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DEFINING RELEVANT IMPLICATION IN A PROPOSITIONALLY QUANTIFIED S4

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Abstract. R. K. Meyer once gave precise form to the question of whether relevant implication can be defined in any modal system, and his answer was 'no'. In the present paper, we extend **S4**, first with propositional quantifiers, to the system **S4**π+; and then with definite propositional descriptions, to the system **S4**$\pi+^{ip}$. We show that relevant implication can in some sense be defined in the modal system **S4**$\pi+^{ip}$, although it cannot be defined in **S4**π+.

§1. **Introduction.** In **S4** and other modal logics, strict implication ($\prec$) is typically defined in terms of necessity ($\Box$) and material implication ($\supset$) thus (see Lewis and Langford [16]):

$$(A \prec B) =_{\mathrm{df}} \Box(A \supset B).$$

It has been argued that $\prec$ better formalises our intuitive notion of implication than does $\supset$. For if we use $\prec$ we avoid some of the "fallacies" of material implication. For example, though the falsehood of p is enough to guarantee the truth of $(p \supset q)$, it is not enough to guarantee the truth of $(p \prec q)$.

Relevance logicians have put forward quite different candidates for formalisations of implication. They have not done so by defining new connectives in classical or modal systems. Rather, they have developed systems of *relevance* logic, whose primitive connectives are typically "classical" &, $\vee$ and $\sim$, together with a "relevant" implicational connective, $\rightarrow$. (There is good reason not to take relevant negation, $\sim$, to be classical. But something like a classical understanding of it is implicit in the development of the earlier and stronger relevance systems.)

Among the better understood of these logics is Anderson and Belnap's **R**. (The *locus classicus* for **R** and its cousins is Anderson and Belnap [1].) The pure theory of $\rightarrow$ in **R** is neither stronger nor weaker than the pure theory of $\prec$ in **S4**: $(p \rightarrow ((p \rightarrow q) \rightarrow q)) \in$ **R** while $(p \prec ((p \prec q) \prec q)) \notin$ **S4**; and $(p \prec (q \prec q)) \in$ **S4** while $(p \rightarrow (q \rightarrow q)) \notin$ **R**.

Can relevant implication be defined in **S4** or in some other modal logic? Meyer [17] gives precise form to a similar question. Noting that $\prec$ is defined above as a modalised truth function, Meyer asks whether we can define $\rightarrow$ as a modalised truth function. For almost all standard relevance logics his answer is 'no'. The present paper's main results are that the $\rightarrow$ of **R** *can* be defined in a natural and independently motivated extension of **S4**, with propositional quantifiers and with

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definite propositional descriptions of the form $\imath pA$; and that the $\rightarrow$ of **R** *cannot* be so defined without the use of definite propositional descriptions. (Edwin Mares suggested that we add definite propositional descriptions to the system.)

We proceed as follows. §2 extends **S4** to the propositionally quantified logic **S4π+**. §3 extends **S4π+** to the logic **S4π+**$^{\imath p}$, by adding definite propositional descriptions. §4 generalises Urquhart's [21] semilattice semantics for $\mathbf{R}_{\rightarrow}$, the pure implicational fragment of **R**. §5 gives precise sense to the claim that the connective $\rightarrow$ of $\mathbf{R}_{\rightarrow}$ can be defined in **S4π+**$^{\imath p}$ or in **S4π+**. §6 proves that the $\rightarrow$ of $\mathbf{R}_{\rightarrow}$ can be defined in **S4π+**$^{\imath p}$. §7 proves that $\rightarrow$ cannot be defined in **S4π+**. §8 generalises and comments on §7's result. And §9 concludes with some open problems and some methodological remarks.

§2. S4π+: S4 with propositional quantifiers, $\forall p$ and $\exists p$. We assume that we have a modal language with a countable set of propositional variables, $\mathrm{PV} = \{p_1, p_2, \dots\}$; connectives $\vee$, $\neg$ and $\Box$; and propositional quantifiers $\forall p$ and $\exists p$. We use p, q, r, ... as meta-linguistic variables ranging over PV, and A, B, C, ... as meta-linguistic variables ranging over formulas.

Here we extend Kripke's [15] semantics for **S4** and its cousins to our propositionally quantified language. A *frame* is a 3-tuple, $\mathrm{F} = (\mathrm{W}, \leq, \mathrm{w}_0)$, where W is a set (of possible worlds); $\leq$ (the accessibility relation) is a binary relation on W; and w_0 (the actual world) $\in$ W. Given a frame, a *proposition* is a subset of W. A *model* is a pair $\mathrm{M} = (\mathrm{F}, \mathrm{V})$, where F is a frame and V is a function assigning a proposition to every propositional variable. Given a model M, a proposition P and a propositional variable p, $\mathrm{M}[\mathrm{P}/p]$ is the model just like M except that it assigns P to p. An **S4**-frame is a frame for which $\leq$ is reflexive and transitive. And an **S4**-model is a model whose underlying frame is an **S4**-frame.

Given a model M and a formula A, the proposition $\mathrm{M}(A)$ assigned by M to A is defined by the following clauses:

(i) $\mathrm{M}(p) = \mathrm{V}(p)$ for $p \in \mathrm{PV}$;
(ii) $\mathrm{M}((B \vee C)) = \mathrm{M}(B) \cup \mathrm{M}(C)$;
(iii) $\mathrm{M}(\neg B) = \mathrm{W} - \mathrm{M}(B)$;
(iv) $\mathrm{M}(\Box B) = \{\mathrm{w} : (\forall \mathrm{w}' \geq \mathrm{w})(\mathrm{w}' \in \mathrm{M}(B))\}$;
(v) $\mathrm{M}(\forall pA) = \bigcap\{\mathrm{M}[\mathrm{P}/p](A) : \mathrm{P} \subseteq \mathrm{W}\}$; and
(vi) $\mathrm{M}(\exists pA) = \bigcup\{\mathrm{M}[\mathrm{P}/p](A) : \mathrm{P} \subseteq \mathrm{W}\}$.

A is *true at the world* w iff $\mathrm{w} \in \mathrm{M}(A)$. A is *true in the model* M iff $\mathrm{w}_0 \in \mathrm{M}(A)$. Kripke [15] proves that if A is quantifier-free, then $A \in$ **S4** iff A is true in every **S4**-model. (See Hughes and Cresswell [9] for soundness and completeness theorems for **K**, **T**, **K4**, **B**, **S4.2**, **S5** and others.)

DEFINITION 1. **S4π+** is the set of formulas true in every **S4**-model.

Fine [4] defines **S4π+** as well as the systems **Kπ+**, **Tπ+**, **K4π+**, **Bπ+**, **S4.2π+**, and **S5π+**. These systems are to be distinguished from **S4π**, **Kπ**, **Tπ**, **K4π**, **Bπ**, **S4.2π**, and **S5π**, which result from adding natural quantification axioms and rules to the unquantified axiomatisations. (See also Bull [2] on **S4π** and **S5π**.) Here are some facts about **Lπ+** and **Lπ**, where **L** ranges over {**S4**, **K**, **T**, **K4**, **B**, **S4.2**, **S5**}.

(1) **Lπ+** and **Lπ** are conservative extensions of **L**, and **Lπ** $\subsetneq$ **Lπ+**.

(2) **S5π+** $=$ **S5π** $+ \exists p(p \,\&\, \forall q(q \supset \Box(p \supset q)))$ and is decidable. See Fine [4] and Kaplan [11].

(3) Fine and Kripke independently showed that **Kπ+**, **Tπ+**, **K4π+**, **Bπ+**, **S4.2π+**, and **S4π+** are recursively isomorphic to full second order classical logic (a slightly weaker result occurs in Fine [4]). The stronger result can be shown by simplifying the intuitionistic strategies of Kremer [12], or as noted by Kremer [14], by adapting the relevance logic strategies used there. A published proof is given by Kaminski and Tiomkin [10].

In the terminology of Kremer [14], borrowed from Henkin [6], the **Lπ+**'s embody the *primary interpretation of propositional quantifiers relative to Kripke's semantics*, while the **Lπ**'s embody the *secondary interpretation*. Henceforth we do not concern ourselves with the **Lπ**'s. It is in an extension of **S4π+** that we will define relevant implication.

§3. S4π+$^{\iota p}$: S4π+ with definite propositional descriptions.

Here we extend the semantics for **S4π+** to give a semantics for expressions of the form ιpA, which are to be read as 'the proposition p such that A'. We add the propositional description operator ιp to the language, together with a grammatical rule: if A is a formula and p is a propositional variable, then ιpA is a formula. We call such formulas *definite propositional descriptions*.

A grammatical aside: in the standard treatments of definite individual descriptions of the form ιxA, ι combines with an individual variable and a formula to form a singular term. In the case of definite *propositional* descriptions, everything is of the same logical type: ι combines with a propositional variable and a formula to form a *formula*. Given its grammatical similarity to $\forall p$ and $\exists p$, we can take ιp to be a new propositional quantifier.

3.1. A first attempt at a semantics for definite propositional descriptions.

In keeping with §2's semantics, we want to provide a clause defining $\mathrm{M}(\iota pA)$, for a model M, a propositional variable p and a formula A. Our treatment of definite propositional descriptions will be *Fregean*, in the sense of Carnap [3], and motivated by Frege [5]: if there is a unique proposition P such that $\mathrm{M}[\mathrm{P}/p] \models A$, then $\mathrm{M}(\iota pA) = \mathrm{P}$; otherwise, $\mathrm{M}(\iota pA)$ is some *designated* proposition.

To make this precise, we make room for designated propositions in our semantics. An *extended frame* (e-frame) is an ordered pair (F, D), where F is a frame, and D, the *designated proposition*, is a subset of W. (F, D) *extends* F. An *extended model* (e-model) is a 3-tuple $\mathrm{N} = (\mathrm{F}, \mathrm{D}, \mathrm{V})$, where (F, D) is an e-frame and (F, V) is a model. $(\mathrm{F}, \mathrm{D}, \mathrm{V})$ *extends* (F, V). **S4**-e-frames and **S4**-e-models are defined in the obvious way. Given an e-model N a proposition P and a propositional variable p, $\mathrm{N}[\mathrm{P}/p]$ is the e-model just like N except that it assigns P to p.

Given an e-model N and a formula A, the proposition, $\mathrm{N}(A)$, assigned by N to A is defined by the following clauses:

(i)–(vii) as in §2, with 'M' everywhere replaced by 'N'; and

(viii) If there is a unique proposition P such that $\mathrm{w}_0 \in \mathrm{N}[\mathrm{P}/p](A)$ then $\mathrm{N}(\iota pA) = \mathrm{P}$. Otherwise $\mathrm{N}(\iota pA) = \mathrm{D}$.

If there is a unique proposition P such that $w_0 \in N[P/p](A)$, then $\imath pA$ is a *proper* description. Otherwise it is *improper*. If N is an e-model, then the formula A is *true at the world* w *iff* $w \in N(A)$. A is *true in the e-model* N iff $w_0 \in N(A)$.

3.2. A problem and a solution. We seem to have added to **S4**π+ a rich theory of definite propositional descriptions. But it is actually quite impoverished. The problem is made clear in Theorem 2.

THEOREM 2. *Suppose that* $N = (W, \leq, w_0, D, V)$ *is an* **S4**-*e-model. Then either* $(\forall w \in W)(w_0 \leq w)$ *or* $(\forall p \in PV)(\forall A)(\imath pA$ *is improper*$)$.

PROOF. Suppose that $N = (W, \leq, w_0, D, V)$ is an **S4**-e-model and that for some $w \in W$, $w_0 \not\leq w$. Assume that $\imath pA$ is proper. So there is a unique proposition, P, such that $w_0 \in N[P/p](A)$. Let $P' = P \cup \{w\}$ and $P'' = P - \{w\}$. By induction on the complexity of formulas we have the following: $(\forall B)(\forall N' = (W, \leq, w_0, D, V'))(\forall w \geq w_0)(w \in N'[P/p](B)$ iff $w \in (N'[P'/p](B) \cap N'[P''/p](B)))$. So w_0 is in both $N[P'/p](A)$ and $N[P''/p](A)$. This contradicts our assumption that $\imath pA$ is proper, since $P' \neq P''$. ⊣

The upshot is that, in most **S4**-e-models, all definite propositional descriptions are improper. The problem can be diagnosed so as to lead to a revision of §3.1's semantics. Suppose that $F = (W, \leq, w_0)$ is a frame in which some world $w \not\geq w_0$. Suppose that (F, D) is an e-frame extending F. Finally, suppose that P and Q are distinct subsets of W, such that $(\forall w \geq w_0)(w \in P$ iff $w \in Q)$. The root of the problem is this:

No **S4**-e-model based on the e-frame F can distinguish between P and Q.

The reason for this is that no **S4**-e-model can tell what is going on at worlds inaccessible to w_0.

This is made precise in Theorem 3.

THEOREM 3. *Suppose that* (F, D), P *and* Q *are as above; that* p *and* q *are distinct propositional variables; that formulas* A *and* B *are the same except that free occurrences of* p *in one might be replaced by free occurrences of* q *in the other; and that* $N = (F, D, V)$ *is an* **S4**-*e-model, where* $V(p) = P$ *and* $V(q) = Q$. *Then* $N \models A$ *iff* $N \models B$.

PROOF. By induction on the complexity of A. ⊣

Our solution is to "identify" propositions that, like P and Q, do not differ with respect to worlds accessible to w_0. This does not affect clauses (i)–(vii) of the definition of $N(A)$ (§3.1) since these clauses are not sensitive to criteria of identity. Clause (viii) is affected. Suppose that at least one proposition "satisfies" A, where A is thought of as a function of p. On the old understanding, $\imath pA$ is improper iff there are two distinct propositions satisfying A. On the new understanding, $\imath pA$ is improper iff there are two non-identified propositions satisfying A.

There is one more thing to iron out: if $\imath pA$ is proper, we still might have several candidates for $N(\imath pA)$, although all of these candidates are identified. We resolve this by choosing the smallest of these candidates. §3.3 makes this precise.

3.3. The semantics for $\imath p$. Suppose $F = (W, \leq, w_0)$ is a frame, (F, D) is an e-frame, and $N = (F, D, V)$ is an e-model. $W_0 =_{df} \{w \in W : w_0 \leq w\}$. Propositions

P and Q are *equivalent* ($P \approx_0 Q$) iff $(P \cap W_0) = (Q \cap W_0)$. The clauses that determine $N(A)$, the proposition assigned by N to the formula A, are

(i)–(vii) as in §3.1; and

(viii) Suppose that there is a proposition P such that $w_0 \in N[P/p](A)$ and for every proposition Q, if $w_0 \in N[Q/p](A)$ then $Q \approx_0 P$. Then $N(\iota pA) = P \cap W_0$. Otherwise $N(\iota pA) = \mathbf{D}$.

Truth at a world and in an e-model are defined as in §3.1.

DEFINITION 4. $\mathbf{S4}\boldsymbol{\pi}+^{\iota p}$ is the set of formulas true in every **S4**-e-model.

We can analogously define $\mathbf{K}\boldsymbol{\pi}+^{\iota p}$, $\mathbf{T}\boldsymbol{\pi}+^{\iota p}$, $\mathbf{K4}\boldsymbol{\pi}+^{\iota p}$, $\mathbf{B}\boldsymbol{\pi}+^{\iota p}$, $\mathbf{S4.2}\boldsymbol{\pi}+^{\iota p}$, and $\mathbf{S5}\boldsymbol{\pi}+^{\iota p}$. To define $\mathbf{K}\boldsymbol{\pi}+^{\iota p}$, $\mathbf{K4}\boldsymbol{\pi}+^{\iota p}$, $\mathbf{T}\boldsymbol{\pi}+^{\iota p}$ and $\mathbf{B}\boldsymbol{\pi}+^{\iota p}$, we alter the definition of W_0, since the accessibility relation is no longer both reflexive and transitive. First we define $w \leq_\omega w'$, in three steps: (1) $w \leq_0 w'$ iff $w = w'$; (2) $w \leq_{n+1} w'$ iff $(\exists w'')(w \leq_n w''$ and $w'' \leq w')$; and (3) $w \leq_\omega w'$ iff $(\exists n)(w \leq_n w'')$. Then $W_0 =_{df} \{w \in W : w_0 \leq_\omega w\}$. Below are some facts about $\mathbf{L}\boldsymbol{\pi}+^{\iota p}$, where **L** ranges over {**K**, **T**, **K4**, **B**, **S4**, **S4.2**, **S5**}.

(1) The following formulas are valid in $\mathbf{L}\boldsymbol{\pi}+^{\iota p}$ in the present sense, but not in the sense of §3.1:

$$(\iota p\forall q\Box(p \supset q) \supset A);\ (A \supset \iota p\forall q\Box(q \supset p));$$
$$(A \equiv \iota p\Box(A \equiv p));\ \iota p(p \,\&\, \forall q(q \supset \Box(p \supset q))).$$

(2) $\mathbf{L}\boldsymbol{\pi}+^{\iota p}$ is a conservative extension of $\mathbf{L}\boldsymbol{\pi}+$. If $\mathbf{L} \neq \mathbf{S5}$ then $\mathbf{L}\boldsymbol{\pi}+^{\iota p}$ is recursively isomorphic to full second order classical logic. Conjecture: $\mathbf{S5}\boldsymbol{\pi}+^{\iota p}$ is decidable.

(3) ιpA is true at a world $w \in W_0$ iff either $w \in P \cap W_0$ where P is the unique proposition, modulo $\approx_0$, such that $w_0 \in N[P/p](A)$; or there is no unique proposition P such that $w_0 \in N[P/p](A)$, and $w \in \mathbf{D}$. This raises two related points. (i) Like $\Box$, ιp is intensional: the truth, at the world w, of ιpA depends on the truth, as p ranges over the propositions, of A *at other worlds*. This is not so for $\forall p$ and $\exists p$. In some sense, ιp is even more intensional than $\Box$: unlike $\Box A$, the truth at w of ιpA may depend upon the truth of A at worlds *that are inaccessible* to w. (ii) There is a *de re-de dicto* issue here. Whether ιpA is true at the world w depends in part on whether ιpA is proper *from the point of view of the actual world* not from the point of view of w. In other words, ignoring, for the moment the designated proposition, ιpA is true at world w iff *de re*, i.e., at the actual world w_0, ιpA is proper and *de dicto*, i.e., at the world w, ιpA is true.

(4) $\mathbf{L}\boldsymbol{\pi}+^{\iota p}$ is not closed under the rule of necessitation, though **L** and $\mathbf{L}\boldsymbol{\pi}+$ are. Consider the minimally true formula: $A = \iota p(p \,\&\, \forall q(q \supset \Box(p \supset q)))$. $A \in \mathbf{L}\boldsymbol{\pi}+^{\iota p}$ but $\Box A \notin \mathbf{L}\boldsymbol{\pi}+^{\iota p}$. The reason: if N is an e-model then $N(A) \cap W_0 = \{w_0\}$.

§4. A generalised semantics for $\mathbf{R}_\rightarrow$. We assume that we have a relevance language with the set, PV, of propositional variables, and with a binary connective, $\rightarrow$. (It is convenient to have the same set PV as for our modal language.) $\mathbf{R}_\rightarrow$ is defined and axiomatised in Anderson and Belnap [1].

Urquhart [21] presents a semilattice semantics for $\mathbf{R}_\rightarrow$. A *semilattice with* 0 is a 3-tuple $\mathbf{L} = (L, \circ, 0)$ where L is a set; $0 \in L$; and $\circ$ is a binary operator on L that is commutative, associative and for which $(\forall u \in L)(0 \circ u = u \circ u = u)$. A *consequence* *model* (c-model) is a semilattice with 0 together with a function $V : PV \to \mathscr{P}(L)$. The proposition $M(A)$ assigned by the model $M = (L, \circ, 0, V)$ to the formula A is defined by the following clauses:

(i) $M(p) = V(p)$ for each $p \in PV$;
(ii) $M(B \to C) = \{u \in L : (\forall v \in L)(v \in M(B) \Longrightarrow u \circ v \in M(C))\}$.

Given a c-model M, the formula A is *true at the point* u iff $u \in M(A)$. A is *true in the c-model* M iff A is true at 0. Urquhart [21] proves that $A \in \mathbf{R}_{\to}$ iff A is true in every c-model.

We can generalise this semantics so that we can interpret formulas of the language of $\mathbf{R}_{\to}$ in the **S4**-e-frames of §3. (An analogous generalisation to that given below can be given for the **S4**-frames and **S4**-models of §2, rather than the **S4**-e-frames and **S4**-e-models of §3. Theorem 6, below, goes through in either case.) In order to so interpret these formulas, we must provide a clause defining $N(B \to C)$ in terms of $N(B)$ and $N(C)$, where N is an **S4**-e-model. First, some definitions.

DEFINITION 5. Suppose that $N = \langle W, \leq, w_0, V\rangle$ is an e-model. The set of *upper bounds* of $u \in W$ and $v \in W$ is $ub(u, v) =_{df} \{w \in W : u \leq w \text{ and } v \leq w\}$. The set of *least upper bounds* of $u \in W$ and $v \in W$ is $lub(u, v) =_{df} \{w \in ub(u, v) : (\forall w' \in ub(u, v))(w \leq w')\}$. The ternary relation Ruvw on W is defined as follows:

Ruvw iff $u, v, w \in W_0$ and either $[(\forall x, y \in W_0)(\exists! z \in lub(x, y))$ and $w \in lub(u, v)]$ or $[\neg(\forall x, y \in W_0)(\exists! z \in lub(x, y))$ and $w \in ub(u, v)]$.

The binary operator, $\to$, on $\mathscr{P}(W)$ is defined as follows, for $P, Q \subseteq W$:

$$(P \to Q) =_{df} \{u \in W_0 : (\forall v, w \in W_0)((v \in P \text{ and } Ruvw) \Longrightarrow w \in Q)\}.$$

(Routley and Meyer [18] make similar use of a ternary relation.)

Given an **S4**-e-model N and a formula A in the language of $\mathbf{R}_{\to}$, the proposition $N(A)$ assigned by N to A is defined by the following clauses:

(i) $N(p) = V(p)$ for each $p \in PV$;
(ii) $N(A \to B) = (N(A) \to N(B))$.

Truth at a world and in an e-model are defined as in §3.1.

THEOREM 6 (Soundness and completeness). *$A \in \mathbf{R}_{\to}$ iff A is true in every* **S4**-*e-model.*

PROOF. ($\Longrightarrow$): By induction on the length of proof of A. ($\Longleftarrow$): The canonical c-model provided by Urquhart [21] is an **S4**-e-model, if we take 0 to be the actual world; if we define $\leq$ so that $u \leq v$ iff $u = (u \circ v)$; and if we let D be any proposition. So if $A \notin \mathbf{R}_{\to}$, then A is false in some **S4**-e-model. ⊣

The fact that the formulas of $\mathbf{R}_{\to}$ can be interpreted in our **S4**-e-frames allows us in §6, below, to define $\to$ in terms of $\neg$, $\vee$, $\Box$, $\forall p$, $\exists p$ and ιp.

§5. Defining one logic's connectives in another logic. Here we make precise the claim that we have defined one logic's connectives in another logic. §5.1 specifies what it is to define a new connective in a given logic. §5.2 partially specifies what it is for that new connective to be "the same as" a particular connective from a different logic.

5.1. Defining new connectives. Consider the following "definitions" of new connectives, in the language of quantified modal logic:

$$(A \supset B) =_{\text{df}} (\neg A \vee B);$$
$$(A \mathbin{-\!\!3} B) =_{\text{df}} \Box(A \supset B).$$
$$@A =_{\text{df}} \forall p \Box(A \supset p).$$

These definitions can be thought of as specifications of functions on the space of formulas. On this interpretation the functions &, $\supset$, $-\!\!3$ and @ are meta-linguistic entities, not new pieces of the object-language's vocabulary.

Not every function on the space of formulas intuitively counts as a connective. Consider the function that takes all formulas with an odd number of propositional variables to p_1 and that takes all formulas with an even number of propositional variables to p_2. We would hardly think of this function as a *connective*. Definition 8, below, characterises those functions that are *connective-like*. Theorem 9 provides an alternate characterisation of connective-like functions.

DEFINITION 7. Suppose that A and $B_1, \ldots, B_n$ are formulas in some propositional language that might or might not have propositional quantifiers, and that $q_1, \ldots, q_n$ are distinct propositional variables. Then $A[B_1/q_1, \ldots, B_n/q_n]$ is the result of simultaneously replacing the free occurrences of the q_i's in A by the B_i's. We assume that there is some systematic way of replacing the bound variables of A so that any free occurrence of a propositional variable in B_i is also free in $A[B_1/q_1, \ldots, B_n/q_n]$.

DEFINITION 8. Suppose that F is an n-ary function on the space of formulas of some propositional language. F is *connective-like* iff for all distinct propositional variables $q_1, \ldots, q_n$ and all formulas $B_1, \ldots, B_n$, $\mathrm{F}(B_1, \ldots, B_n) = \mathrm{F}(q_1, \ldots, q_n)[B_1/q_1, \ldots, B_n/q_n]$.

THEOREM 9. *Suppose that* F *is an n-ary function on the space of formulas of some propositional language. Then* F *is connective-like iff there are distinct propositional variables* $q_1, \ldots, q_n$, *and there is a formula* A *all of whose free propositional variables are among the* q_i *and for which, for all formulas* $B_1, \ldots, B_n$, *we have* $\mathrm{F}(B_1, \ldots, B_n) = A[B_1/q_1, \ldots, B_n/q_n]$.

THEOREM 10 (Composability theorem). *If* F *is a connective-like n-ary function and* $\mathrm{G}_1, \ldots, \mathrm{G}_n$ *are connective-like m-ary functions, then the m-ary function* $\mathrm{F}(\mathrm{G}_1, \ldots, \mathrm{G}_n)$ *is connective-like.*

In §6 we will define $\rightarrow$ in **S4π**$+^{\iota p}$. This will implicitly rely on the composability theorem.

5.2. Identifying connectives across logics. Suppose F is a binary connective-like function on the space of formulas in the language of **S4π**$+^{\iota p}$ and that F is intended to represent $\rightarrow$. Given this intention, F induces the following one-one map, G, from formulas of the relevance language to formulas of the modal language:

$$\mathrm{G}(p) = p, \text{ for propositional variables, } p; \text{ and}$$
$$\mathrm{G}(A \rightarrow B) = \mathrm{F}(\mathrm{G}(A), \mathrm{G}(B)).$$

Intuitively, the formula A in the language of $\mathbf{R}_{\rightarrow}$ is identified with the formula $\mathrm{G}(A)$ in the language of $\mathbf{S4\pi}+^{\prime p}$. And the function F succeeds in representing $\rightarrow$ iff

$$A \in \mathbf{R}_{\rightarrow} \text{ iff } \mathrm{G}(A) \in \mathbf{S4\pi}+^{\prime p}.$$

Definitions 11 and 12 generalise these intuitions.

DEFINITION 11. Suppose that $\mathbf{L}$ and $\mathbf{L}'$ are propositional logics, with or without propositional quantifiers, formulated for languages with the set, PV, of propositional variables. Suppose that the only primitive connective of $\mathbf{L}$ is the n-ary connective $*$. Let the $\mathbf{L}$-formulas ($\mathbf{L}'$-formulas) be the formulas in the language of $\mathbf{L}$ ($\mathbf{L}'$). Suppose that F is an n-ary function on the space of $\mathbf{L}'$-formulas. Let G be that function from $\mathbf{L}$-formulas to $\mathbf{L}'$-formulas for which

$$\mathrm{G}(p) = p, \text{ for } p \in \mathrm{PV}; \text{ and}$$
$$\mathrm{G}(*(B_1, \dots, B_n)) = \mathrm{F}(\mathrm{G}(B_1), \dots, \mathrm{G}(B_n)).$$

F *succeeds in representing the* $*$ *of* $\mathbf{L}$ *in* $\mathbf{L}'$ iff for every $\mathbf{L}$-formula A, $A \in \mathbf{L}$ iff $\mathrm{G}(A) \in \mathbf{L}'$.

DEFINITION 12. Suppose that $\mathbf{L}$, $\mathbf{L}'$ and $*$ are as in Definition 11. *The* $*$ *of* $\mathbf{L}$ *is definable in* $\mathbf{L}'$ iff there is an n-ary function on the space of $\mathbf{L}'$-formulas that succeeds in representing the $*$ of $\mathbf{L}$ in $\mathbf{L}'$.

We can now give a precise statement of our main results:

THEOREM 13. *The* $\rightarrow$ *of* $\mathbf{R}_{\rightarrow}$ *is definable in* $\mathbf{S4\pi}+^{\prime p}$. (See §6.)

THEOREM 14. *The* $\rightarrow$ *of* $\mathbf{R}_{\rightarrow}$ *is not definable in* $\mathbf{S4\pi}+$. (See §7.)

§6. Theorem 13: defining relevant implication in $\mathbf{S4\pi}+^{\prime p}$.

6.1. Table 1: preliminary definitions of connectives. Tables 1 and 2 define a host of connectives in the language of $\mathbf{S4\pi}+^{\prime p}$. The headings of the last two columns of Table 1 require some explanation.

Given an **S4**-e-model N and a formula A, we can think of A as playing two semantic roles: it *names* the proposition $\mathrm{N}(A)$; and it *makes a claim* about the model. For example, the formula $\Box(\neg p \vee q)$ *names* $\mathrm{N}(\Box(\neg p \vee q))$. Furthermore it *says that* $(\mathrm{N}(p) \cap \mathrm{W}_0) \subseteq (\mathrm{N}(q) \cap \mathrm{W}_0)$. For, for any **S4**-e-model N, $\mathrm{N} \models \Box(\neg p \vee q)$ iff $(\mathrm{N}(p) \cap \mathrm{W}_0) \subseteq (\mathrm{N}(q) \cap \mathrm{W}_0)$. Kremer [14] takes advantage of the same ideas.

In light of §4.2, we take A and B to name the same proposition if they name equivalent propositions. Given an equivalence class $\{\mathrm{P}, \mathrm{Q}, \mathrm{R}, \dots\}$ of propositions, it is useful to focus on a particular representative of that class, namely $\mathrm{P} \cap \mathrm{W}_0 (= \mathrm{Q} \cap \mathrm{W}_0 = \mathrm{R} \cap \mathrm{W}_0 \dots)$. Henceforth, we think of the formula A as naming not $\mathrm{N}(A)$, but $\mathrm{N}_0(A) = \mathrm{N}(A) \cap \mathrm{W}_0$.

Finally, the table's blank entries are those of no particular interest.

6.2. Table 2: further definitions of connectives. Table 2 omits the fourth column of Table 1. In Table 2, we use the following abbreviations, where $*$ is any $n+1$-place connective:

TABLE 1. Preliminary definitions of connectives.

Definiendum	Definiens	What the definiens says: N $\models$ definiens iff	What the definiens names: N_0 (definiens) =
$(A \& B)$	$(A \vee B)$	$\mathrm{N} \models A$ and $\mathrm{N} \models B$	$\mathrm{N}_0(A) \cap \mathrm{N}_0(B)$
$\diamond A$	$\Box A$	$(\exists \mathrm{w} \in \mathrm{W}_0)(\mathrm{w} \in \mathrm{N}_0(A))$	$\{\mathrm{w} \in \mathrm{W}_0 : \exists \mathrm{w}' \geq \mathrm{w})(\mathrm{w}' \in \mathrm{N}_0(A))\}$
F	$\forall pp$		$\emptyset$
T	$\exists pp$		W_0

TABLE 2. Further definitions of connectives.

Definiendum	Definiens	What the definiens says: N $\models$ definiens iff
$(A \supset B)$	$(\neg A \vee B)$	if $\mathrm{N} \models A$ then $\mathrm{N} \models B$
$(A \equiv B)$	$((A \supset B) \& (B \supset A))$	$\mathrm{N} \models A$ iff $\mathrm{N} \models B$
$(A \mathbin{-\!\!3} B)$	$\Box(A \supset B)$	$\mathrm{N}_0(A) \subseteq \mathrm{N}_0(B)$
$(A = B)$	$((A \mathbin{-\!\!3} B) \& (B \mathbin{-\!\!3} A))$	$\mathrm{N}_0(A) = \mathrm{N}_0(B)$
$(A \in B)$	$(\neg A \mathbin{-\!\!3} \mathrm{F}) \& (A \mathbin{-\!\!3} B) \& \forall p(\neg((p \& A) \mathbin{-\!\!3} \mathrm{F}) \supset (A \mathbin{-\!\!3} p))$	$\mathrm{N}_0(A) = \{\mathrm{w}\}$ where $\mathrm{w} \in \mathrm{N}_0(B)$
$(A \leq B)$	$(A \in T) \& (B \in T) \& (A \mathbin{-\!\!3} \diamond B)$	$\mathrm{N}_0(A) = \{\mathrm{w}\}$ and $\mathrm{N}_0(B) = \{\mathrm{w}'\}$, where $\mathrm{w} \leq \mathrm{w}'$
$C\,\mathrm{ub}(A, B)$	$(A \leq C) \& (B \leq C)$	$\mathrm{N}_0(A) = \{\mathrm{w}\}, \mathrm{N}_0(B) = \{\mathrm{w}'\}$, and $\mathrm{N}_0(C) = \{\mathrm{w}''\}$ where $\mathrm{w}'' \in \mathrm{ub}(w, \mathrm{w}')$
$C\,\mathrm{lub}(A, B)$	$C\,\mathrm{ub}(A, B) \& \forall p(p\,\mathrm{ub}(A, B) \supset C \leq p)$	$\mathrm{N}_0(A) = \{\mathrm{w}\}, \mathrm{N}_0(B) = \{\mathrm{w}'\}$, and $\mathrm{N}_0(C) = \{\mathrm{w}''\}$ where $\mathrm{w}'' \in \mathrm{lub}(\mathrm{w}, \mathrm{w}')$
$\mathrm{R}(A, B, C)$	$[(\forall p \in T)\ (\forall q \in T)\ (\exists r\,\mathrm{lub}(p, q))\ (\forall s\,\mathrm{lub}(p, q))\ (s = r) \&\ C\,\mathrm{lub}(A, B)] \vee [\neg(\forall p \in T)\ (\forall q \in T)\ (\exists r\,\mathrm{lub}(p, q))\ (\forall s\,\mathrm{lub}(p, q))\ (s = r)\ \&\ C\,\mathrm{ub}(A, B)]$	$\mathrm{N}_0(A) = \{\mathrm{w}\}, \mathrm{N}_0(B) = \{\mathrm{w}'\}$, and $\mathrm{N}_0(C) = \{\mathrm{w}''\}$ where $\mathrm{R\,w\,w'\,w''}$
$C\sharp(A, B)$	$(\forall p \in T)(p \in C \equiv \forall q(\forall r \in A)(\mathrm{R}(p, r, q) \supset q \in B))$	$\mathrm{N}_0(C) = (\mathrm{N}_0(A) \rightarrow \mathrm{N}_0(B))$
$(A \rightarrow B)$	$\iota p(p\sharp(A, B))$	

$(\exists p * (A_1, \dots, A_n))B$ abbreviates $\exists p(p * (A_1, \dots, A_n) \,\&\, B)$; and
$(\forall p * (A_1, \dots, A_n))B$ abbreviates $\forall p(p * (A_1, \dots, A_n) \supset B)$,

where $\supset$ is defined in the first row.

6.3. The definition of relevant implication. The last row of Table 2 gives the definition of relevant implication. That this definition succeeds is indicated in Corollary 16, from which Theorem 13 (§5.3) follows immediately.

LEMMA 15. *If* N *is any e-frame and if A and B are formulas in the language of* **S4π+**$^{\iota p}$, *then* $\mathrm{N}_0(A \to B) = (\mathrm{N}_0(A) \to \mathrm{N}_0(B))$.

PROOF. By a straightforward appeal to the definitions in Tables 1 and 2. ⊣

COROLLARY 16. *The binary function* $\to$ *(on formulas in the language of* **S4π+**$^{\iota p}$*) defined in Table 2 succeeds in representing the* $\to$ *of* $\mathbf{R}_{\to}$ *in* **S4π+**$^{\iota p}$.

§7. Theorem 14: Relevant implication is not definable in S4π+. Theorem 14 (§5.3) is a corollary to Lemma 19, below.

LEMMA 17. *If* $A \in$ **S4π+** *then, for every* **S4***-model, A is true at every world.*

LEMMA 18. *Suppose that* M *is a model; p is a propositional variable; A and B are* **S4π+***-formulas; and* $\mathrm{M}(A) = \mathrm{P}$. *Then* $\mathrm{M}[\mathrm{P}/p](B) = \mathrm{M}(B[A/p])$.

LEMMA 19. *Suppose that* $*$ *is a binary connective-like function on the space of* **S4π+***-formulas, such that for every propositional variable, p,* $(p * p) \in$ **S4π+**. *Then, for any distinct propositional variables p and q,* $((p * p) * (q * q)) \in$ **S4π+**.

PROOF. Suppose that p and q are distinct variables. Let $\mathrm{M} = (\mathrm{W}, \leq, \mathrm{w}_0, \mathrm{V})$ be an **S4**-model. We will show that $((p * p) * (q * q))$ is true in M. Choose propositional variables r and s that are distinct and distinct from p and q. Let $\mathrm{M}' = \mathrm{M}[\mathrm{W}/s]$ and $\mathrm{M}'' = \mathrm{M}[\mathrm{W}/s][\mathrm{W}/r]$. By Lemma 17, $\mathrm{M}(q * q) = \mathrm{M}'(p * p) = \mathrm{M}''(r * r) = \mathrm{W}$. By Lemma 18, $\mathrm{M}''(r * r) = \mathrm{M}''(r * s) = \mathrm{M}'((p * p) * s) = \mathrm{M}((p * p) * (q * q))$. So $\mathrm{M}((p * p) * (q * q)) = \mathrm{W}$. So $((p * p) * (q * q))$ is true in M. ⊣

PROOF OF THEOREM 14 (§5.3). If $\to$ is definable in **S4π+** then $((p \to p) \to (q \to q)) \in$ **S4π+**, since $(p \to p) \in \mathbf{R}_{\to}$. But $((p \to p) \to (q \to q)) \notin \mathbf{R}_{\to}$. ⊣

§8. Generalising Theorem 14. Given the generality of the proof of Lemma 19 we can generalise Theorem 14: a broad range of relevant and nearly relevant $\to$'s are not definable in any of a broad range of propositionally quantified modal logics (Theorem 20).

Suppose that **L** is a *normal* modal logic, as characterised in Hughes and Cresswell [9]. All of the unquantified modal logics that we have mentioned are normal. Define

> **Lπ+** $=_{\mathrm{df}}$ $\{A : A$ is a formula in the language of **S4π+** and A is validated by every model structure that validates all of the formulas in **L**$\}$.

(The definition of **S4π+** here is slightly different but equivalent to the definition in §2. Similarly, the present definition of **Kπ+**, **Tπ+**, **K4π+**, **S4.2π+**, **Bπ+** and **S5π+** is equivalent to the definitions in Fine [4].)

THEOREM 20. *Suppose that* **I** *is a set of sentences in a quantified or unquantified propositional language with an implicational connective* $\rightarrow$, *and that* $(A \rightarrow A) \in$ **I** *for every sentence* A, *and that, for some propositional variables* p *and* q, $((p \rightarrow p) \rightarrow (q \rightarrow q)) \notin$ **I**. (**I** could be any relevance logic, or one of numerous strong pseudo-relevance logics like **RM** or even **RM3**, which Meyer [17] calls "just a shade off being classical".) *Then the* $\rightarrow$ *of* **I** *cannot be defined in* $\mathbf{L}\pi+$.

Theorem 20 is in some ways broader and in some ways narrower than the result of Meyer [17]. Theorem 20 is broader than Meyer [17]. First, Meyer considers only definitions of $\rightarrow$ as modalised truth functions, while Theorem 20 concerns definitions much more broadly conceived. Second, Meyer considers defining $\rightarrow$ in unquantified modal systems, while we consider defining $\rightarrow$ in propositionally quantified modal systems. And third, Meyer places stronger restrictions on the behaviour of the newly defined connective $\rightarrow$: in effect, he insists that, for a definition to have succeeded, the $\neg \vee \& \rightarrow$ fragment of the modal logic **L** be equal to the $\sim \vee \& \rightarrow$ fragment of the relevance logic, where $\neg$ and $\sim$ are identified. But we insist only that the $\rightarrow$ fragments be equal.

Theorem 20 is narrower than Meyer [17]. Meyer puts very weak conditions on the propositional modal system **L**: they are (1) that **L** contain all of the classical propositional tautologies; and (2) that once $\rightarrow$ has been defined, if $((A \rightarrow B)\&(B \rightarrow A)) \in$ **L** then, for any formula C and any propositional variable p, $(C[A/p] \rightarrow C[B/p]) \in$ **L**. We insist that **L** be a normal modal system, although we do not insist on Meyer's condition (2). Open question: can our conditions on **L** be weakened?

§9. Concluding remarks.

9.1. Technical remarks. (1) The proof of Theorem 13 goes through for $\mathbf{S4.2}\pi+^{\prime p}$ as well as for $\mathbf{S4}\pi+^{\prime p}$. Open question: is $\rightarrow$ definable in $\mathbf{K}\pi+^{\prime p}$, $\mathbf{T}\pi+^{\prime p}$, $\mathbf{K4}\pi+^{\prime p}$, $\mathbf{B}\pi+^{\prime p}$, or $\mathbf{S5}\pi+^{\prime p}$? Conjecture: yes in $\mathbf{K}\pi+^{\prime p}$, $\mathbf{T}\pi+^{\prime p}$, $\mathbf{K4}\pi+^{\prime p}$ and $\mathbf{B}\pi+^{\prime p}$; and no in $\mathbf{S5}\pi+^{\prime p}$. Open question: is $\rightarrow$ is definable in any $\mathbf{L}\pi$, as opposed to $\mathbf{L}\pi+$? Conjecture: no.

(2) The relevance logic with the closest affinity to **S4** has been taken to be Anderson and Belnap's **E** rather than **R**. Indeed, the pure implicational fragment for **E** is motivated in part by a natural deduction system which combines a natural deduction system for the $\multimap\!\!3$ of **S4** with a natural deduction system for the $\rightarrow$ of **R**. (Anderson and Belnap [1, pp. 1–26].) Furthermore, Urquhart's [21] semantics for $\mathbf{E}_{\rightarrow}$ combines the semilattice semantics for $\mathbf{R}_{\rightarrow}$ with Kripke's semantics for **S4**. But we have relied upon a semantic connection between **R** and **S4** rather than between **E** and **S4**. Open question: is the $\rightarrow$ of **E** definable in $\mathbf{S4}\pi+^{\prime p}$? Conjecture: no.

(3) Our treatment of definite propositional descriptions is motivated by a Fregean treatment of definite individual descriptions. But other treatments of definite individual descriptions have been proposed. (Carnap [3] considers the proposals of Russell [19] and [20] and of Hilbert and Bernays [7], and compares them to the Fregean proposal.) These proposals *agree* regarding formulas that contain only *proper* descriptions. In our definition of $\rightarrow$ (§6.2, Table 2) the only pertinent definite propositional descriptions are *proper in every model*. So our definition of $\rightarrow$

should work in the presence of *any* well-motivated treatment of definite propositional descriptions, since any such treatment will agree with our Fregean treatment in the pertinent cases.

9.2. Methodological remarks. Many issues upon which we have touched call out for further elaboration. To what extent does $\mathbf{S4\pi+}$ capture the notion of propositional quantification? Can the notion of definite propositional description be fruitfully elaborated? Can a general theory of connective definition and connective identification across logics be developed?

Regarding this last question, we note (with a little dismay) that, although the $\to$ fragment of $\mathbf{S4\pi+}^{\iota p}$ is equal to $\mathbf{R}_{\to}$, things do not work out so nicely with larger fragments. $(\mathbf{S4\pi+}^{\iota p})_{\to\&} = \mathbf{R}_{\to\&}$, but $(\mathbf{S4\pi+}^{\iota p})_{\to\&\vee} \neq \mathbf{R}_{\to\&\vee}$. So, while we have expressed, in $\mathbf{S4\pi+}^{\iota p}$, relevant implication in isolation from other connectives, we have not expressed its interactions with all of the other connectives.

That $(\mathbf{S4\pi+}^{\iota p})_{\to\&\vee} \neq \mathbf{R}_{\to\&\vee}$ is only damaging to the extent that the $\vee$ of $\mathbf{S4\pi+}^{\iota p}$ can be identified with the $\vee$ of $\mathbf{R}$. But the use of the same symbol '$\vee$' is not enough to identify the two connectives. We may identify them at an intuitive level: both connectives are meant to formalise the same pre-theoretic notion of disjunction. But this is not sufficient grounds for identifying two formal connectives. For the claim that we can have rival formalisations of pre-theoretic notions suggests that the same intuitive notion can be formalised by *distinct* formal connectives. What is wanting here is a more complete theory of cross-logic identification of connectives.

While we are on the topic, we note (with a little more dismay) the way $\to$ interacts, in $\mathbf{S4\pi+}^{\iota p}$, with $\Box$. The following are, as expected, theorems: $(\Box A \to A)$, $(\Box A \to \Box\Box A)$ and $\Box(A \to B) \to (\Box A \to \Box B)$. But, though $(p \to p)$ is a theorem, $\Box(p \to p)$ is not. Indeed, there are models in which $(p \to p)$ is true only at the actual world. This calls for further thought.

Despite these drawbacks, we maintain that it is interesting that the pure theory of relevant implication can be represented, in a strong and interesting sense, in an independently motivated extension of **S4**.

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