A Small Counterexample in Intuitionistic Dynamic Topological Logic

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Let $\mathcal{L}$ be an intuitionistic language with a set PV of propositional variables; connectives &, $\vee$, $\rightarrow$, and $\sim$; and temporal modalities $\bigcirc$, and $*$. Let a *dynamic topological system* be an ordered pair, $\langle X,\ f\rangle$, where X is a topological space and f is a continuous function on X. Let a *dynamic topological model* is an ordered triple $M = \langle X,\ f,\ V\rangle$ where $\langle X,\ f\rangle$ is a dynamic topological system and $V : PV \rightarrow \mathcal{P}(X)$ is a valuation function assigning an *open* subset of X to each propositional variable. V is extended to all formulas as follows:

$V(A \vee B) = V(A) \cup V(B)$;

$V(A\ \&\ B) = V(A) \cap V(B)$;

$V(\sim A) = Int(X - V(A))$, where Int is topological interior;

$V(A \rightarrow B) = Int((X - V(A)) \cup V(B))$;

$V(\bigcirc A) = f^{-1}(V(A))$; and

$V(*A) = Int(\cap_{n\geq 0}\ f^{-n}(V(A)))$.

In the last clause we take the *topological interior* of $\cap_{n\geq 0}\ f^{-n}(V(A))$ in order to ensure that the set $V(A)$ is open for each formula A. We define standard validity relations:

A is *validated by* $M = \langle X,\ f,\ V\rangle$ ($M \models A$) iff $V(A) = X$.

A is *valid* ($\models A$) iff $M \models A$ for every dynamic topological model M.

In ordinary ω-time temporal logic, three principles governing $\bigcirc$ and $*$ are

$*A \rightarrow \bigcirc *A$ and $*\bigcirc A \rightarrow \bigcirc *A$ and $*A \rightarrow **A$.

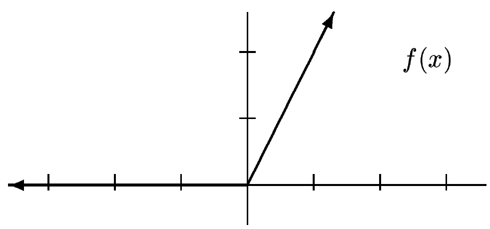
The main point of this note is that

$\not\models *p \to \bigcirc *p$ and $\not\models *\bigcirc p \to \bigcirc *p$ and $\not\models *p \to **p$.

where p is a propositional variable. To see this let $M = \langle \mathbb{R},\ f,\ V \rangle$ where f is defined as follows,

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 2x, & x \geq 0 \end{cases}$$

and where $V(p)$ is the open set $(-\infty,\ 1)$.



Note:

$$V(p) = (-\infty,\ 1)$$

$$f^{-1}(V(p)) = (-\infty,\ \tfrac{1}{2})$$

$$f^{-2}(V(p)) = (-\infty,\ \tfrac{1}{4})$$

$$f^{-n}(V(p)) = (-\infty,\ \tfrac{1}{2^n})$$

$$\cap_n\ f^{-n}(V(p)) = (-\infty,\ 0]$$

$$V(*p) = Int(\cap_{n\geq 0}\ f^{-n}(V(p))) = (-\infty,\ 0)$$

$$V(*\bigcirc p) = Int(\cap_{n\geq 1}\ f^{-n}(V(p))) = (-\infty,\ 0)$$

$$V(\bigcirc *p) = \emptyset$$

$$V(**p) = \emptyset$$

Thus

$$V(*p \to \bigcirc *p) = (0,\ \infty) = V(*\bigcirc p \to \bigcirc *p) = V(*p \to **p) \neq \mathbb{R}$$

Thus, as desired,

$\not\models *p \to \bigcirc *p$ and $\not\models *\bigcirc p \to \bigcirc *p$ and $\not\models *p \to **p$.