Chapter 10
Strong Completeness of S4 for the Real Line

Philip Kremer

Second Reader

R. Goldblatt
Victoria University of Wellington

Abstract In the topological semantics for modal logic, S4 is well-known to be complete for the rational line and for the real line: these are special cases of S4's completeness for any dense-in-itself metric space. The construction used to prove completeness can be slightly amended to show that S4 is not only complete, but strongly complete, for the rational line. But no similarly easy amendment is available for the real line. In an earlier paper, we proved a general theorem: S4 is strongly complete for any dense-in-itself metric space. Strong completeness for the real line is a special case. In the current paper, we give a proof of strong completeness tailored to the special case of the real line: the current proof is simpler and more accessible than the proof of the more general result, and involves slightly different techniques. We proceed in two steps: first, we show that S4 is strongly complete for the space of finite and infinite binary sequences, equipped with a natural topology; and then we show that there is an interior map from the real line onto this space.

Key words: Modal logic, topological semantics, strong completeness, real line

10.1 Introduction

It is my honour to contribute to this festschrift for Alasdair Urquhart. The first logic course I ever took was a course on relevance logic with Alasdair in 1985: as a result, I

Philip Kremer
Department of Philosophy, University of Toronto, e-mail: philip.kremer@utoronto.ca

saw a completeness proof for the relevance logic R before I ever saw one for classical logic. Ever since, I've maintained an interest in alternatives to, and extensions of, classical logic—especially in completeness results.

In the topological semantics for modal logic ([9, 10, 12]), S4 is well-known to be complete for the class of all topological spaces, as well as for a number of particular topological spaces, notably the rational line, $\mathbb{Q}$, and the real line, $\mathbb{R}$. The results for $\mathbb{Q}$ and $\mathbb{R}$ are special cases of the fact that S4 is complete for any dense-in-itself metric space: see [12], Theorem XI, 9.1, which is derived from [9, 10]. It is customary to strengthen completeness to *strong* completeness, i.e., the claim that any consistent set of formulas is satisfiable at some point in the space in question. As long as the language is countable, the construction used to prove completeness can be slightly amended to show that S4 is not only complete, but strongly complete, for $\mathbb{Q}$ (see [6]). But no similarly easy amendment is available for $\mathbb{R}$: until [6], the questions of strong completeness for $\mathbb{R}$ was open.

In [6], we prove that S4 is strongly complete for *any* dense-in-itself metric space—and therefore for $\mathbb{R}$. In the current paper, we give a proof of strong completeness tailored to the special case of $\mathbb{R}$. This proof is useful for at least two reasons. First, since the proof in the current paper is tailored to a special case, it is simpler and more accessible than the proof in [6], avoiding many of the bells and whistles needed there for the more general claim. In particular, we can bypass all mention of ultrafilters and of algebraic semantics. We believe that it usefully clarifies matters to work through a simplified proof of a special case, before considering a more general case. Second, in proving Lemma 10.5, below, we use a different technique than the proof of the same lemma, Lemma 6.1, in [6]. The proof technique used here might be generalized or adapted in ways that the proof technique in [6] cannot, either for logics stronger than S4 or for logics that extend S4 with additional vocabulary such as the universal modality $\forall$ or the difference modality $\neq$: [5], for example, leaves a number of open questions in this area for which the techniques here might be useful.

Completeness for any given dense-in-itself metric space X is typically proved by showing that any finite rooted reflexive transitive Kripke frame is the image of an interior map from X. When $X = \mathbb{Q}$, strengthening completeness to *strong* completeness is accomplished by slightly amending the construction to show that any *countable* rooted reflexive transitive Kripke frame is the image of an interior map from $\mathbb{Q}$. But this strategy is not generalizable: because of the Baire Category Theorem, the countable rooted reflexive transitive Kripke frame $\langle \mathbb{N} \leq \rangle$, for example, is *not* the image of any interior map from $\mathbb{R}$ (I owe this observation to Guram Bezhanishvili, David Gabelaia, and Valentin Shehtman): see [6], Section 3, for details.

To show that S4 is strongly complete for $\mathbb{R}$, we proceed in two steps. First we show that S4 is strongly complete for the space $2^{\leq\omega}$ of finite and infinite binary sequences, equipped with a natural topology: see Section 10.3. We call $2^{\leq\omega}$ the *infinite binary tree with limits* and Lando [8] calls it the *complete binary tree*. Then we show that there is an interior map from $\mathbb{R}$ onto $2^{\leq\omega}$: see Section 10.4. Thus S4 is strongly complete for $\mathbb{R}$. In fact, we proceed by showing that there's an interior map from the open unit interval, $\mathcal{I} = (0, 1)$ onto $2^{\leq\omega}$: this suffices since there are many interior maps from $\mathbb{R}$ onto $\mathcal{I}$. We note that Lando [8] already constructs an interior map from the closed unit interval $[0, 1]$ onto $2^{\leq\omega}$: see [8], Section 5.4. Our construction is quite similar, but is simpler because Lando's project requires her to track not only topological properties of the map but also measure-theoretic properties.

Note: while this chapter was in production, Robert Goldblatt contacted me about the infinite binary tree with limits. While browsing casually through [3] (published in 1990), he had chanced upon both a definition of this tree and a construction of an interior map from the closed interval $[-1/2, 1/2]$ onto this tree: it follows that there is an interior map from $\mathbb{R}$ onto it, see [3, 1.749 and 1.74(10)].

10.2 Basics

We begin by fixing notation and terminology. We assume a propositional language with a countable set *PV* of propositional variables; standard Boolean connectives $\wedge$, $\vee$ and $\neg$; and one modal operator, $\Box$. A finite set of formulas is *consistent* iff either it is empty or the negation of the conjunction of the formulas in it is not a theorem of S4; and an infinite set of formulas is *consistent* iff every finite subset is consistent.

A *Kripke frame* is an ordered pair $\langle X, R\rangle$, where X is a nonempty set and $R \subseteq X \times X$. We somewhat imprecisely identify X with $\langle X, R\rangle$, letting context or fiat determine R. A Kripke frame X is reflexive [transitive] iff R is: for the rest of this paper, we assume that all Kripke frames are reflexive and transitive. A Kripke frame is *rooted* iff $(\exists r \in W)(\forall w \in W)(rRw)$. A subset O of X is *open* iff $(\forall x, y \in X)(x \in O \,\&\, xRy \Rightarrow y \in O)$. A subset C of X is *closed* iff $X - C$ is open. The *interior* of a set $S \subseteq X$ is the largest open subset of S: $Int(S) =_{\text{df}} \{x \in S : \forall y \in X, xRy \Rightarrow y \in S\}$. The *closure* of a set $S \subseteq X$ is the smallest closed superset of S: $Cl(S) =_{\text{df}} X - Int(X - S)$. A *topological space* is an ordered pair $\langle X, \tau\rangle$, where X is a nonempty set and $\tau \subseteq \wp(X)$ is a topology on X. We somewhat imprecisely identify X with $\langle X, \tau\rangle$, letting context or fiat determine τ. Thus, for example, we identify $\mathbb{R}$ with $\langle \mathbb{R}, \tau_{\mathbb{R}}\rangle$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. We assume the basics of point-set topology, in particular the notion of the interior and closure, $Int(S)$ and $Cl(S)$, of a subset S of a topological space.

A *Kripke model* [*topological model*] is an ordered pair $M = \langle X, V\rangle$, where X is a Kripke frame [topological space] and $V : PV \longrightarrow \wp(X)$. We use the term *model* to cover Kripke models and topological models. For any model $M = \langle X, V\rangle$, V is extended to all formulas as follows: $V(\neg A) = X - V(A)$; $V(A \wedge B) = V(A) \cap V(B)$; $V(A \vee B) = V(A) \cup V(B)$; and $V(\Box A) = Int(V(A))$. If Γ is a nonempty set of formulas, then $V(\Gamma) =_{\text{df}} \bigcap_{A\in\Gamma} V(A)$; if Γ is empty, then $V(\Gamma) =_{\text{df}} X$.

Suppose that Γ is a set of formulas. If X is a Kripke frame or topological space and $x \in X$, then we say that Γ is *satisfiable at x in X* iff there is some model $M = \langle X, V\rangle$ such that $x \in V(\Gamma)$; and we say that Γ is *satisfiable in X* iff Γ is satisfiable at some x in X. We say that S4 is *complete* for X iff every finite consistent set of formulas is satisfiable in X, and *strongly complete* for X iff every consistent set of formulas is satisfiable in X.

The following completeness theorem follows from [12], Theorem XI, 9.1, (vii), which itself derived from [9, 10]:

Theorem 10.1. S4 *is complete for* $\mathbb{R}$.

Theorem 10.1 is well-known: there are new and more accessible proofs in [1, 2, 11]. The current paper's main result is a special case of the main theorem, Theorem 1.2, in [6]:

Theorem 10.2. S4 *is* strongly *complete for* $\mathbb{R}$.

Before we prove Theorem 10.2, we recall the standard notion of an *interior map*. A function from a topological space or Kripke frame to a topological space or Kripke frame is *continuous* iff the preimage of every open set is open; is *open* iff the image of every open set is open; and is an *interior map* iff it is continuous and open. Suppose that $M = \langle X, V\rangle$ and $M' = \langle X', V'\rangle$ are models, and that f is a surjective interior map from X onto X'. Then f is an *interior map from M onto M'* iff, for every $p \in PV$ and $x \in X$, $x \in V(p)$ iff $f(x) \in V'(p)$. The following lemma and corollary are standard:

Lemma 10.1. *If f is an interior map from $M = \langle X, V\rangle$ onto $M' = \langle X', V'\rangle$, then for every formula B and $x \in X$, $x \in V(B)$ iff $f(x) \in V'(B)$.*

Corollary 10.1. *Suppose that each of X and X' is a Kripke frame or topological space, and that there is an interior map from X onto X'. Then if Γ is satisfiable in X' then Γ is satisfiable in X.*

Given Corollary 10.1, we can divide the work of proving Theorem 10.2 into two parts. The first part is mainly logical: we show that S4 is strongly complete for the space $2^{\leq\omega}$ of finite and infinite binary sequences, equipped with a natural topology (Lemma 10.5). The second part is purely topological: we show that there's an interior map from $\mathbb{R}$ onto $2^{\leq\omega}$.

10.3 The space $2^{\leq\omega}$

For each $n \geq 0$, let 2^n be the set of binary sequences (sequences of 0's and 1's) of length n. Let $2^{<\omega} =_{\text{df}} \bigcup_{n=0}^{\infty} 2^n$, i.e., $2^{<\omega}$ is the set of finite binary sequences. We write *length*(b) for the length of $b \in 2^{<\omega}$. Let 2^ω be the set of infinite binary sequences of order type ω. And let $2^{\leq\omega} =_{\text{df}} 2^{<\omega} \cup 2^\omega$. We use Λ for the empty binary sequence, i.e., the binary sequence of length 0. We use b, b', etc., to range over $2^{<\omega}$; $\mathbf{b}, \mathbf{b}'$, etc., to range over 2^ω; and $\boldsymbol{b}, \boldsymbol{b}'$, etc., to range over $2^{\leq\omega}$. If $b \in 2^{<\omega}$ and $\boldsymbol{b} \in 2^{\leq\omega}$, then we write $b^\frown\boldsymbol{b}$ for b concatenated with $\boldsymbol{b}$. We write $b0$ and $b1$ for $b^\frown\langle 0\rangle$ and $b^\frown\langle 1\rangle$. Given any $\mathbf{b} \in 2^\omega$ and any $n \in \mathbb{N}$, the finite binary sequence $\mathbf{b}|_n$ is the initial segment of length n of $\mathbf{b}$. Thus $\mathbf{b}|_0 = \Lambda$. Given $b \in 2^{<\omega}$ and $\boldsymbol{b} \in 2^{\leq\omega}$, we say $b \leq \boldsymbol{b}$ iff b is an initial segment of $\boldsymbol{b}$ and $b < \boldsymbol{b}$ iff both $b \leq \boldsymbol{b}$ and $b \neq \boldsymbol{b}$. We also use '$\leq$' for $\leq$ restricted to $2^{<\omega}$.

We identify $2^{<\omega}$ with the *infinite binary tree*, i.e., the countably infinite rooted transitive reflexive Kripke frame $\langle 2^{<\omega}, \leq \rangle$. We can think of an infinite binary sequence $\mathbf{b} \in 2^{\omega}$ as the *limit* of the branch of finite sequences $\mathbf{b}|_0$, $\mathbf{b}|_1$, $\mathbf{b}|_2$, ... Accordingly, we think of $2^{\leq\omega}$ as the infinite binary tree *with limits*.

For any $b \in 2^{<\omega}$, it will be useful to define two related sets: $\leq(b) =_{\text{df}} \{b' \in 2^{<\omega} : b \leq b'\}$ and $\leq^*(b) =_{\text{df}} \{\boldsymbol{b}' \in 2^{\leq\omega} : b \leq \boldsymbol{b}'\}$. We impose a natural topology on $2^{\leq\omega}$, by taking as a basis all the sets of the form $\leq^*(b)$, where $b \in 2^{<\omega}$. (Nick Bezhanishvili pointed out to me that this is the Scott topology on $2^{\leq\omega}$: See [15], p. 95, for a definition of the Scott topology on any partially ordered set.) The main task of the current section is to prove that S4 is strongly complete for $2^{\leq\omega}$ – see Lemma 10.5.

The following result, due originally to Dov Gabbay and independently discovered by Johan van Benthem, is well-known; for a proof see [4], Theorem 1:

Lemma 10.2. *Any finite rooted reflexive transitive Kripke frame is the image of* $2^{<\omega}$ *under some interior map.*

Together with the fact that any finite consistent set Γ of formulas is satisfiable in some finite rooted reflexive transitive Kripke frame, Lemma 10.2 entails that S4 is complete for $2^{<\omega}$. Lemma 10.2 can be strengthened to

Lemma 10.3. *Any countable rooted reflexive transitive Kripke frame is the image of* $2^{<\omega}$ *under some interior map.*

Proof. This is Lemma 3.3 in [6]. Unfortunately, the proof there is incorrect. Here, we reproduce, almost verbatim, the corrected proof in [7]: see p. 451, proof of (ii.b). Suppose that $\langle W, R \rangle$ is a countable Kripke frame with root r. We will, in effect, unravel $\langle W, R \rangle$ into $2^{<\omega}$. For each $w \in W$, let $R(w) = \{w' \in W : wRw'\}$ and let $succ_0(w), succ_1(w), succ_2(w), \ldots$ be an enumeration of $R(w)$ in which every member of $R(w)$ occurs infinitely often. We also need a function $zero : 2^{<\omega} \to \mathbb{N}$, defined as follows: $zero(\Lambda) = 0$; $zero(b0) = zero(b) + 1$; and $zero(b1) = 0$. Note that $zero(b)$ is simply the number of uninterrupted occurrences of 0 at the end of b: e.g., $zero(001101000) = 3$, $zero(100001) = 0$, and $zero(000100) = 2$. Now we define the surjective interior map $\varphi : 2^{<\omega} \to W$ recursively as follows: $\varphi(\Lambda) = r$; $\varphi(b0) = \varphi(b)$; and $\varphi(b1) = succ_{zero(b)}(\varphi(b))$.

We have to check that φ is a surjective interior map. **Surjectivity.** To see that φ is surjective, suppose that $w \in W$. Then $w = succ_n(r)$ for some $n \in \mathbb{N}$. Let 0^n be the sequence of n 0's. And note that $\varphi(0^n 1) = succ_{zero(0^n)}(r) = succ_n(r) = w$. **Continuity.** It suffices to show that the preimage of $R(w)$ is open for every $w \in W$: note that the preimage of $R(w)$ is $\bigcup_{\varphi(b)=w} \leq(b)$. **Openness.** It suffices to show that the image of $\leq(b)$ is open for every $b \in 2^{<\omega}$: note that the image of $\leq(b)$ is $R(\varphi(b))$.

Together with the fact that any consistent set Γ of formulas is satisfiable in some countable rooted reflexive transitive Kripke frame, Lemma 10.3 entails

Lemma 10.4. S4 *is strongly complete for* $2^{<\omega}$.

The remainder of this section uses Lemma 10.4 to prove

Lemma 10.5. S4 *is strongly complete for* $2^{\leq\omega}$.

Proof. This is Lemma 6.1 in [6]. Here we give quite a different proof that bypasses the dependence, useful in [6] but unnecessary here, on ultrafilters and on algebraic semantics. Let Γ be a consistent set of formulas. Given Lemma 10.4, Γ is satisfiable in $2^{<\omega}$. So there is a Kripke model $M = \langle 2^{<\omega}, V\rangle$ such that $V(\Gamma) \neq \emptyset$. We will define a $V^* : PV \to 2^{\leq\omega}$ and show that, in the topological model $M^* = \langle 2^{\leq\omega}, V^*\rangle$, we have $V^*(\Gamma) \neq \emptyset$.

First, we assign sets $\Delta_{\boldsymbol{b}}$ and $\Sigma_{\boldsymbol{b}}$ of formulas to each $\boldsymbol{b} \in 2^{\leq\omega}$. If $\boldsymbol{b} \in 2^{<\omega}$ then $\Delta_{\boldsymbol{b}} = \Sigma_{\boldsymbol{b}} =_{\text{df}} \{A : \boldsymbol{b} \in V(A)\}$. Note, if $\boldsymbol{b} \in 2^{<\omega}$, then $\Sigma_{\boldsymbol{b}}$ is consistent; $\Sigma_{\boldsymbol{b}}$ is also *complete* in the following sense: for every formula A, either $A \in \Sigma_{\boldsymbol{b}}$ or $\neg A \in \Sigma_{\boldsymbol{b}}$. If $\boldsymbol{b} \in 2^{\omega}$, then let $\Delta_{\boldsymbol{b}} =_{\text{df}} \bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} \Sigma_{\boldsymbol{b}|_m} = \{A : (\exists n \in \mathbb{N})(\forall m \geq n)(\boldsymbol{b}|_m \in V(A))\}$. Note that $\Delta_{\boldsymbol{b}}$ is consistent, so that we can let $\Sigma_{\boldsymbol{b}}$ be any complete consistent superset of $\Delta_{\boldsymbol{b}}$.

For $p \in PV$, define $V^*(p) = \{\boldsymbol{b} \in 2^{\leq\omega} : p \in \Sigma_{\boldsymbol{b}}\}$. Now we show that, for every formula A,

$$\text{for every } \boldsymbol{b} \in 2^{\leq\omega}, \boldsymbol{b} \in V^*(A) \text{ iff } A \in \Sigma_{\boldsymbol{b}}. \tag{10.1}$$

The proof is by induction on the construction of A. If $A \in PV$ then (10.1) follows from the definition of $V^*(A)$; and if A is of the form $\neg B$, $(B \wedge C)$ or $(B \vee C)$, then (10.1) follows from the fact that each $\Sigma_{\boldsymbol{b}}$ is consistent and complete. So suppose that A is of the form $\Box B$ and make the inductive hypothesis that

$$\text{for every } \boldsymbol{b} \in 2^{\leq\omega}, \boldsymbol{b} \in V^*(B) \text{ iff } B \in \Sigma_{\boldsymbol{b}}. \tag{10.2}$$

We want to show,

$$\text{for every } \boldsymbol{b} \in 2^{\leq\omega}, \boldsymbol{b} \in V^*(\Box B) \text{ iff } \Box B \in \Sigma_{\boldsymbol{b}}. \tag{10.3}$$

Proof of ($\Rightarrow$). Choose $\boldsymbol{b} \in 2^{\leq\omega}$ and assume that $\boldsymbol{b} \in V^*(\Box B)$. So there is some $b' \in 2^{<\omega}$ such that $\boldsymbol{b} \in {\leq^*}(b') \subseteq V^*(B)$. So, for every $b'' \in {\leq}(b') = 2^{<\omega} \cap {\leq^*}(b')$, we have $B \in \Sigma_{b''}$, by (10.2). So ${\leq}(b') \subseteq V(B)$, by the definition of the Σ_b. So ${\leq}(b') \subseteq V(\Box B)$, by the definition of $V(\Box B)$. If $\boldsymbol{b} \in 2^{<\omega}$, then $\boldsymbol{b} \in {\leq}(b') \subseteq V(\Box B)$, so that $\Box B \in \Sigma_{\boldsymbol{b}}$, by the definition of $\Sigma_{\boldsymbol{b}}$. On the other hand, suppose that $\boldsymbol{b} \notin 2^{<\omega}$. So $\boldsymbol{b} \in 2^{\omega}$. Since $b' \leq \boldsymbol{b}$, we have $b' = \boldsymbol{b}|_n$, for some $n \in \mathbb{N}$. So $\boldsymbol{b}|_n \in {\leq}(b') \subseteq V(\Box B)$. So $\boldsymbol{b}|_m \in V(\Box B)$, for every $m \geq n$. So $\Box B \in \Sigma_{\boldsymbol{b}|_m}$, for every $m \geq n$. So $\Box B \in \Delta_{\boldsymbol{b}}$. So $\Box B \in \Sigma_{\boldsymbol{b}}$, as desired.

Proof of ($\Leftarrow$). Choose $\boldsymbol{b} \in 2^{\leq\omega}$ and assume that $\Box B \in \Sigma_{\boldsymbol{b}}$. We consider two cases: (i) $\boldsymbol{b} \in 2^{<\omega}$, and (ii) $\boldsymbol{b} \in 2^{\omega}$. In Case (i) $\boldsymbol{b} \in V(\Box B)$, by the definition of $\Sigma_{\boldsymbol{b}}$. So, for every $b' \in {\leq}(\boldsymbol{b})$, we have $b' \in V(B)$. So, for every $b' \in {\leq}(\boldsymbol{b})$, we have $B \in \Sigma_{b'}$. So, for every $\boldsymbol{b}' \in {\leq^*}(\boldsymbol{b})$, we have $B \in \Delta_{\boldsymbol{b}'}$. So, for every $\boldsymbol{b}' \in {\leq^*}(\boldsymbol{b})$, we have $B \in \Sigma_{\boldsymbol{b}'}$. So, by (10.2), for every $\boldsymbol{b}' \in {\leq^*}(\boldsymbol{b})$, we have $\boldsymbol{b}' \in V^*(B)$. So $\boldsymbol{b} \in V^*(\Box B)$, as desired.

In Case (ii), $\neg\Box B \notin \Delta_{\boldsymbol{b}}$. So there is some $m \in \mathbb{N}$ such that $\Box B \in \Sigma_{\boldsymbol{b}|_m}$. So $\boldsymbol{b}|_m \in V(\Box B)$. So, for every $b' \in 2^{<\omega}$, if $\boldsymbol{b}|_m \leq b'$ then $b' \in V(B)$. So, for every $b' \in 2^{<\omega}$, if $\boldsymbol{b}|_m \leq b'$ then $B \in \Sigma_{b'}$. But then, by the definition of $\Delta_{\mathbf{b}''}$ for $\mathbf{b}'' \in 2^{\omega}$, we have for every $\mathbf{b}'' \in 2^{\omega}$, if $\boldsymbol{b}|_m \leq \mathbf{b}''$ then $B \in \Delta_{\mathbf{b}''} \subseteq \Sigma_{\mathbf{b}''}$. So, for every $\boldsymbol{b}^* \in 2^{\leq\omega}$, if $\boldsymbol{b}|_m \leq \boldsymbol{b}^*$ then $B \in \Sigma_{\boldsymbol{b}^*}$. So, by (10.2), for every $\boldsymbol{b}^* \in \leq^*(\boldsymbol{b}|_m)$, $\boldsymbol{b}^* \in V^*(B)$. So $\boldsymbol{b} \in \leq^*(\boldsymbol{b}|_m) \subseteq V^*(\Box B)$, as desired.

Given (10.1), to see that Γ is satisfiable in $2^{\leq\omega}$, simply choose $b \in 2^{<\omega}$ with $b \in V(\Gamma)$. Note: $\Gamma \subseteq \Sigma_b$, so that $b \in V^*(\Gamma)$, by (10.1).

10.4 An interior map from $\mathcal{I} = (0, 1)$ onto $2^{\leq\omega}$

Our remaining work is purely topological: we want to prove

Lemma 10.6. *There is an interior map from* $\mathbb{R}$ *onto* $2^{\leq\omega}$.

Let $\mathcal{I} = (0, 1)$ be the open unit interval. As noted in the introductory remarks, it suffices to prove

Lemma 10.7. *There is an interior map from* $\mathcal{I}$ *onto* $2^{\leq\omega}$.

As noted above, Lando [8] already constructs an interior map from the closed unit interval $[0, 1]$ onto $2^{\leq\omega}$. The following construction is similar but simpler, because Lando's project requires her to track not only topological properties of the map but also measure-theoretic properties. In particular, for measure-theoretic reasons, Lando uses 'thick' Cantor sets, where we only use Cantor sets. (These constructions were discovered independently, around the same time, around 2011.)

We prove Lemma 10.7 by partitioning $\mathcal{I}$ into nonempty pairwise disjoint sets $X_{\boldsymbol{b}}$, one for each $\boldsymbol{b} \in 2^{\leq\omega}$. We then define $\mathbf{F} : \mathcal{I} \to 2^{\leq\omega}$ as follows: $\mathbf{F}(x) =$ the unique $\boldsymbol{b} \in 2^{\leq\omega}$ such that $x \in X_{\boldsymbol{b}}$. The trick is to do this in such a way that $\mathbf{F}$ is a surjective interior map.

First, some preliminaries. For subsets of $\mathcal{I}$, we interpret interior, Int, and closure, Cl, as relativized to $\mathcal{I}$. Let C be the Cantor set without the endpoints 0 and 1. So C is the set of all real numbers that have a ternary expansion of the form $0.a_1a_2a_3 \ldots a_k \ldots$ where each a_k is either 0 or 2, and where not all the a_k's are 0 (so that $0 \notin C$) and not all the a_k's are 2 (so that $1 \notin C$): we will find it useful to represent real numbers as ternary expansions. Figure 10.1 pictorially represents C, which is closed (in the space $\mathcal{I}$). C can be got from progressively deleting open intervals from

Fig. 10.1 The Cantor set, C, without the endpoints 0 and 1.

$\mathcal{I} = (0, 1)$ as follows: delete the open interval $(0.1, 0.2)$, which is the middle third of

$\mathcal{I}$, leaving $(0, 0.1] \cup [0.2, 1)$. Then delete the middle thirds of each of these: delete the open interval $(0.01, 0.02)$ from $(0, 0.1]$ and delete the open interval $(0.21, 0.22)$ from $[0.2, 1)$: this leaves $(0, 0.01] \cup [0.02, 0.1] \cup [0.2, 0.21] \cup [0.22, 1)$. More precisely, a *middle third* is any open interval of the form $(0.a_1a_2 \ldots a_n1, 0.a_1a_2 \ldots a_n2)$, where $n \geq 0$ and where $a_k = 0$ or 2 for all $k \leq n$. It is well-known that if we take what's left undeleted after we carry out this process of deleting middle thirds *ad infinitum*, then we get $C = \mathcal{I} - \bigcup\{J : J \text{ is a middle third}\}$.

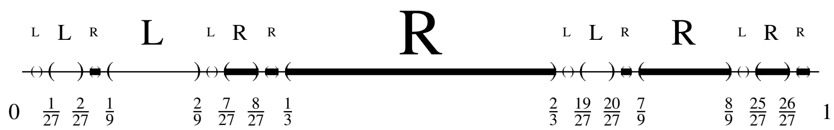

Fig. 10.2 Labelling deleted middle thirds with L and R. The labels appear above the labelled middle thirds: for clarity, we have written the labels of larger middle thirds in larger fonts. The set $\mathcal{R}$ is represented by thicker lines.

Label the deleted middle thirds with L and R, for *left* and *right*, as in Figure 10.2. And let $\mathcal{L}$ be the union of the middle thirds labeled L, and $\mathcal{R}$ be the union of the middle thirds labeled R.

Now suppose that $J \subseteq \mathcal{I}$ is an open interval. Let $f_J : \mathcal{I} \to J$ be the unique increasing linear function from $\mathcal{I}$ onto J. We define $\mathcal{L}(J)$, $\mathcal{R}(J)$, and $C(J)$ as the images under f_J of $\mathcal{L}$, $\mathcal{R}$, and C respectively. Thus $\mathcal{L}(J)$, $\mathcal{R}(J)$, and $C(J)$ are copies of $\mathcal{L}$, $\mathcal{R}$, and C, respectively. Finally, suppose that $O \subseteq \mathcal{I}$ is open. We say that an open interval $J \subseteq O$ is a *maximal open interval in* O iff, for any open interval $J' \subseteq O$, if $J \cap J' \neq \emptyset$ then $J' \subseteq J$. Note that O is the disjoint union of the maximal open intervals in O. We define

$$\mathcal{L}(O) = \bigcup_{J \text{ is a maximal open interval in } O} \mathcal{L}(J), \tag{10.4}$$

and similarly for $\mathcal{R}(O)$ and $C(O)$. So $\mathcal{L}(O)$ is the union of copies of $\mathcal{L}$, and similarly for $\mathcal{R}(O)$ and $C(O)$. Note the following:

Lemma 10.8. *1. $\mathcal{L}(O)$, $\mathcal{R}(O)$, and $C(O)$ are pairwise disjoint;*
2. $\mathcal{L}(O)$ and $\mathcal{R}(O)$ are open;
3. $O = \mathcal{L}(O) \dot\cup \mathcal{R}(O) \dot\cup C(O)$; and
4. $Cl(\mathcal{L}(O)) - \mathcal{L}(O)$
$= Cl(\mathcal{R}(O)) - \mathcal{R}(O)$
$= Cl(C(O))$
$= Cl(O) - (\mathcal{L}(O) \cup \mathcal{R}(O))$.
5. If J is a maximal open interval in O, then $J \cap C(O)$ is nonempty.

If $S \subseteq \mathcal{I}$ and there is some open interval $J \subseteq S$, then we define the *width* of S as follows: $width(S) = sup\{length(J) : J$ is an open interval and $J \subseteq S\}$. Note the following:

Lemma 10.9. *If O is an open subset of $\mathcal{I}$, then $width(\mathcal{R}(O)) = width(O)/3$ and $width(\mathcal{L}(O)) = width(O)/9$.*

Our next task is to define nonempty open $O_b \subseteq \mathcal{I}$ and other nonempty sets $X_b \subseteq \mathcal{I}$ for each $b \in 2^{<\omega}$, and also to define nonempty sets $X_{\mathbf{b}} \subseteq \mathcal{I}$ for each $\mathbf{b} \in 2^{\omega}$. Once this has been done, we will have a partition of $\mathcal{I}$ into sets $X_{\boldsymbol{b}}$ for each $\boldsymbol{b} \in 2^{\leq\omega}$. We will define $\mathbf{F} : 2^{\leq\omega} \to \mathcal{I}$ as follows: $\mathbf{F}(\boldsymbol{b})$ = the unique $x \in \mathbb{R}$ such that $x \in X_{\boldsymbol{b}}$. And we will show that $\mathbf{F}$ is a surjective interior map.

Define the O_b, for $b \in 2^{<\omega}$, recursively as follows:

$$O_\Lambda =_{\mathrm{df}} \mathcal{I} \tag{10.5}$$

$$O_{b0} =_{\mathrm{df}} \mathcal{L}(O_b) \tag{10.6}$$

$$O_{b1} =_{\mathrm{df}} \mathcal{R}(O_b) \tag{10.7}$$

For $b \in 2^{<\omega}$, we define $X_b =_{\mathrm{df}} C(O_b)$: If $b = \Lambda$, then X_b is simply C, the Cantor set without endpoints; and if b is some other finite binary sequence, then X_b is a union of infinitely many copies of C. Note that each of $\mathcal{L}(O_{\boldsymbol{b}})$ and $\mathcal{R}(O_{\boldsymbol{b}})$ is open in $\mathcal{I}$; that each of $\mathcal{L}(O_{\boldsymbol{b}})$, $\mathcal{R}(O_{\boldsymbol{b}})$, and $C(O_b)$ is nonempty; and that, $(\forall b \in 2^{<\omega})(O_b = O_{b0} \dot\cup O_{b1} \dot\cup X_b)$. Note the following facts about the O_b and the X_b:

Lemma 10.10. *1. X_b and O_b are nonempty, for each $b \in 2^{<\omega}$.*
2. O_b is open, for each $b \in 2^{<\omega}$.
3. If $b \leq b'$ then $X_{b'} \subseteq O_{b'} \subseteq O_b$.
4. If $b < b'$ then $X_b \cap X_{b'} = X_b \cap O_{b'} = \emptyset$.
5. If $b' \not\leq b \not\leq b'$ then $O_b \cap O_{b'} = \emptyset$.
6. If $b \not\leq b'$ then $O_b \cap X_{b'} = \emptyset$.
7. If $b \neq b'$ then $X_b \cap X_{b'} = \emptyset$.
8. $width(O_b) \leq 1/3^{length(b)}$.

Lemma 10.11. $(\forall b, b' \in 2^{<\omega})(b \leq b' \Rightarrow Cl(X_b) \subseteq Cl(X_{b'}))$.

Proof. The fact that $(\forall b \in 2^{\omega})(Cl(X_b) \subseteq Cl(X_{b0}))$ follows immediately from the following, for any $b \in 2^{\omega}$:

1. $O_{b0} = O_{b00} \dot\cup O_{b01} \dot\cup X_{b0}$ (Lemma 10.8, item 3),
2. $Cl(X_{b0}) = Cl(O_{b0}) - (O_{b00} \cup O_{b01})$ (Lemma 10.8, item 4), and
3. $Cl(X_b) = Cl(O_{b0}) - O_{b0}$ (Lemma 10.8, item 4).

Similarly $(\forall b \in 2^{\omega})(Cl(X_b) \subseteq Cl(X_{b1}))$. This suffices for the lemma.

For $\mathbf{b} \in 2^{\omega}$, define $X_{\mathbf{b}} =_{\mathrm{df}} \bigcap_{n\in\mathbb{N}} O_{\mathbf{b}|_n}$.

Lemma 10.12. $\mathcal{I} = \dot\bigcup_{\boldsymbol{b}\in 2^{\leq\omega}} X_{\boldsymbol{b}}$.

Proof. The $X_{\boldsymbol{b}}$ are pairwise disjoint, by Lemma 10.10. To see that $\mathcal{I} = \bigcup_{\boldsymbol{b} \in 2^{\leq \omega}} X_{\boldsymbol{b}}$, suppose that $x \in \mathcal{I}$, but suppose that $x \notin X_b$ for any $b \in 2^{<\omega}$. It suffices to find a $\mathbf{b} \in 2^{\omega}$ such that $x \in X_{\mathbf{b}}$: we will inductively define $b_n \in 2^{<\omega}$, each of length n, so that $b_0 \leq b_1 \leq \ldots \leq b_n \leq b_{n+1} \leq \ldots$, and so that $x \in O_{b_n}$ for each n. Let $b_0 = \Lambda$, the empty sequence. Assume that $x \in O_{b_n}$. Then $x \in O_{b_n 0} \dot{\cup} O_{b_n 1} \dot{\cup} X_{b_n}$. But $x \notin X_{b_n}$. So x is a member of exactly one of $O_{b_n 0}$ and $O_{b_n 1}$. Let b_{n+1} be whichever of $b_n 0$ and $b_n 1$ is such that $x \in O_{b_{n+1}}$. Note that each b_n has length n, that $b_0 \leq b_1 \leq \ldots \leq b_n \leq b_{n+1} \leq \ldots$ and that $x \in O_{b_n}$ for each n. Let $\mathbf{b}$ be the unique member of 2^{ω} such that $\mathbf{b}|_n = b_n$. Then note that $x \in \bigcap_n O_{\mathbf{b}|_n} = X_{\mathbf{b}}$, as desired.

Given Lemma 10.12, every $x \in \mathcal{I}$ is in exactly one of the $X_{\boldsymbol{b}}$. Let $\mathbf{F}(x) =_{\mathrm{df}}$ the unique $\boldsymbol{b} \in 2^{\leq \omega}$ such that $x \in X_{\boldsymbol{b}}$. Our final task is to show that $\mathbf{F}$ is a surjective interior map. This follows from Lemma 10.13 ($\mathbf{F}$ is continuous) and Corollary 10.3 ($\mathbf{F}$ is an open surjection), below. Notation: for $S \subseteq 2^{\leq \omega}$, we use $Preimg(S)$ for the preimage of S under $\mathbf{F}$; and for $S \subseteq \mathcal{I}$, we use $Img(S)$ for the image of S under $\mathbf{F}$.

Lemma 10.13. $\mathbf{F}$ *is continuous.*

Proof. Recall that the the sets of the form $\leq^*(b)$, where $b \in 2^{<\omega}$, form a basis for our topology on $2^{\leq \omega}$. Also recall that O_b is open in $\mathcal{I}$, by Lemma 10.10, item 2. So it suffices to show that $Preimg(\leq^*(b)) = O_b$, for any $b \in 2^{<\omega}$.

Choose $b \in 2^{<\omega}$. We show, in turn, that (1) $O_b \subseteq Preimg(\leq^*(b))$ and that (2) $Preimg(\leq^*(b)) \subseteq O_b$.

For (1), choose $x \in O_b$. To show that $x \in Preimg(\leq^*(b))$, it suffices to show that $b \leq \mathbf{F}(x)$. Suppose, for a reductio, that $b \nleq \mathbf{F}(x)$. We consider two cases. (Case 1) $\mathbf{F}(x) \in 2^{<\omega}$. Then $O_b \cap X_{\mathbf{F}(x)} = \emptyset$, since $b \nleq \mathbf{F}(x)$ and by Lemma 10.10, item 6. But then, since $x \in X_{\mathbf{F}(x)}$, we have $x \notin O_b$, a contradiction. (Case 2) $\mathbf{F}(x) \in 2^{\omega}$. Let $n = |b|$. Then $b \neq \mathbf{F}(x)|_n$, since $b \nleq \mathbf{F}(x)$. So $b \nleq \mathbf{F}(x)|_n \nleq b$, since b and $\mathbf{F}(x)|_n$ are of the same length. So $O_b \cap O_{\mathbf{F}(x)|_n} = \emptyset$, Lemma 10.10, item 5. But $x \in X_{\mathbf{F}(x)} = \bigcap_{k \in \mathbb{N}} O_{\mathbf{F}(x)|_k} \subseteq O_{\mathbf{F}(x)|_n}$. So $x \notin O_b$, a contradiction.

For (2), choose $x \in Preimg(\leq^*(b))$. Then $\mathbf{F}(x) \in \leq^*(b)$, so that $b \leq \mathbf{F}(x)$. Recall that $x \in X_{\mathbf{F}(x)}$: so, to show that $x \in O_b$, it suffices to show that $X_{\mathbf{F}(x)} \subseteq O_b$. If $\mathbf{F}(x) \in 2^{<\omega}$ then $X_{\mathbf{F}(x)} \subseteq O_b$, by Lemma 10.10, item 3. Suppose, on the other hand, that $\mathbf{F}(x) \in 2^{\omega}$. Since $b \leq \mathbf{F}(x)$, we get $b = \mathbf{F}(x)|_k$ for some $k \in \mathbb{N}$. Thus $X_{\mathbf{F}(x)} = \bigcap_{n \in \mathbb{N}} O_{\mathbf{F}(x)|_n} \subseteq O_{\mathbf{F}(x)|_k} = O_b$.

Lemma 10.14. *Suppose that* $J \subseteq \mathcal{I}$ *is an open interval,* $b \in Img(J) \cap 2^{<\omega}$, $b' \in 2^{<\omega}$ *and* $b \leq b'$. *Then* $b' \in Img(J)$.

Proof. Choose $x \in J$ with $\mathbf{F}(x) = b$. Then $x \in X_b$. So $x \in Cl(X_{b'})$, by Lemma 10.11. So there is some $y \in X_{b'} \cap J$. So $b' \in Img(J)$, since $\mathbf{F}(y) = b'$.

Lemma 10.15. *Suppose that* $J \subseteq \mathcal{I}$ *is an open interval,* $b \in Img(J) \cap 2^{<\omega}$, $\mathbf{b}' \in 2^{\omega}$ *and* $b \leq \mathbf{b}'$. *Then* $\mathbf{b}' \in Img(J)$.

Proof. Let $n = length(b)$, so that $b = \mathbf{b}'|_n$. We will now inductively choose open intervals $J_0, J_1, ... \subseteq J \cap O_b$ and points $x_0 \in J_0, x_1 \in J_1, \ldots$ so that $F(x_k) = \mathbf{b}'|_{n+k}$, for each $k \geq 0$.

First, choose $x_0 \in J$ such that $\mathbf{F}(x_0) = b = \mathbf{b}'|_n$. Since $x_0 \in J \cap O_b$, we can choose an open interval J_0 so that $x_0 \in J_0$ and $Cl(J_0) \subseteq J \cap O_b$. Suppose that we have chosen an open interval J_k and a point $x_k \in J_k$ with $F(x_k) = \mathbf{b}'|_{n+k}$. Then $\mathbf{b}'|_{n+k} \in Img(J_k)$. So $\mathbf{b}'|_{n+k+1} \in Img(J_k)$, by Lemma 10.14. So there is an $x_{k+1} \in Img(J_k)$ with $\mathbf{F}(x_{k+1}) = \mathbf{b}'|_{n+k+1}$. Note that $x_{k+1} \in X_{\mathbf{b}'|_{n+k+1}} \subseteq O_{\mathbf{b}'|_{n+k+1}}$. So $x_{k+1} \in J_k \cap O_{\mathbf{b}'|_{n+k+1}}$. So we can choose an open interval J_{k+1} with $x_{k+1} \in J_{k+1}$ and $Cl(J_{k+1}) \subseteq J_k \cap O_{\mathbf{b}'|_{n+k+1}}$.

Note: $Cl(J_{k+1}) \subseteq J_k$ for each $k \geq 0$. So $\langle Cl(J_k)\rangle_k$ is a decreasing sequence of closed intervals. So $\bigcap_k Cl(J_k)$ is nonempty. Also, $\bigcap_k Cl(J_k) \subseteq J$ and $\bigcap_k Cl(J_k) \subseteq \bigcap_k O_{\mathbf{b}'|_{n+k}}$. So there is a point $x \in \bigcap_k Cl(J_k) \subseteq J \cap X_{\mathbf{b}'}$. So $\mathbf{F}(x) = \mathbf{b}'$ and $x \in J$. So $\mathbf{b}' \in Img(J)$.

Lemma 10.16. *Suppose that $J \subseteq \mathcal{I}$ is an open interval and $\mathbf{b} \in Img(J) \cap 2^{\omega}$. Then there is a $b' \in Img(J) \cap 2^{<\omega}$ with $b' \leq \mathbf{b}$.*

Proof. Suppose that $J \subseteq \mathcal{I}$ is an open interval and $\mathbf{b} \in Img(J) \cap 2^{\omega}$. Choose $x \in J$ with $\mathbf{F}(x) = \mathbf{b}$, and choose a positive real number d so that $(x-d, x+d) \subseteq J$. Choose $n \in \mathbb{N}$ with $1/3^n < d$ and let $b' = \mathbf{b}|_n \in 2^{<\omega}$. Note that $x \in O_{b'} \cap (x-d, x+d)$; also, $width(O_{b'}) < d$, by Lemma 10.10, item 8. Let J' be any maximal open interval in $O_{b'}$ with $x \in J'$, and note two things about J': (1) J' has length $\leq width(O_{b'}) < d$, since J' is an open interval and $J' \subseteq O_{b'}$; and (2) $J' \cap C(O_{b'})$ is nonempty, by Lemma 10.8, item 5. By (2), there is an $x' \in J' \cap X_{b'}$, and by (1) $J' \subseteq (x-d, x+d)$. So $x' \in J$ and $\mathbf{F}(x') = b'$. So $b' \in Img(J)$.

Corollary 10.2. *$Img(J)$ is open in $2^{\leq\omega}$, for every interval $J \subseteq \mathcal{I}$.*

Proof. It suffices to show that $Img(J) = \bigcup_{b \in Img(J)\cap 2^{<\omega}} \leq^*(b)$. So consider any interval $J \subseteq \mathcal{I}$. We will show, in turn, that (1) $\bigcup_{b \in Img(J)\cap 2^{<\omega}} \leq^*(b) \subseteq Img(J)$ and that (2) $Img(J) \subseteq \bigcup_{b \in Img(J)\cap 2^{<\omega}} \leq^*(b)$.

For (1), by Lemmas 10.14 and 10.15, if $b \in Img(J) \cap 2^{<\omega}$ then $\leq^*(b) \subseteq Img(J)$. So $\bigcup_{b \in Img(J)\cap 2^{<\omega}} \leq^*(b) \subseteq Img(J)$.

For (2), note that if $\boldsymbol{b} \in Img(J)$, then there is some $b' \leq \boldsymbol{b}$ such that $b' \in Img(J) \cap 2^{<\omega}$: this follows from Lemma 10.16 if $\boldsymbol{b} \in 2^{\omega}$; and it is trivial if $\boldsymbol{b} \in 2^{<\omega}$, since we can just let $b' = \boldsymbol{b}$. Thus, if $\boldsymbol{b} \in Img(J)$ then there exists $b' \in 2^{<\omega}$ with $\boldsymbol{b} \in \leq^*(b') \subseteq Img(J)$. Thus $Img(J) \subseteq \bigcup_{b \in Img(J)\cap 2^{<\omega}} \leq^*(b)$.

Corollary 10.3. $\mathbf{F}$ *is an open surjection.*

Proof. The openness of $\mathbf{F}$ follows immediately from Corollary 10.2. It remains to show that $\mathbf{F}$ is surjective – equivalently, that $\boldsymbol{b} \in Img(\mathcal{I})$, for every $\boldsymbol{b} \in 2^{\leq\omega}$. By Corollary 10.2, $Img(\mathcal{I})$ is open and therefore upwardly closed under $\leq$. Also, $\Lambda \in Img(\mathcal{I})$, since $\Lambda = \mathbf{F}(x)$ for any $x \in X_\Lambda = C(\mathcal{I}) = C$. So, $\boldsymbol{b} \in Img(\mathcal{I})$, for every $\boldsymbol{b} \in 2^{\leq\omega}$, since $\Lambda \leq \boldsymbol{b}$ and $Img(\mathcal{I})$ is upwardly closed under $\leq$.

Acknowledgements Thanks to the audience at the Ninth International Tbilisi Symposium on Language, Logic and Computation (2011) in Kutaisi, Georgia, for listening to me present the more general paper, [6]. Special thanks to each of David Gabelaia, Nick Bezhanishvili, Roman

Kontchakov and Mamuka Jibladze, for indulging me by letting me explain the proof in detail in the case considered by this paper, $\mathbb{R}$. Also, a big thanks to Robert Goldblatt, for carefully reading a draft of this paper and for very useful comments.

References

[1] Aiello, M., van Benthem, J., and Bezhanishvili, G. (2003). Reasoning about space: the modal way. *J. Logic Comput.*, 13(6):889–920.

[2] Bezhanishvili, G. and Gehrke, M. (2005). Completeness of S4 with respect to the real line: revisited. *Ann. Pure Appl. Logic*, 131(1-3):287–301.

[3] Freyd, P. J. and Scedov, A. (1990). *Categories, Allegories*. North-Holland, Amsterdam.

[4] Goldblatt, R. (1980). Diodorean modality in Minkowski spacetime. *Studia Logica*, 39(2-3):219–236.

[5] Goldblatt, R. and Hodkinson, I. (2019). Strong completeness of modal logics over 0-dimensional metric spaces. *Rev. Symb. Log.*, pages 1–22. Published online, 24 October 2019.

[6] Kremer, P. (2013). Strong completeness of S4 for any dense-in-itself metric space. *Rev. Symb. Log.*, 6(3):545–570.

[7] Kremer, P. (2014). Quantified modal logic on the rational line. *Rev. Symb. Log.*, 7(3):439–454.

[8] Lando, T. (2012). Completeness of $S4$ for the Lebesgue measure algebra. *J. Philos. Logic*, 41(2):287–316.

[9] McKinsey, J. C. C. (1941). A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology. *J. Symbolic Logic*, 6:117–134.

[10] McKinsey, J. C. C. and Tarski, A. (1944). The algebra of topology. *Ann. of Math. (2)*, 45:141–191.

[11] Mints, G. and Zhang, T. (2005). A proof of topological completeness for $S4$ in $(0,1)$. *Ann. Pure Appl. Logic*, 133(1-3):231–245.

[12] Rasiowa, H. and Sikorski, R. (1963). *The Mathematics of Metamathematics*. Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warsaw.

[13] Tarski, A. (1938). Der Aussagenkalkül und die Topologie. *Fundamenta Mathematicae*, 31:103–134.

[14] van Benthem, J., Bezhanishvili, G., ten Cate, B., and Sarenac, D. (2006). Multimodal logics of products of topologies. *Studia Logica*, 84(3):369–392.

[15] Vickers, S. (1989). *Topology via logic*, volume 5 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge.