} = Int_1 \circ Int_2$. So $\mathbb{Q} \times \mathbb{N} \vDash A'$, since $\Box$ does not occur in A'. So $A' \in [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$, by Theorem 2.3. So $A \in (\mathrm{S4} \otimes [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}) + (\Box p \equiv \Box_1 \Box_2 p)$, as desired.

3 Proof of Theorem 2.3

3.1 p-morphisms

Suppose that $\mathcal{X} = \langle X, \tau \rangle$ and $\mathcal{Y} = \langle Y, \sigma \rangle$ are topological spaces and that $f : X \to Y$. We say that f is *continuous* iff the preimage of every open set is open; that f is *open* iff the image of every open set is open; that f is a *homeomorphism* iff f is a continuous open bijection; and that f is a *topological p-morphism* (or simply *p-morphism*) iff f is a continuous open surjection. Since we are identifying any reflexive transitive uniframe $\mathcal{U}$ with the Alexandrov space $\mathcal{X}_{\mathcal{U}}$, we can meaningfully talk about continuous functions, etc., from a topological space $\mathcal{X}$ to a reflexive transitive uniframe $\mathcal{U}$. The proof of following theorem is standard:

Theorem 3.1. *For any topological spaces $\mathcal{X}$ and $\mathcal{Y}$,*

1. *if there is a homeomorphism from $\mathcal{X}$ to $\mathcal{Y}$ then, for every formula A of $\mathcal{L}$, $\mathcal{X} \vDash A$ iff $\mathcal{Y} \vDash A$; and*

2. *if there is a topological p-morphism from $\mathcal{X}$ to $\mathcal{Y}$ then, for every formula A of $\mathcal{L}$, if $\mathcal{X} \vDash A$ then $\mathcal{Y} \vDash A$.*

Suppose that $\mathcal{X} = \langle X, \tau_1, \tau_2 \rangle$ and $\mathcal{Y} = \langle Y, \sigma_1, \sigma_2 \rangle$ are bispaces and that $f : X \to Y$. We say that f is *1-continuous* [*2-continuous*] iff the preimage of every 1-open [2-open] set is 1-open [2-open]; and we say that f is *1-open* [*2-open*] iff the image of every 1-open [2-open] set is 1-open [2-open]. We say that f is *bicontinuous* [*biopen*] iff f is both 1- and 2-continuous [-open]. We give similar definitions of the following: *1-* and *2-homeomorphism* and *1-* and *2-p-morphism*; and *bihomeomorphism* and *bi-p-morphism.* Just as we

[10] Given any unimodal logic L formulated in $\mathcal{L}$ and any bimodal logic L′ formulated in $\mathcal{L}_{12}$, define the trimodal logic $\mathrm{L} \otimes \mathrm{L}'$, formulated in $\mathcal{L}_{\Box 12}$, in the obvious way.

are identifying any reflexive transitive uniframe $\mathcal{U}$ with the Alexandrov space $\mathcal{X}_{\mathcal{U}}$, we can identify any bireflexive bitransitive biframe $\mathcal{B} = \langle W, R_1, R_2 \rangle$ with the biAlexandrov bispace $\mathcal{X}_{\mathcal{B}} = \langle W, \tau_1, \tau_2 \rangle$, where for $i = 1, 2$, $O \in \tau_i$ iff $\forall w \in O, \forall w' \in W(wR_iw' \Rightarrow w' \in O)$. Thus, we can meaningfully talk about bicontinuous functions, etc., from a bispace $\mathcal{X}$ to a bireflexive bitransitive biframe $\mathcal{B}$. The proof of following theorem is standard:

Theorem 3.2. *For any bispaces $\mathcal{X}$ and $\mathcal{Y}$,*

1. *if there is a bihomeomorphism from $\mathcal{X}$ to $\mathcal{Y}$ then, for every formula A of $\mathcal{L}_{12}$, $\mathcal{X} \models A$ iff $\mathcal{Y} \models A$; and*
2. *if there is a bi-p-morphism from $\mathcal{X}$ to $\mathcal{Y}$ then, for every formula A of $\mathcal{L}_{12}$, if $\mathcal{X} \models A$ then $\mathcal{Y} \models A$.*

3.2 Completeness of S4 wrt $\mathbb{Q}$

The core of many proofs that S4 is complete wrt the topological space $\mathbb{Q}$ is the construction, for every finite rooted (see page 4) uniframe $\mathcal{U}$, of a p-morphism from $\mathbb{Q}$ to $\mathcal{U}$ or of some other dense linear ordering without endpoints to $\mathcal{U}$. Given such a construction, the completeness argument goes as follows. Suppose that A is a formula of $\mathcal{L}$ and that $A \notin$ S4. By the finite frame property for S4, there is some finite rooted uniframe $\mathcal{U}$ such that $\mathcal{U} \not\models A$. Since there is a p-morphism from $\mathbb{Q}$ to $\mathcal{U}$, we conclude that $\mathbb{Q} \not\models A$, by Theorem 3.1. QED. (See, e.g., [21] and [12].)

In fact, we can generalize most of the constructions in the literature to construct, for every *countable* rooted uniframe $\mathcal{U}$, a a p-morphism from $\mathbb{Q}$ to $\mathcal{U}$:

Theorem 3.3. *For every countable rooted uniframe $\mathcal{U}$, there is a p-morphism from $\mathbb{Q}$ to $\mathcal{U}$.*

Proof. There are any number of constructions of p-morphisms from $\mathbb{Q}$ to arbitrary finite rooted uniframes: these are often generalizable to countable roote uniframes. See the appendix in [11]. ■

We introduce some notions we will use below. An *open $\mathbb{Q}$-interval* is any set of the form $\mathbb{Q} \cap (a, b)$, where (a, b) is some open interval in $\mathbb{R}$. We write $(a, b)_{\mathbb{Q}}$ for this open $\mathbb{Q}$-interval. Note that a or b could be irrational: if both a and b are irrational then we say that $(a, b)_{\mathbb{Q}}$ is an *irrational interval*:

Note that every irrational interval is both an open and a closed subset of $\mathbb{Q}$. Suppose that $\mathcal{U} = \langle W, R\rangle$ is a countable rooted uniframe, and that f is a p-morphism from $\mathbb{Q}$ to $\mathcal{U}$. If $w \in W$, we say that an open $\mathbb{Q}$-interval I is a *w-interval* iff the image of I under f is $[w] =_{\text{df}} \{w' \in W : wRw'\}$. We will make use of the following lemma below:

Lemma 3.4. *Suppose that f is a p-morphism from $\mathbb{Q}$ to a uniframe $\mathcal{U} = \langle W, R\rangle$. Then for every $x \in \mathbb{Q}$, there is an irrational $f(x)$-interval I such that $x \in I$.*

Proof. Choose any $x \in \mathbb{Q}$, and let $w = f(x)$. Since $[w]$ is an open subset of W and since f is continuous, there is a $\mathbb{Q}$-interval J such that $x \in J$ and $f(y) \in [w]$ for every $y \in J$. Let I be any irrational $\mathbb{Q}$-interval such that $x \in I \subseteq J$. Then $x \in I$ and $f(y) \in [w]$ for every $y \in I$. Let O be the image of I under f: then $w = f(x) \in O$ and $O \subseteq [w]$. Since f is an open function, the image of I is open, so that O is open. So $O = [w]$. ■

3.3 Expanding relativized product frames

A biframe $\mathcal{B} = \langle W, R_1, R_2\rangle$ is a *subframe* of a biframe $\mathcal{B}' = \langle W', R'_1, R'_2\rangle$ iff $W' \subseteq W$ and $R_1 = R'_1 \cap (W \times W)$ and $R_2 = R'_2 \cap (W \times W)$. Following [14] and [13], we say that a biframe $\mathcal{B} = \langle W, R_1, R_2\rangle$ is an *expanding relativized product frame* (ERPF) iff there are uniframes $\mathcal{U}_1 = \langle W_1, S_1\rangle$ and $\mathcal{U}_2 = \langle W_1, S_2\rangle$ such that

- $\mathcal{B}$ is a subframe of $\mathcal{U}_1 \times \mathcal{U}_2$, and
- for all $\langle w_1, w_2\rangle \in W$ and $w \in W_1$, if $w_1 S_1 w$ then $\langle w, w_2\rangle \in W$.

Figure 3 represents a product space $\mathcal{U}_1 \times \mathcal{U}_2$ together with a subspace which is an ERPF. Again, following [14] and [13], define EX to be the class of all ERPF's; and for normal unimodal logics L_1 and L_2, define the logic $(\mathrm{L}_1 \times \mathrm{L}_2)^{\mathsf{EX}}$ as follows:

$$(\mathrm{L}_1 \times \mathrm{L}_2)^{\mathsf{EX}} =_{\text{df}} \{A : \mathcal{B} \models A \text{ for each ERPF } \mathcal{B} \text{ which is a subframe of some } \mathcal{U}_1 \times \mathcal{U}_2 \text{ where } \mathcal{U}_1 \models \mathrm{L}_1 \text{ and } \mathcal{U}_1 \models \mathrm{L}_2\}$$

The following is a special case of Theorem 6 in [14], where $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}$ is the e-commutator of L_1 and L_2 defined on page 11, above:

Theorem 3.5. $(\mathrm{S4} \times \mathrm{S5})^{\mathsf{EX}} = [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$.

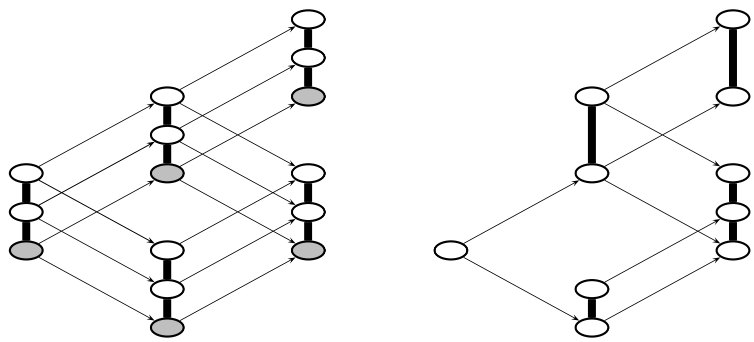

Figure 3: To the left is an example of a product space of the form $\mathcal{U}_1 \times \mathcal{U}_2 = \langle W_1 \times W_2, S_1', S_2' \rangle$. S_1' is the reflexive transitive closure of the relation given by the thin diagonal arrows, and S_2' is the reflexive transitive symmetric closure of the relation given by the thick vertical lines. A copy of $\mathcal{U}_1$ is shaded in grey. To the right is an expanding relativized product frame that is a subframe of $\mathcal{U}_1 \times \mathcal{U}_2$.

Theorem 3.5 is, in effect, a soundness and completeness theorem for the logic $[\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ wrt the following subclass of EX:

$$\mathsf{EX}' =_{\mathrm{df}} \{\mathcal{B} \in \mathsf{EX} : \mathcal{B} \text{ is a subspace of } \mathcal{U}_1 \times \mathcal{U}_2 \text{ where } \mathcal{U}_1 \text{ is reflexive and transitive and } \mathcal{U}_2 \text{ is reflexive, transitve and symmetric}\}.$$

If $\mathcal{B} = \langle W, R_1, R_2 \rangle$ is any biframe, say that $r \in W$ is a *root* of $\mathcal{B}$ iff $\forall w \in W$, $\exists w' \in W$, rR_1w' & $w'R_2w$. And say that $\mathcal{B}$ is *rooted* iff $\mathcal{B}$ has a root. Example: the ERPF represented on the right side of Figure 3 is rooted. By standard methods, we can show that $[\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ is complete wrt the class of rooted biframes in EX'. By the methods in Section 5 of [5], we can do even better. Define

$$\mathsf{EX}'' =_{\mathrm{df}} \{\mathcal{B} \in \mathsf{EX}' : \mathcal{B} \text{ is } \textit{countable} \text{ and rooted}\}.$$

Theorem 3.6. *$A \in [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ iff $\mathcal{B} \models A$ for every $\mathcal{B} \in \mathsf{EX}''$.*

3.4 The proof of Theorem 2.3

Recall that we want to prove the ($\Rightarrow$) direction of Theorem 2.3. Given Theorems 3.2 and 3.6, it suffices to prove the following:

Theorem 3.7. *For each $\mathcal{B} \in \mathsf{EX}''$, there is a bi-p-morphism from $\mathbb{Q} \times \mathbb{N}$ to $\mathcal{B}$.*

Proof. Suppose that $\mathcal{B} = \langle W, R_1, R_2 \rangle \in \mathsf{EX}''$. It will suffice to find a countable set X and a a bi-p-morphism from $\mathbb{Q} \times \mathcal{X}$ to $\mathcal{B}$, where $\mathcal{X} = \langle X, \tau \rangle$ and τ is the trivial topology on $\mathcal{X}$. Since $\mathcal{B}$ is countable and rooted, there is

1. a countable rooted reflexive transitive uniframe $\mathcal{U}_1 = \langle W_1, S_1 \rangle$ with root $r_1 \in W_1$, and
2. a countable *universal* uniframe $\mathcal{U}_2 = \langle W_2, S_2 \rangle$, such that
3. $\mathcal{B}$ is a subframe of $\mathcal{U}_1 \times \mathcal{U}_2 = \langle W_1 \times W_2, S_1', S_2' \rangle$; and
4. for some $r_2 \in W_2$,
   (a) $\langle r_1, r_2 \rangle$ is a root of $\mathcal{B}$, and
   (b) $W_1 \times \{r_2\} \subseteq W$; and
5. $\forall v \in W_2,\ \exists u \in W_1,\ \langle u, v \rangle \in W$.

For each $u \in W_1$, recall the definition $[u] =_{\text{df}} \{w \in W_1 : uS_1w\}$. For each $\langle u, v \rangle \in W$, note that the set $(W_1 \times \{r_2\}) \cup ([u] \times \{v\}) = ([r_1] \times \{r_2\}) \cup ([u] \times \{v\})$ is 1-open in $\mathcal{B}$.

Let f be a p-morphism from $\mathbb{Q}$ to $\mathcal{U}_1$ (see Theorem 3.3). Let $X = \{\langle x, v \rangle \in \mathbb{Q} \times W_2 : \langle f(x), v \rangle \in W\}$. Note that X is countable. Let τ be the trivial topology on X, i.e., $\tau = \{\emptyset, X\}$; and let $\mathcal{X} = \langle X, \tau \rangle$. Shortly, we define a function $F : \mathbb{Q} \times X \to W$. First, for every $x \in \mathbb{Q}$, choose an irrational $f(x)$-interval I_x such that $x \in I_x$ (see Lemma 3.4). Define F as follows:

$$F(\langle y, \langle x, v \rangle \rangle) = \begin{cases} \langle f(y), v \rangle \text{ if } y \in I_x \\ \langle f(y), r_2 \rangle \text{ if } y \notin I_x \end{cases}$$

We must ensure that $F(\langle y, \langle x, v \rangle \rangle) \in W$ whenever $y \in \mathbb{Q}$ and $\langle f(x), v \rangle \in W$. If $y \notin I_x$, then $F(\langle y, \langle x, v \rangle \rangle) = \langle f(y), r_2 \rangle \in W$, since $W_1 \times \{r_2\} \subseteq W$. Suppose, on the other hand, that that $y \in I_x$. Note that $f(x)S_1f(y)$, since I_x is an $f(x)$-interval and since $y \in I_x$. So $\langle f(y), v \rangle \in W$, by the second clause of the definition of expanding relativized product frames and by the fact that $\langle f(x), v \rangle \in W$.

All we have left is to prove that F is a bi-p-morphism from $\mathbb{Q} \times \mathcal{X}$ to $\mathcal{B}$, i.e., that F is bicontinuous, biopen and onto.

F **is onto.** Suppose that $\langle u, v\rangle \in W$. Choose any $x \in \mathbb{Q}$ such that $f(x) = u$. Recall that $x \in I_x$. So $F(x, \langle x, v\rangle) = \langle f(x), v\rangle = \langle u, v\rangle$.

F **is 1-continuous.** Suppose that O is a 1-open subset of W, i.e., O is closed under the relation R_1. We want to show that the preimage of O under F, say O', is a 1-open subset of $\mathbb{Q} \times \mathcal{X}$. We can assume that O is of the form $[u] \times \{v\}$ for some $u \in W_1$ and $v \in W_2$ with $\langle u, v\rangle \in W$, since sets of this form form a basis for the topology on $\mathcal{B}$ induced by the relation R_1. Let O'' be the preimage of the set $[u] \subseteq W_1$ under the p-morphism f. The set $O'' \subseteq \mathbb{Q}$ is open, since f is continuous. Given the definition of F,

$$
\begin{aligned}
\langle y, \langle x, w\rangle\rangle \in O' \quad & \text{iff} \quad F(\langle y, \langle x, w\rangle\rangle) \in [u] \times \{v\} \\
& \text{iff} \quad \begin{cases} \langle f(y), w\rangle \in [u] \times \{v\} \ \&\ y \in I_x, \text{ or} \\ \langle f(y), r_2\rangle \in [u] \times \{v\} \ \&\ y \notin I_x; \end{cases} \\
& \text{iff} \quad \begin{cases} f(y) \in [u] \ \&\ w = v \ \&\ y \in I_x, \text{ or} \\ f(y) \in [u] \ \&\ v = r_2 \ \&\ y \notin I_x; \end{cases} \\
& \text{iff} \quad \begin{cases} y \in O'' \ \&\ w = v \ \&\ y \in I_x, \text{ or} \\ y \in O'' \ \&\ v = r_2 \ \&\ y \notin I_x; \end{cases} \\
& \text{iff} \quad \begin{cases} y \in O'' \cap I_x \ \&\ w = v, \text{ or} \\ y \in O'' - I_x \ \&\ v = r_2. \end{cases}
\end{aligned}
$$

So if $v = r_2$, then

$$O' \quad = \bigcup_{\langle x, r_2\rangle \in X} (O'' \cap I_x) \times \{\langle x, r_2\rangle\} \cup \bigcup_{\langle x, w\rangle \in X} (O'' - I_x) \times \{\langle x, w\rangle\}$$

And if $v \neq r_2$, then

$$O' \quad = \bigcup_{\langle x, v\rangle \in X} (O'' \cap I_x) \times \{\langle x, v\rangle\}$$

To show that O' is 1-open, it is enough to show that, whether or not $v = r_2$, O' is a union of 1-open subsets of $\mathbb{Q} \times X$. Recall that I_x is both open and closed in $\mathbb{Q}$, for every $x \in \mathbb{Q}$. So $O'' \cap I_x$ and $O'' - I_x$ are open in $\mathbb{Q}$, for every $x \in \mathbb{Q}$. So the following are 1-open in $\mathbb{Q} \times X$ for any $\langle x, w\rangle \in X$: $(O'' \cap I_x) \times \{\langle x, w\rangle\}$ and $(O'' - I_x) \times \{\langle x, w\rangle\}$. So O' is a union of 1-open subsets of $\mathbb{Q} \times X$.

F **is 1-open.** Suppose that O is a 1-open subset of $\mathbb{Q} \times \mathcal{X}$. We want to show that the image of O under F, say O', is a 1-open subset of W. We can assume that O is of the form $I \times \{\langle x, v\rangle\}$ for some open $\mathbb{Q}$-interval I and some $\langle x, v\rangle \in X$ since sets of this form form a basis for the horizontal topology on $\mathbb{Q} \times \mathcal{X}$.

Let $O_1 = I \cap I_x$ and $O_2 = I - I_x$. Note that each O_i is open in $\mathbb{Q}$, since I_x is both open and closed in $\mathbb{Q}$; and that $I = O_1 \cup O_2$. Let O_i^* be the image of O_i under f: O_i^* is open since f is a p-morphism. And let O_i' be the image of $O_i \times \{\langle x, v\rangle\}$ under F. It will suffice to show that each O_i' is open since $O' = O_1' \cup O_2'$.

Re O_1'. It will suffice to show that $O_1' = O_1^* \times \{v\}$. To see that $O_1' \subseteq O_1^* \times \{v\}$, suppose that $\langle y, \langle z, w\rangle\rangle \in O_1 \times \{\langle x, v\rangle\}$. Then $y \in I \cap I_x$ and $\langle z, w\rangle = \langle x, v\rangle$. So $F(\langle y, \langle z, w\rangle\rangle) = \langle f(y), v\rangle \in O_i^* \times \{v\}$, as desired. To see that $O_1^* \times \{v\} \subseteq O_1'$, suppose that $\langle w, v\rangle \in O_1^* \times \{v\}$. Then $w \in O_1^*$, so that $w = f(c)$ for some $c \in O_1 = I \cap I_x$. So $F(\langle c, \langle x, v\rangle\rangle) = \langle f(c), v\rangle = \langle w, v\rangle$. So $\langle w, v\rangle \in O_1'$, as desired.

Re O_2'. It will suffice to show that $O_2' = O_2^* \times \{r_2\}$. To see that $O_2' \subseteq O_2^* \times \{r_2\}$, suppose that $\langle y, \langle z, w\rangle\rangle \in O_2 \times \{\langle x, v\rangle\}$. Then $y \in I - I_x$ and $\langle z, w\rangle = \langle x, v\rangle$. So $F(\langle y, \langle z, w\rangle\rangle) = \langle f(y), r_2\rangle \in O_2^* \times \{r_2\}$, as desired. To see that $O_2^* \times \{r_2\} \subseteq O_2'$, suppose that $\langle w, r_2\rangle \in O_2^* \times \{r_2\}$. Then $w \in O_2^*$, so that $w = f(c)$ for some $c \in O_2 = I - I_x$. So $F(\langle c, \langle x, v\rangle\rangle) = \langle f(c), r_2\rangle = \langle w, r_2\rangle$. So $\langle w, r_2\rangle \in O_2'$, as desired.

F **is 2-continuous.** For this it will suffice to show that, for every $y \in \mathbb{Q}$ and every $\langle z, u\rangle, \langle x, v\rangle \in X$, we have

$$F(y, \langle z, u\rangle) \;\; R_2 \;\; F(y, \langle x, v\rangle).$$

Note: $F(y, \langle z, u\rangle) = \langle f(y), w\rangle$, where $w = r_2$ or u and $\langle f(y), w\rangle \in W \subseteq W_1 \times W_2$. Likewise, $F(y, \langle x, v\rangle) = \langle f(y), w'\rangle$, where $w' = r_2$ or v and $\langle f(y), w'\rangle \in W \subseteq W_1 \times W_2$. Now, $\langle f(y), w\rangle S_2' \langle f(y), w'\rangle$. So $\langle f(y), w\rangle R_2 \langle f(y), w'\rangle$, since $\mathcal{B}$ is a subframe of $\mathcal{U}_1 \times \mathcal{U}_2$. So $F(y, \langle z, u\rangle) R_2 F(y, \langle x, v\rangle)$, as desired.

F **is 2-open.** For this it will suffice to show that, for every $y \in \mathbb{Q}$, every $\langle x, v\rangle \in X$ and every $\langle w_1, w_2\rangle \in W$,

$$\text{if } F(\langle y, \langle x, v\rangle\rangle)\ R_2\ \langle w_1, w_2\rangle \text{ then } \exists\langle x', v'\rangle \in X,\ F(\langle y, \langle x', v'\rangle\rangle) = \langle w_1, w_2\rangle.$$

So suppose that $F(\langle y, \langle x, v\rangle\rangle)\ R_2\ \langle w_1, w_2\rangle$. Then $f(y) = w_1$. Note that $F(\langle y, \langle y, w_2\rangle\rangle) = \langle f(y), w_2\rangle$, since $y \in I_y$. Let $x' = y$ and $v' = w_2$. Then $F(\langle y, \langle x', v'\rangle\rangle) = \langle f(y), w_2\rangle = \langle w_1, w_2\rangle$, as desired. ■

4 The bimodal logic of products of arbitrary spaces and Alexandrov spaces

Van Benthem *et al* [21] note that, since Alexandrov spaces are identified with reflexive transitive frames (see subsection 1.3), the bimodal logic of the products of Alexandrov spaces is simply the frame product S4×S4. They ask what the bimodal logic is of the products of Alexandrov spaces with arbitrary topological spaces, conjecturing that it is $(\text{S4} \otimes \text{S4}) + com_{\subset} + chr$. Their conjecture follows from its mirror image, i.e., that the bimodal logic of the products of arbitrary spaces with Alexandrov spaces is $(\text{S4}\otimes\text{S4})+com_{\supset}+chr$. In this section, we prove this last claim.

For another way to put this, let $\mathfrak{Top}$ be the class of all topological spaces and let $\mathfrak{Alex}$ be the class of Alexandrov spaces. For any classes $\mathfrak{X}$ and $\mathfrak{Y}$ of topological spaces, let $\mathfrak{X} \times \mathfrak{Y} =_{\text{df}} \{\mathcal{X} \times \mathcal{Y} : \mathcal{X} \in \mathfrak{X}\ \&\ \mathcal{Y} \in \mathfrak{Y}\}$. And for any class $\mathfrak{X}$ of bitopological spaces and any formula A of $\mathcal{L}_{12}$, say that $\mathfrak{X} \vDash A$ iff $\mathcal{X} \vDash A$, for every $\mathcal{X} \in \mathfrak{X}$. Then we have

Theorem 4.1. *$A \in [\text{S4}, \text{S4}]^{\mathsf{EX}}$ iff $\mathfrak{Top} \times \mathfrak{Alex} \vDash A$.*

We leave soundness, i.e., the ($\Rightarrow$) direction of the biconditional, to the reader. Completeness, i.e., the ($\Leftarrow$) direction, follows from the completeness of $[\text{S4}, \text{S4}]^{\mathsf{EX}}$ for the space $\mathbb{Q} \times \mathbb{N}^*$, where $\mathbb{Q}$ is the rational line equipped with the standard topology and where $\mathbb{N}^*$ is the set of finite sequences of natural numbers, equipped with the following Alexandrov topology: a set $O \subseteq \mathbb{N}^*$ is open iff, for each $\overline{a}, \overline{b} \in \mathbb{N}^*$, if $\overline{a} \in O$ and $\overline{a}$ is an initial segment of $\overline{b}$, then $\overline{b} \in O$:

Theorem 4.2. *If $\mathbb{Q} \times \mathbb{N}^* \vDash A$ then $A \in [\text{S4}, \text{S4}]^{\mathsf{EX}}$.*

Proof. Some notation. For $\overline{a} \in \mathbb{N}^*$, we write $\overline{a} = \overline{a}_0, \ldots, \overline{a}_{ln(\overline{a})-1}$, where $ln(\overline{a})$ is the length of $\overline{a}$. For $\overline{a}, \overline{b} \in \mathbb{N}^*$, we write $\overline{a}\overline{b}$ for $\overline{a}$ concatenated with $\overline{b}$. And for $\overline{a} \in \mathbb{N}^*$ and $n \in \mathbb{N}$, we write $\overline{a}n$ for $\overline{a}$ concatenated with n. We use Λ for the empty sequence in $\mathbb{N}^*$. We define $S =_{\text{df}} \{\langle\overline{a}, \overline{b}\rangle : \overline{a}$ is an initial segment of $\overline{b}\}$. Note that S is a reflexive transitive relation, and that our topology on $\mathbb{N}^*$ is the Alexandrov topology induced by S. Finally, if $\langle u, v\rangle$ is any ordered pair (usually of worlds), we define $\mathit{lft}(\langle u, v\rangle) =_{\mathrm{df}} u$ and $\mathit{rt}(\langle u, v\rangle) =_{\mathrm{df}} v$.

Suppose that $A \notin [\mathrm{S4}, \mathrm{S4}]^{\mathsf{EX}}$. It is a special case of Theorem 6 in [14] that $[\mathrm{S4}, \mathrm{S4}]^{\mathsf{EX}} = (\mathrm{S4} \times \mathrm{S4})^{\mathsf{EX}}$. So there is some reflexive transitive unary frame $\mathcal{U}_1 = \langle W_1, S_1\rangle$ and some reflexive transitive unary frame $\mathcal{U}_2 = \langle W_2, S_2\rangle$ and some expanding relativized product frame, $\mathcal{B} = \langle W, R_1, R_2\rangle$, that is a subframe of $\mathcal{U}_1 \times \mathcal{U}_2 = \langle W_1 \times W_2, S_1', S_2'\rangle$, such that $\mathcal{B} \not\models A$. We can assume that both $\mathcal{U}_1$ and $\mathcal{U}_2$ are countable and rooted, with roots r_1 and r_2, respectively, with $\langle r_1, r_2\rangle \in W$. We can also assume that, for every $w_2 \in W_2$, there is a $w_1 \in W_1$ such that $\langle w_1, w_2\rangle \in W$. To see that $\mathbb{Q} \times \mathbb{N}^* \not\models A$, it suffices to specify a bi-p-morphism from $\mathbb{Q} \times \mathbb{N}^*$ to $\mathcal{B}$.

Let

$$\langle x_0, v_0\rangle, \langle x_1, v_1\rangle, \langle x_2, v_2\rangle, \ldots, \langle x_n, v_n\rangle, \ldots$$

be an enumeration of $\mathbb{Q} \times W_2$. And let f be a p-morphism from $\mathbb{Q}$ to $\mathcal{U}_1$ (see Theorem 3.3). For every $\overline{a} \in \mathbb{N}^*$, we will shortly define a function $f_{\overline{a}} : \mathbb{Q} \to W_1 \times W_2$. First, for every $x \in \mathbb{Q}$, choose an irrational $f(x)$-interval I_x such that $x \in I_x$ (see Lemma 3.4 and the proof of Theorem 3 in subsection 3.4). The important thing is that each I_x is both open and closed.

Now we define the $f_{\overline{a}}$'s, by recursion on the construction of $\overline{a}$. First define

$$f_{\Lambda}(x) \quad = \quad \langle f(x), r_2\rangle, \text{ for each } x \in \mathbb{Q}.$$

Assuming that $f_{\overline{a}}$ has been defined, define $f_{\overline{a}n}$ as follows:

$$f_{\overline{a}n}(x) \quad = \quad \begin{cases} \langle f(x), v_n\rangle, \text{ if } \langle f(x_n), v_n\rangle \in W \ \&\ \mathit{rt}(f_{\overline{a}}(x_n)) S_2 v_n \ \&\ x \in I_{x_n} \\ f_{\overline{a}}(x), \text{ otherwise.} \end{cases}$$

We now state and prove three useful claims about the $f_{\overline{a}}$'s.

Claim 1. Each $f_{\overline{a}} : \mathbb{Q} \to W \subseteq W_1 \times W_2$. *Proof.* By induction on the length of $\overline{a}$. In the base case ($\overline{a} = \Lambda$), it suffices to note that $\langle r_1, r_2\rangle \in W$, that $r_1 S_1 f(x)$ for each $x \in \mathbb{Q}$, and that $\mathcal{B}$ is an expanding relativized product frame. For the inductive step, our inductive hypothesis (IH) is that $f_{\overline{a}} : \mathbb{Q} \to W$. We want to show that $f_{\overline{a}n} : \mathbb{Q} \to W$. So choose $x \in \mathbb{Q}$ and consider $f_{\overline{a}n}(x)$. If either $\langle f(x_n), v_n\rangle \notin W$ or not $\mathit{rt}(f_{\overline{a}}(x_n)) S_2 v_n$ or $x \in I_{x_n}$, then $f_{\overline{a}n}(x) = f_{\overline{a}}(x) \in W$, as desired. So suppose that $\langle f(x_n), v_n\rangle \in W \ \&\ \mathit{rt}(f_{\overline{a}}(x_n)) S_2 v_n \ \&\ x \in I_{x_n}$. Since I_{x_n} is an $f(x_n)$-interval, we have $f(x_n) S_1 f(x)$. Thus, since $\mathcal{B}$ is an *expanding* relativized product model, we have $f_{\overline{a}n}(x) = \langle f(x), v_n\rangle \in W$, as desired.

Claim 2. Each $f_{\overline{a}} : \mathbb{Q} \to W$ is a continuous function from $\mathbb{Q}$ to the unary frame $\mathcal{U}' = \langle W, R_1 \rangle$. *Proof.* By induction on the length of $\overline{a}$. In the base case ($\overline{a} = \Lambda$), the continuity of f from $\mathbb{Q}$ to $\mathcal{U}_1$ suffices to guarantee the continuity of f_Λ from $\mathbb{Q}$ to $\mathcal{U}'$. For the inductive step, our inductive hypothesis (IH) is that $f_{\overline{a}}$ is a continuous function. We want to show that $f_{\overline{a}n}$ is a continuous function.

Suppose that O is an open subset of the unary frame $\mathcal{U}'$, i.e., O is closed under the relation R_1. We want to show that the preimage of O under $f_{\overline{a}n}$, say O', is an open subset of $\mathbb{Q}$. We can assume that O is of the form $[u] \times \{v\}$ for some $u \in W_1$ and $v \in W_2$ with $\langle u, v \rangle \in W$, since sets of this form form a basis for the topology on $\mathcal{U}'$ induced by the relation R_1. (Here, $[u] =_{\text{df}} \{u' \in W_1 : uS_1u'\}$.) Let O'' be the preimage of the set $[u] \subseteq W_1$ under the p-morphism f. And let O''' be the preimage of the set $[u] \times \{v\}$ under the function $f_{\overline{a}}$. The set $O'' \subseteq \mathbb{Q}$ is open, since f is continuous; and the set $O''' \subseteq \mathbb{Q}$ is open by (IH).

Then, given the definition of $f_{\overline{a}n}$, for every $x \in \mathbb{Q}$, we have

$$
\begin{aligned}
& x \in O' \\
\text{iff}\quad & f_{\overline{a}n}(x) \in [u] \times \{v\} \\
\text{iff}\quad & \begin{cases} \langle f(x), v_n \rangle \in [u] \times \{v\} \ \&\ \langle f(x_n), v_n \rangle \in W \ \&\ rt(f_{\overline{a}}(x_n))S_2v_n \ \&\ x \in I_{x_n} \text{ or} \\ f_{\overline{a}}(x) \in [u] \times \{v\} \ \&\ (\langle f(x_n), v_n \rangle \notin W \text{ or } \neg(rt(f_{\overline{a}}(x_n))S_2v_n) \text{ or } x \notin I_{x_n}) \end{cases} \\
\text{iff}\quad & \begin{cases} f(x) \in [u] \ \&\ v_n = v \ \&\ \langle f(x_n), v_n \rangle \in W \ \&\ rt(f_{\overline{a}}(x_n))S_2v_n \ \&\ x \in I_{x_n} \text{ or} \\ x \in O''' \ \&\ (\langle f(x_n), v_n \rangle \notin W \text{ or } \neg(rt(f_{\overline{a}}(x_n))S_2v_n) \text{ or } x \notin I_{x_n}) \end{cases} \\
\text{iff}\quad & \begin{cases} x \in O'' \ \&\ v_n = v \ \&\ \langle f(x_n), v_n \rangle \in W \ \&\ rt(f_{\overline{a}}(x_n))S_2v_n \ \&\ x \in I_{x_n} \text{ or} \\ x \in O''' \ \&\ (\langle f(x_n), v_n \rangle \notin W \text{ or } \neg(rt(f_{\overline{a}}(x_n))S_2v_n) \text{ or } x \notin I_{x_n}) \end{cases} \\
\text{iff}\quad & \begin{cases} x \in O'' \ \&\ v_n = v \ \&\ \langle f(x_n), v_n \rangle \in W \ \&\ rt(f_{\overline{a}}(x_n))S_2v_n \ \&\ x \in I_{x_n} \text{ or} \\ x \in O''' \ \&\ \langle f(x_n), v_n \rangle \notin W \text{ or } \neg(rt(f_{\overline{a}}(x_n))S_2v_n)) \text{ or} \\ x \in O''' \ \&\ x \notin I_{x_n} \end{cases} \\
\text{iff}\quad & \begin{cases} x \in O'' \cap I_{x_n} \ \&\ v_n = v \ \&\ \langle f(x_n), v_n \rangle \in W \ \&\ rt(f_{\overline{a}}(x_n))S_2v_n \text{ or} \\ x \in O''' \ \&\ \langle f(x_n), v_n \rangle \notin W \text{ or } \neg(rt(f_{\overline{a}}(x_n))S_2v_n)) \text{ or} \\ x \in O''' - I_{x_n} \end{cases}
\end{aligned}
$$

So either $O' = (O'' \cap I_{x_n}) \cup (O''' - I_{x_n})$ or $O' = O'''$ or $O' = O''' - I_{x_n}$. In any case, O' is open, as desired.

Claim 3. Each $f_{\overline{a}} : \mathbb{Q} \to W$ is an open function from $\mathbb{Q}$ to the unary frame $\mathcal{U}' = \langle W, R_1 \rangle$. *Proof.* By induction on the length of $\overline{a}$. In the base case, the openness of f from $\mathbb{Q}$ to $\mathcal{U}_1$ suffices to guarantee the openness of $f_\wedge$ from $\mathbb{Q}$ to $\mathcal{U}'$. For the inductive step, our inductive hypothesis (IH) is that $f_{\overline{a}}$ is an open function. We want to show that $f_{\overline{a}n}$ is an open function. If either $\langle f(x_n), v_n \rangle \notin W$ or $\neg(rt(f_{\overline{a}}(x_n))S_2 v_n)$, then $f_{\overline{a}n} = f_{\overline{a}}$, and we are done. So suppose that $\langle f(x_n), v_n \rangle \in W$ and $rt(f_{\overline{a}}(x_n))S_2 v_n$.

Suppose that O is an open subset of $\mathbb{Q}$. We want to show that the image of O under $f_{\overline{a}n}$, say O', is an open subset of the unary frame $\mathcal{U}' = \langle W, R_1 \rangle$. Let $O_1 = O \cap I_{x_n}$ and $O_2 = O - I_{x_n}$. Note that each O_i is open in $\mathbb{Q}$, since I_{x_n} is both open and closed in $\mathbb{Q}$; and that $O = O_1 \cup O_2$. Let O'_i be the image of O_i under $f_{\overline{a}n}$. It will suffice to show that each O'_i is open since $O' = O'_1 \cup O'_2$. *Re* O'_1. Let $O^*_1 \subseteq W_1$ be the image of O_1 under f. O^*_1 is open since f is an open function. Note that $f_{\overline{a}n}(x) = \langle f(x), v_n \rangle \in W$ for every $x \in O_1$. So $O'_1 = O^*_1 \times \{v_n\}$, which is open in $\mathcal{U}'$. *Re* O'_2. Note that $f_{\overline{a}n}(x) = f_{\overline{a}}(x)$ for every $x \in O_2$. So O'_2 is open by IH.

Summary. Each $f_{\overline{a}} : \mathbb{Q} \to W$ is a continuous open function from $\mathbb{Q}$ to the unary frame $\mathcal{U}' = \langle W, R_1 \rangle$.

We add one more claim, which has an easy inductive proof:

Claim 4. For each $f_{\overline{a}} : \mathbb{Q} \to W$ and each $x \in \mathbb{Q}$, we have $lft(f_{\overline{a}}(x)) = f(x)$.

Now define

$$F(x, \overline{a}) = f_{\overline{a}}(x).$$

All we have left is to prove that F is a bi-p-morphism from $\mathbb{Q} \times \mathbb{N}^*$ to $\mathcal{B}$, i.e., that F is bicontinuous, biopen and onto. First note that F is 1-continuous and 1-open, since each $f_{\overline{a}}$ is both continuous and open from $\mathbb{Q}$ to $\mathcal{U}_1$. We now prove that F is onto, 2-continuous, and 2-open.

F is onto. Suppose that $\langle u, v \rangle \in W$. Choose any $x \in \mathbb{Q}$ such that $f(x) = u$. And choose n so that $x_n = x$ and $v_n = v$. We will identify n with the singleton sequence with one member, n. Note that $\langle f(x_n), v_n \rangle \in W$; and $rt(f_\wedge(x_n))S_2 v_n$, since $rt(f_\wedge(x_n)) = r_2$; and $x_n \in I_{x_n}$. So $F(x, n) = f_n(x) =$

$\langle f(x), v_n\rangle = \langle f(x_n), v_n\rangle = \langle u, v\rangle$, as desired.

F **is 2-continuous.** For this it will suffice to show that, for every $x \in \mathbb{Q}$ and every $\overline{a}, \overline{b} \in \mathbb{N}^*$, if $\overline{a} S \overline{b}$, then

$$F(\langle x, \overline{a}\rangle) \;\; R_2 \;\; F(\langle x, \overline{b}\rangle).$$

For this, in turn, it will suffice to show that, for every $x \in \mathbb{Q}$ and every $\overline{a} \in \mathbb{N}^*$ and $n \in \mathbb{N}$,

$$f_{\overline{a}}(x) \;\; R_2 \;\; f_{\overline{a}n}(x).$$

This comes to the same thing as

$$f_{\overline{a}}(x) \;\; S_2' \;\; f_{\overline{a}n}(x),$$

since $\mathcal{B}$ is a subframe of $\mathcal{U}_1 \times \mathcal{U}_2$. So choose $x \in \mathbb{Q}$, $\overline{a} \in \mathbb{N}^*$ and $n \in \mathbb{N}$. If either $\langle f(x_n), v_n\rangle \notin W$ or $\neg(rt(f_{\overline{a}}(x_n))S_2 v_n)$ or $x \notin I_{x_n}$, then $f_{\overline{a}n}(x) = f_{\overline{a}}(x)$, and we are done. So assume that $\langle f(x_n), v_n\rangle \in W$ & $rt(f_{\overline{a}}(x_n))S_2 v_n$ & $x \in I_{x_n}$. Then $f_{\overline{a}n}(x) = \langle f(x), v_n\rangle$. So $f_{\overline{a}}(x) S_2' f_{\overline{a}n}(x)$, given that $lft(f_{\overline{a}}(x)) = f(x)$ (by Claim 4) and $rt(f_{\overline{a}}(x_n))S_2 v_n$.

F **is 2-open.** For this it will suffice to show that, for every $x \in \mathbb{Q}$, every $\overline{a} \in \mathbb{N}^*$ and every $\langle u, v\rangle \in W$, if $F(\langle x, \overline{a}\rangle)R_2\langle u, v\rangle$ then for some $\overline{b} \in \mathbb{N}^*$, we have $\overline{a} S \overline{b}$ (i.e., $\overline{a}$ is an initial segment of $\overline{b}$) and

$$F(\langle x, \overline{b}\rangle) = \langle u, v\rangle.$$

So suppose that $F(\langle x, \overline{a}\rangle)R_2\langle u, v\rangle$. So $F(\langle x, \overline{a}\rangle)S_2'\langle u, v\rangle$, since $\mathcal{B}$ is a subspace of $\mathcal{U}_1 \times \mathcal{U}_2$. So $f_{\overline{a}}(x)S_2'\langle u, v\rangle$. So $f(x) = lft(f_{\overline{a}}(x)) = u$ and $rt(f_{\overline{a}}(x))S_2 v$. Choose $n \in \mathbb{N}$ so that $x_n = x$ and $v_n = v$, and let $\overline{b} = \overline{a}n$. Clearly, $\overline{a} S \overline{b}$. Also note that $F(\langle x, \overline{b}\rangle) = f_{\overline{a}n}(x) = \langle f(x), v_n\rangle$, by the definition of $f_{\overline{a}n}(x)$, $= \langle u, v\rangle$. ■

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