Quantified S4 in the Lebesgue measure algebra with a constant countable domain

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Abstract

Define quantified S4, QS4 [first-order S4, FOS4], by combining the axioms and rules of inference of propositional S4 with the axioms and rules of classical first order logic without identity [with identity]. In the 1950's, Rasiowa and Sikorski extended the algebraic semantics for propositional S4 to a constant-domain algebraic semantics for QS4, and showed that QS4 is sound and complete for this semantics. Recently, Lando has extended the algebraic semantics for propositional S4 to an expanding-domain algebraic semantics FOS4. Her main result is that FOS4 is complete for an algebra of particular interest, the Lebesgue measure algebra, with expanding countable domains. In the current paper, we show that QS4, without identity, is complete for the Lebesgue measure algebra with a *constant* countable domain. One takeaway is that measure-theoretic semantics might need varying domains to handle identity but does not need them to handle quantification.

Keywords: Quantified modal logic, topological semantics, algebraic semantics, Lebesgue measure algebra.

The three dominant semantics for the propositional modal logic S4 are the algebraic semantics, which generalizes the topological semantics, which in effect generalizes the Kripke semantics. The algebraic semantics, in particular, interprets the modal language in *topological Boolean algebras* also known as *interior algebras*, i.e., Boolean algebras equipped with an interior operator. Define quantified S4, QS4, by combining the axioms and rules of

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inference of S4 with the axioms and rules of classical first order logic without identity. In [15], Rasiowa and Sikorski extend the algebraic and topological semantics to QS4, with a constant domain for the quantifiers. It is well-known that, to get completeness in the Kripke semantics for QS4, we need expanding domains, i.e., domains that expand along the accessibility relation. In particular, Kripke semantics with constant domains validates the converse Barcan formula, $(\forall \mathsf{x}\Box \mathsf{Px} \supset \Box\forall \mathsf{x}\mathsf{Px})$, even though it is not a theorem of QS4. Define first-order S4, FOS4, by extending QS4 with standard axioms for identity. In all three semantics, to invalidate non-theorems such as $\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} \neq \mathsf{y} \supset \Box \mathsf{x} \neq \mathsf{y})$, we need varying domains (or some device to that effect) as well as some device for handling contingent nonidentity.

Beginning with [13] and [14], completeness results have been given not only for general classes of algebras, topological spaces, or Kripke frames, but for particular cases. Thus, propositional S4 is complete not only for the class of all topological spaces, but also for particular spaces such as the real line, the rational line, and Cantor space – indeed, for any dense-in-itself metric space ([13, 14, 15]). We cite four more recent results for propositional S4: S4 is complete for the binary tree conceived of as a Kripke frame ([6]); S4 is complete for the remainder space $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone space of the Boolean algebra $\mathcal{P}(\mathbb{N})$ ([1]); S4 is complete for the infinite binary tree with limits, equipped with a natural topology ([12, 8]); and, of particular interest to the current paper, S4 is complete for the Lebesgue measure algebra, $\mathcal{M}$, i.e., the algebra of Lebesgue-measurable subsets of the real interval [0,1] modulo sets of Lebesgue-measure zero, equipped with a suitable interior operator ([10]).

For QS4 and FOS4, there are fewer case-specific results. In the constant-domain semantics of [15], QS4 is complete for the rational line with a countable domain ([9]). And Lando [11] extends the constant-domain algebraic semantics for QS4 of [15] to an expanding-domain algebraic semantics for FOS4 (without constants and function symbols), and shows that FOS4 is complete for $\mathcal{M}$ with expanding countable domains. Thus, trivially, QS4 (without constants and function symbols) is complete for $\mathcal{M}$ with expanding countable domains. Our main result here: QS4 is complete for $\mathcal{M}$ with a *constant* countable domain. This gives a positive answer to Open Question 10.5 in [11].

Lando uses "measure-theoretic semantics" for algebraic semantics restricted to algebras, like $\mathcal{M}$, that are measure algebras as well as topological Boolean algebras. One takeaway from the current paper is that measure-theoretic semantics might need varying domains to handle identity but does not need them to handle quantification. In this way, measure-theoretic semantics is like both algebraic semantics (of which it is a special case) and topological semantics, and unlike Kripke semantics. Not surprising, but it's nice to have a proof.

1 Preliminaries

Let $\mathcal{L}$ be a quantified modal language with a countably infinite set Var of variables; disjoint countable sets Pred_n of n-ary predicate symbols, for each $n \geq 1$; a set Names of names; disjoint countable sets Func_n of n-ary function symbols, for each $n \geq 1$; connective &, $\vee$ and $\neg$; a modal operator $\Box$; a quantifier $\forall$; and parentheses. We write $\Diamond A$ for $\neg\Box\neg A$, $(A \supset B)$ for $(\neg A \vee B)$, and $\exists \mathsf{x} A$ for $\neg\forall\mathsf{x}\neg A$. Let $\mathsf{Pred} = \bigcup_n \mathsf{Pred}_n$ and $\mathsf{Func} = \bigcup_n \mathsf{Func}_n$; we assume that Pred is nonempty. Note that $\mathcal{L}$ has no equals sign. If A is a formula, t is a term, and x is a variable, then $[\mathsf{t}/\mathsf{x}]A$ is the result of replacing every free occurrence of x in A with t. We say that t is *substitutible for* x *in* A iff no free occurrence of x in A is in the scope of any bound variable y, where y occurs in t. Given any set D, *D-terms*, *D-formulas* and *D-sentences* are terms, formulas and sentences in the language $\mathcal{L}(D)$, which is the result of expanding the language $\mathcal{L}$ so that every member of the set D is a name of $\mathcal{L}$. (Here we assume that $D \cap S = \emptyset$, if $S = \mathsf{Var}$, Pred, Names or Func.) It will be useful to let $\mathsf{Term}(D)$ be the set of closed D-terms. Note that, given any D-formula A, any variable x and any $d \in D$, the D-sentence $[d/\mathsf{x}]A$ is the result of replacing every occurrence of x in A with d. We reserve the unprefixed expressions 'formula(s)' and 'sentence(s)' for formulas and sentences in the original language $\mathcal{L}$.

Let QS4 be the logic axiomatized in $\mathcal{L}$ as follows:

- Axioms and axiom schemes:
  - every instance, in $\mathcal{L}$, of a theorem of propositional S4;
  - $(\forall \mathsf{x} A \to A[\mathsf{t}/\mathsf{x}])$, where the term t is substitutible for x in A;
  - $\forall \mathsf{x}(A \to B) \to (\forall \mathsf{x} A \to \forall x B)$; and
  - $A \to \forall \mathsf{x} A$, where x does not occur free in A.
- Rules: modus ponens, necessitation and universal generalization.

Note: It is well-known that ($\Box\forall\mathsf{x}\mathsf{Px} \supset \forall\mathsf{x}\Box\mathsf{Px}$) $\in$ QS4, but ($\forall\mathsf{x}\Box\mathsf{Px} \supset \Box\forall\mathsf{x}\mathsf{Px}$) $\notin$ QS4.

2 Algebraic semantics (constant domain)

The semantics in this section is a terminological/notational variant of the constant-domain algebraic semantics for quantified modal logic found in [15], liberalized a bit – see Remark 2.1. We assume familiarity with Boolean algebras and topological spaces. We often denote the top element of a Boolean algebra by 1, and the bottom by 0, and we always assume that $1 \neq 0$. We denote meet, join, and complementation by $\wedge$, $\vee$ and $-$. A Boolean algebra $\mathcal{A}$ is *complete* [σ-*complete*] iff it is closed under arbitrary joins [countable joins]: i.e., iff every $S \subseteq \mathcal{A}$ [every countable $S \subseteq \mathcal{A}$] has a least upper bound (lub) in $\mathcal{A}$, denoted $\bigvee S$ or $\bigvee_{a \in S} a$. It is obvious that this definition could have been given in terms of meets and greatest lower bounds (glb's), since a Boolean algebra $\mathcal{A}$ is complete [σ-complete] iff it is closed under arbitrary meets [countable meets]: i.e., iff every $S \subseteq \mathcal{A}$ [every countable $S \subseteq \mathcal{A}$] has a glb in $\mathcal{A}$, denoted $\bigwedge S$ or $\bigwedge_{a \in S} a$. A *topological Boolean algebra* (TBA) is a Boolean algebra $\mathcal{A}$ equipped with an *interior* operator I on $\mathcal{A}$ that satisfies, for any $a, b \in \mathcal{A}$,

$$\begin{array}{ll} \text{(I1)} & I1 = 1 \\ \text{(I2)} & Ia \leq a \\ \text{(I3)} & I(a \wedge b) = Ia \wedge Ib \\ \text{(I4)} & IIa = Ia \end{array}$$

A standard example is to start with any topological space X; take the Boolean algebra $\mathcal{P}(X)$ where joins, meets, and complements are intersections, unions, and set-theoretic complements; and let the interior operator I be topological interior. We follow [11] in denoting this TBA $\mathcal{B}(X)$. Note that $\mathcal{B}(X)$ is complete.

A *predicate TBA* is an ordered pair $\langle \mathcal{A}, D \rangle$, where $\mathcal{A}$ is a TBA and D is a nonempty set – thought of as a domain for the quantifiers. A *valuation over* $\langle \mathcal{A}, D \rangle$ is a function

$$V : \mathsf{Pred} \cup \mathsf{Names} \cup \mathsf{Func} \to \bigcup_{n \geq 1} \mathcal{A}^{D^n} \cup D \cup (\bigcup_{n \geq 1} D^{D^n})$$

such that

- $V(\mathsf{P}) : D^n \to \mathcal{A}$ for every $\mathsf{P} \in \mathsf{Pred}_n$,
- $V(\mathsf{c}) \in D$ for every $\mathsf{c} \in \mathsf{Names}$, and
- $V(\mathsf{f}) : D^n \to D$ for every $\mathsf{f} \in \mathsf{Func}_n$.

Given a valuation V over $\langle \mathcal{A}, D \rangle$, we define $\mathit{Val}(\mathsf{t}) \in D$ for every closed D-term t: $\mathit{Val}(d) = d$, for $d \in D$; $\mathit{Val}(\mathsf{c}) = V(\mathsf{c})$, for $\mathsf{c} \in \mathsf{Names}$; and $\mathit{Val}(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n) = V(\mathsf{f})(\mathit{Val}(\mathsf{t}_1), \ldots, \mathit{Val}(\mathsf{t}_n))$, for $\mathsf{f} \in \mathsf{Func}_n$ and D-terms t_1, $\ldots$, t_n.

An algebraic *model* is an ordered triple $\mathfrak{A} = \langle \mathcal{A}, D, V \rangle$, where $\langle \mathcal{A}, D \rangle$ is a predicate TBA and V is a valuation such that the sixth clause in the following definition of $\mathit{Val}(A)$ for each D-sentence A is feasible, i.e., the definiens $\bigwedge_{d \in D} \mathit{Val}([d/\mathsf{x}]A)$ always exists:

$$\begin{array}{rcl}
\mathit{Val}(\mathsf{P}t_1 \ldots t_n) & = & V(\mathsf{P})(\mathit{Val}(t_1), \ldots, \mathit{Val}(t_n)) \\
\mathit{Val}(A \,\&\, B) & = & \mathit{Val}(A) \wedge \mathit{Val}(B) \\
\mathit{Val}(A \vee B) & = & \mathit{Val}(A) \vee \mathit{Val}(B) \\
\mathit{Val}(\neg A) & = & -\mathit{Val}(A) \\
\mathit{Val}(\Box A) & = & I\,\mathit{Val}(A) \\
\mathit{Val}(\forall \mathsf{x} A) & = & \bigwedge_{d \in D} \mathit{Val}([d/\mathsf{x}]A)
\end{array}$$

Remark 2.1. In the main presentation of the algebraic semantics for QS4 in [15] and elsewhere, algebraic models are based on *complete* TBAs, rather than on TBAs in general: this ensures that the sixth clause in the definition of $\mathit{Val}(A)$ is always feasible, regardless of the domain D and the valuation V. As noted in [15], pp. 235–236, "the hypothesis that $\mathcal{A}$ is complete was assumed only in order to assure that all infinite operations appearing in the inductive definition of [$\mathit{Val}(A)$] are feasible... However, it may happen that, for a given [valuation V], ... all infinite operations appearing in the definition of [$\mathit{Val}(A)$] are feasible, in spite of the fact that ... $\mathcal{A}$ is incomplete." We follow [15], p. 236, in allowing algebraic models based on incomplete TBAs.

A sentence A of $\mathcal{L}$ is *valid* in an algebraic model $\mathfrak{A} = \langle \mathcal{A}, D, V \rangle$ iff $\mathit{Val}(A) = 1$. A is *valid* in a predicate TBA $\langle \mathcal{A}, D \rangle$ iff A is valid in every algebraic model $\langle \mathcal{A}, D, V \rangle$. A is *valid* in a TBA $\mathcal{A}$ iff A is valid in $\langle \mathcal{A}, D \rangle$ for every domain D. A is *valid* in a class of [predicate] TBAs iff it is valid in every [predicate] TBA in that class. QS4 is *sound* for $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$, $\langle \mathcal{A}, D, V \rangle$] iff every theorem of QS4 (in $\mathcal{L}$) is valid in $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$, $\langle \mathcal{A}, D, V \rangle$]. Ditto for soundness for a class of [predicate] TBAs. QS4 is *complete* for $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$,

$\langle\mathcal{A}, D, V\rangle$] iff every sentence of $\mathcal{L}$ valid in $\mathcal{A}$ [$\langle\mathcal{A}, D\rangle$, $\langle\mathcal{A}, D, V\rangle$] is a theorem of QS4. Ditto for completeness for a class of [predicate] TBAs. Finally, QS4 is complete for $\mathcal{A}$ with a constant countable domain iff there is some countable domain D such that QS4 is complete for $\langle\mathcal{A}, D\rangle$.

We can now define the topological semantics as a special case of the algebraic semantics. In these terms, a *topological model* is simply an algebraic model of the form $\langle\mathcal{B}(X), D, V\rangle$, where X is a topological space. We will inaccurately write $\langle X, D, V\rangle$ instead of $\langle\mathcal{B}(X), D, V\rangle$, and talk about a sentence A being valid in the topological space X instead of in $\mathcal{B}(X)$, and QS4 being sound, complete, etc., for X [with a countable domain] instead of $\mathcal{B}(X)$ [with a countable domain]. It will be useful to also define a *predicate topological space* (henceforth, *predicate space*) to be an ordered pair $\langle X, D\rangle$, where X is a topological space and D is a nonempty domain. It will also be useful to define $\mathfrak{A}, x \Vdash A$, for any topological model $\mathfrak{A} = \langle X, D, V\rangle$ and any D-sentence A as follows:

$$\begin{array}{rcl}
\mathfrak{A}, x \Vdash \mathsf{Pt}_1 \ldots \mathsf{t}_n & \text{iff} & x \in V(\mathsf{P})(\mathit{Val}(\mathsf{t}_1), \ldots, \mathit{Val}(\mathsf{t}_n)), \text{ where } \mathsf{P} \in \mathsf{Pred}_n \\
\mathfrak{A}, x \Vdash \neg A & \text{iff} & \mathfrak{A}, x \nVdash A \\
\mathfrak{A}, x \Vdash (A \;\&\; B) & \text{iff} & \mathfrak{A}, x \Vdash A \text{ and } \mathfrak{A}, x \Vdash B \\
\mathfrak{A}, x \Vdash (A \vee B) & \text{iff} & \mathfrak{A}, x \Vdash A \text{ or } \mathfrak{A}, x \Vdash B \\
\mathfrak{A}, x \Vdash \Box A & \text{iff} & \text{for some open } O \subseteq X, \\
& & x \in O \text{ and for every } x' \in O, \mathfrak{A}, x' \Vdash A \\
\mathfrak{A}, x \Vdash \forall \mathsf{x} A & \text{iff} & \text{for every } d \in D, \mathfrak{A}, x \Vdash [d/\mathsf{x}]A
\end{array}$$

Note that $\mathit{Val}(A) = \{x \in X : \mathfrak{A}, x \Vdash A\}$.

It is routine to show that QS4 is sound for any [predicate] TBA. As for completeness, an early result in [15] is that there is some set X of irrationals such that QS4 is complete for X with a countable domain (Chapter XI, Proposition 10.2 (v)). More recently, the main result of [9] is that QS4 is complete, indeed strongly complete,[1] for $\mathbb{Q}$ with a countable domain. We define the *Lebesgue measure algebra* $\mathcal{M}$ in Section 2.1, below. Our main result is

Theorem 2.2. QS4 *is complete for* $\mathcal{M}$ *with a countable domain.*

Note that this result is both weaker and stronger than the main result in [11], that FOS4 is complete for $\mathcal{M}$ with expanding countable domains: the result is weaker since it only concerns the identity-free fragment of FOS4 (though,

[1]See Section 7 for a definition of strong completeness.

unlike [11], we also have names and function symbols); but it is stronger because here we have completeness in the *constant-domain* semantics of [15], rather than in an expanding-domain semantics.

We close this section with a definition of $\mathcal{M}$ and related algebras; and with a discussion of subalgebras.

2.1 The Lebesgue measure algebra, $\mathcal{M}$

Above, we characterized $\mathcal{M}$ as the algebra of Lebesgue-measurable subsets of [0,1], modulo sets of Lebesgue measure zero, a characterization taken verbatim from [10]. An alternate characterization is of $\mathcal{M}$ as the algebra of Borel subsets of [0,1], modulo sets of measure zero – see [11]. More generally, we will define an algebra $\mathcal{M}(X)$ for any Borel subset X of $[0,1]$ of measure 1.

For any such X, let $\mathit{Borel}(X)$ be the Boolean algebra of sets that are Borel in X. Note that each set that is Borel in X is also Borel in $[0,1]$ and hence Lebesgue-measurable. Let $\mathit{Null}(X) = \{N \in \mathit{Borel}(X) : \lambda(N) = 0\}$, where λ is the Lebesgue measure. Thus, $\mathit{Null}(X)$ is a σ-ideal in $\mathit{Borel}(X)$. Consider the quotient algebra, $\mathcal{Q}(X) = \mathit{Borel}(X)/\mathit{Null}(X)$, which can be defined as follows. First, for $S, S' \in \mathit{Borel}(X)$, say that $S \sim S'$ iff $((S - S') \cup (S' - S)) \in \mathit{Null}(X)$. The relation $\sim$ is an equivalence relation. Let $|S| =_{\mathrm{df}} \{S' \in \mathit{Borel}(X) : S' \sim S\}$. The quotient algebra, $\mathcal{Q}(X)$, is the set of equivalences classes, with joins, meets and complements defined in terms of the underlying sets:

$$\begin{aligned} |S| \wedge |S'| &= |S \cap S'| \\ |S| \vee |S'| &= |S \cup S'| \\ -|S| &= |X - S| \end{aligned}$$

It is easy to check that these operations are well-defined, and that $\mathcal{Q}(X)$ is a Boolean algebra, with top and bottom elements $|X|$ and $|\emptyset|$. We can define a measure on $\mathcal{Q}(X)$ in the obvious way: for any Borel $S \subseteq X$, let $\mu(|S|) = \lambda(S)$, where λ is the Lebesgue measure. It can be checked that $\mathcal{Q}(X)$ together with μ is a measure algebra.[2] Note that $\mathcal{Q}(X)$ is a *complete* Boolean algebra: the proof is the same as the proof of Proposition 3.8 in [10], with $\mathcal{M}$ there replaced by $\mathcal{Q}(X)$.

[2]A *measure algebra* is a Boolean algebra $\mathcal{A}$ together with a countably additive positive measure, i.e., a function $\mu : \mathcal{A} \to [0,1]$ such that $\mu(0) = 0$; $\mu(1) = 1$; $\mu(a) > 0$ if $a \neq 0$; and $\mu(\bigwedge_{a \in S} a) = \sum_{a \in S} \mu(a)$, where $S \subseteq \mathcal{A}$ is countable and its members are disjoint, i.e., $a \wedge b = 0$ whenever $a, b \in S$ and $a \neq b$.

The TBA $\mathcal{M}(X)$ is the quotient algebra $\mathcal{Q}(X)$, equipped with an interior operator defined as follows:

$$I|S| \quad = \quad \bigvee\{|O| : |O| \leq |S| \text{ and } O \text{ is open in } X\}$$

The completeness of $\mathcal{Q}(X)$ assures us that the meet in the definition of $I|S|$ exists. By Proposition 3.12 in [10], I is an interior operator. (The proof in [10] is given only when $X = [0,1]$, but goes through in the general case.) By Proposition 11.2 in [11], the TBA $\mathcal{M}(X)$ is isomorphic to the TBA $\mathcal{M}([0,1])$ (see page 13, below, for a definition of isomorphic TBAs). Finally, we simply define $\mathcal{M}$ as $\mathcal{M}([0,1])$.

2.2 Subalgebras

A TBA $\mathcal{A}'$ is a *subalgebra* of $\mathcal{A}$, in symbols $\mathcal{A}' \sqsubseteq \mathcal{A}$, iff $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{A}'$ is closed under finite meets, finite joins, complements, and interiors (consequently, $0, 1 \in \mathcal{A}'$). Note that $\mathcal{A}'$ is then a TBA in its own right. Suppose that $\mathcal{A}$ is a complete TBA and $\mathcal{A}' \sqsubseteq \mathcal{A}$: $\mathcal{A}'$ is *regular* [σ-*regular*] iff for every subset [every countable subset] $S \subseteq \mathcal{A}'$, we have $\bigvee_{a \in S} a \in \mathcal{A}'$, where $\bigvee_{a \in S} a$ is the lub of S in $\mathcal{A}$.[3] Note that if $\mathcal{A}'$ is a regular [σ-regular] subalgebra of $\mathcal{A}$ and S is an arbitrary [countable] subset of $\mathcal{A}'$, then $\bigvee_{a \in S} a$ is then also the lub of S in $\mathcal{A}'$, so that $\mathcal{A}'$ is complete [σ-complete]. Also note that this definition could equivalently be given in terms of glb's instead of lub's.

The following lemma will be useful:

Lemma 2.3. *Suppose that $\mathfrak{A} = \langle \mathcal{A}, D, V \rangle$ is an algebraic model, that $\mathcal{A}' \sqsubseteq \mathcal{A}$, and that $Val(A) \in \mathcal{A}'$ for every D-sentence A. Then $\mathfrak{A}' = \langle \mathcal{A}', D, V \rangle$ is also an algebraic model, and $Val'(A) = Val(A)$ for every D-sentence A, where $Val'(A)$ is the value of A calculated in $\mathfrak{A}'$.*

Proof. It suffices to show that $Val'(A)$ is defined and indeed that $Val'(A) = Val(A)$, for every D-sentence A. Suppose not, and let A be a D-sentence of least complexity (fewest connectives and quantifiers) such that either $Val'(A)$ is undefined or $Val'(A) \neq Val(A)$. Clearly, A is not atomic nor of the form

[3] Definitions of *regular* subalgebras are usually given for Boolean algebras and not for topological Boolean algebras, but clearly we can apply these definitions to the latter. The definition of *regular* we are using is from [16]; the literature contains non-equivalent definitions, e.g., the definition in [5]: there $\mathcal{A}'$ is called *regular* iff whenever a subset S of $\mathcal{A}'$ has a lub a in $\mathcal{A}'$, a is the lub of S in $\mathcal{A}$ as well. Note that, if a subalgebra is regular in the sense of [16], then it is also regular in the sense of [5] but not vice versa.

$B \mathbin{\&} C$, $B \vee C$, $\neg B$, or $\Box B$. So A is of the form $\forall x B$. For any $S \subseteq \mathcal{A}'$, we will use $\bigwedge S$ for the glb in $\mathcal{A}$ of members of S, if it exists; and $\bigwedge' S$ for the glb in $\mathcal{A}'$ of members of S, if it exists. Note that $Val(A) = \bigwedge_{d \in D} Val([d/\mathsf{x}]B)$. Our task is to show that $\bigwedge'_{d \in D} Val'([d/\mathsf{x}]B)$ exists and $Val(A) = \bigwedge'_{d \in D} Val'([d/\mathsf{x}]B)$.

Since A is a D-sentence of *least* complexity such that either $Val'(A)$ is undefined or $Val'(A) \neq Val(A)$, we have $Val'([d/\mathsf{x}]B) = Val([d/\mathsf{x}]B)$ for every $d \in D$. Note $Val(A) \in \mathcal{A}'$, by assumption. Also, $Val(A)$ is the glb, in $\mathcal{A}$, of the set $\{ Val([d/\mathsf{x}]B) : d \in D\}$; and $\{ Val([d/\mathsf{x}]B) : d \in D\} \subseteq \mathcal{A}'$. So $Val(A)$ is the glb, in $\mathcal{A}'$, of the set $\{ Val([d/\mathsf{x}]B) : d \in D\} = \{ Val'([d/\mathsf{x}]B) : d \in D\}$. So $Val(A) = \bigwedge'_{d \in D} Val'([d/\mathsf{x}]B) = Val'(A)$. $\Box$

Given an algebraic model $\mathfrak{A} = \langle \mathcal{A}, D, V \rangle$, it will be useful below to define a subalgebra $\mathcal{A}_{val}$ of $\mathcal{A}$ as follows: $\mathcal{A}_{val} =_{\text{df}} \{ Val(A) : A \text{ is a } D\text{-sentence}\}$. It is easy to check that $\mathcal{A}_{val}$ is indeed a subalgebra of $\mathcal{A}$. Note that the valuation V is a valuation over $\langle \mathcal{A}_{val}, D \rangle$ as well as over $\langle \mathcal{A}, D \rangle$. By Lemma 2.3, $\mathfrak{A}_{val} =_{\text{df}} \langle \mathcal{A}_{val}, D, V \rangle$ is an algebraic model, and the value of any D-sentence is the same in $\mathfrak{A}_{val}$ as in $\mathfrak{A}$.

3 Kripke semantics

Prospectus. In a slightly different setting, [11] achieves the completeness of FOS4 for $\mathcal{M}$ with expanding countable domains by a series of completeness transfers: from the class of countable rooted Kripke frames to a particular countable rooted Kripke frame, the infinite binary tree; from the infinite binary tree to a complete subalgebra of $Borel(X_0)$ for a particularly useful $X_0 \subseteq [0, 1]$; from that subalgebra to $\mathcal{M}(X_0)$; and finally from $\mathcal{M}(X_0)$ to $\mathcal{M}$. We will proceed in similar fashion, with some necessary adjustments. Because we are working in an algebraic semantics with constant domains, we will also adapt a strategy from [9], used there to show that QS4 is complete for the rational line $\mathbb{Q}$ with a constant countable domain. To get this all off the ground, we start with the Kripke semantics.

A *Kripke frame* is an ordered pair $\mathcal{K} = \langle W, R \rangle$, where W is a nonempty set and R is a reflexive transitive relation on W.[4] We say that $r \in W$ is a *root* of $\mathcal{K}$ iff $\forall w \in W, rRw$. We say that $\mathcal{K}$ is *rooted* iff $\mathcal{K}$ has at least one root. Given $w \in W$, $R(w) =_{\text{df}} \{w' \in W : wRw'\}$. A set $S \subseteq W$ is *open* iff $(\forall w \in S)(\forall w' \in W)(\text{ if } wRw' \text{ then } w' \in S)$. Note that the open subsets of a

[4] We include reflexivity and transitivity since we are interested in quantified S4.

Kripke frame form a topology, for which the family $\{R(w)\}_{w \in W}$ form a basis: the resulting topological space is an *Alexandrov* space, i.e., the intersection of arbitrarily many open sets is open. It is easy to show that every Alexandrov space is thus related to a Kripke frame. Henceforth, we will simply identify each Kripke frame with the related Alexandrov space.

A [rooted] *predicate frame* is an ordered triple $\mathbf{K} = \langle W, R, D \rangle$, where $\mathcal{K} = \langle W, R \rangle$ is a [rooted] frame and D is a family, $(D_w)_{w \in W}$, of nonempty domains indexed by possible world in W, such that $wRw' \Rightarrow D_w \subseteq D_{w'}$. This last clause is a requirement that the domains be *expanding* along the accessibility relation. We say that $\mathbf{K}$ is *countable* iff W is countable and each D_w is countable. Let $D_W =_{\text{df}} \bigcup_{w \in W} D_w$.

A *frame model* is an ordered quartuple $\mathfrak{A} = \langle W, R, D, V \rangle$, where $\mathbf{K} = \langle W, R, D \rangle$ is a predicate frame, and

$$V : \mathsf{Pred} \cup \mathsf{Names} \cup \mathsf{Func} \to \bigcup_{n \geq 1} \mathcal{P}(W)^{{D_W}^n} \cup D_W \cup (\bigcup_{n \geq 1} {D_W}^{{D_W}^n})$$

is such that

- $V(\mathsf{P}) : {D_W}^n \to \mathcal{P}(W)$ for every $\mathsf{P} \in \mathsf{Pred}_n$,
- if $w \in V(\mathsf{P})(d_1, \ldots, d_n)$ then $d_1, \ldots, d_n \in D_w$, for every $\mathsf{P} \in \mathsf{Pred}_n$, every $w \in W$, and every $d_1, \ldots, d_n \in D_W$.
- $V(\mathsf{c}) \in D_w$ for every $\mathsf{c} \in \mathsf{Names}$ and every $w \in W$, and
- $V(\mathsf{f}) : {D_W}^n \to D_W$ for every $\mathsf{f} \in \mathsf{Func}_n$, and
- $V(\mathsf{f})(d_1, \ldots, d_n) \in D_w$ for every $\mathsf{f} \in \mathsf{Func}_n$, every $w \in W$, and every $d_1, \ldots, d_n \in D_w$.

We say that $\mathfrak{A}$ is *based on* $\mathbf{K}$.

Suppose that $\mathfrak{A} = \langle W, R, D, V \rangle$ is a frame model. We define $\mathit{Val}(\mathsf{t}) \in D_W$ for every closed D_W-term t exactly as in the algebraic semantics. Next, we define $\mathfrak{A}, w \Vdash A$, for each $w \in W$ and each D_w-sentence A as follows:

$$\begin{array}{rcl}
\mathfrak{A}, w \Vdash \mathsf{P}\mathsf{t}_1 \ldots \mathsf{t}_n & \text{iff} & w \in V(\mathsf{P})(\mathit{Val}(\mathsf{t}_1), \ldots, \mathit{Val}(\mathsf{t}_n)), \text{ where } \mathsf{P} \in \mathsf{Pred}_n \\
\mathfrak{A}, w \Vdash \neg A & \text{iff} & \mathfrak{A}, w \nVdash A \\
\mathfrak{A}, w \Vdash (A \,\&\, B) & \text{iff} & \mathfrak{A}, w \Vdash A \text{ and } \mathfrak{A}, w \Vdash B \\
\mathfrak{A}, w \Vdash (A \vee B) & \text{iff} & \mathfrak{A}, w \Vdash A \text{ or } \mathfrak{A}, w \Vdash B \\
\mathfrak{A}, w \Vdash \Box A & \text{iff} & \text{for every } w' \in W, \text{ if } wRw' \text{ then } \mathfrak{A}, w' \Vdash A \\
\mathfrak{A}, w \Vdash \forall \mathsf{x} A & \text{iff} & \text{for every } d \in D_w, \mathfrak{A}, w \Vdash [d/\mathsf{x}]A
\end{array}$$

For any D_W-sentence A, it will be useful below to define $val(A) = \{w \in W : \mathfrak{A}, w \Vdash A\}$.

A sentence A of $\mathcal{L}$ is *valid* in $\mathfrak{A}$ iff $\mathfrak{A}, w \Vdash A$ for every $w \in W$. A is *valid* in a predicate frame $\langle W, R, D\rangle$ iff A is valid in $\langle W, R, D, V\rangle$ for every valuation V. A is *valid* in a Kripke frame $\langle W, R\rangle$ iff A is valid in $\langle W, R, D\rangle$ for every domain D. A is *valid* in a class of predicate frames iff A is valid for every member of that class. QS4 is *sound* for $\langle W, R\rangle$ [$\langle W, R, D\rangle$, $\langle W, R, D, V\rangle$] iff every theorem of QS4 (in $\mathcal{L}$) is valid in $\langle W, R\rangle$ [$\langle W, R, D\rangle$, $\langle W, R, D, V\rangle$]. QS4 is *sound* for a class of predicate frames iff every theorem of QS4 (in $\mathcal{L}$) is valid for every predicate frame in that class. QS4 is *complete* for $\langle W, R\rangle$ [$\langle W, R, D\rangle$, $\langle W, R, D, V\rangle$] iff every sentence of $\mathcal{L}$ valid in $\langle W, R\rangle$ [$\langle W, R, D\rangle$, $\langle W, R, D, V\rangle$] is a theorem of QS4. QS4 is *complete* for a class of predicate frames iff every sentence of $\mathcal{L}$ valid in that class is a theorem of QS4. Finally, QS4 is complete for $\langle W, R\rangle$ with expanding countable domains iff there is some family $D = \{D_w\}_{w\in W}$ of countable domains such that QS4 is complete for $\langle W, R, D\rangle$.

The following theorem is well-known (see, e.g., [7] or [4]):

Theorem 3.1. QS4 *is sound for the class of predicate frames; and complete for the class of countable rooted predicate frames.*[5]

4 Completeness/validity transferring maps

A function from any topological space (including any Kripke frame) to another is *continuous* iff the preimage of every open set is open; is *open* iff the image of every open set is open; and is an *interior map* iff it is continuous and open. Say that a topological space is an interior image of another if there is an interior map from the latter onto the former. It is well-known that, in the topological semantics for a propositional as opposed to a quantified modal language, surjective interior maps are useful for transferring validity and hence completeness from one [class of] topological space[s] to another. In particular, suppose that a topological space Y is an interior image of X. It is routine to show, in the propositional case, that if a sentence A is valid in X then A is valid in Y. Thus, if S4 is complete for Y, then it is complete for X. Note that this transfers completeness backwards, from the range of the

[5] Both [7] and [4] state this theorem for languages without names and function symbols, but the result extends to languages with names and function symbols.

interior map to the domain. More generally, suppose that every topological space in some class of spaces is an interior image of X. Again, it is routine to show, in the propositional case, that if a sentence A is valid in X then it is valid in that class of spaces. Thus, if S4 is complete for that class, then it is complete for X.

In this section, we generalize this completeness transferring phenomenon, from the propositional to the quantified context, in three ways: the first two relatively easy, and the third somewhat more involved.

4.1 Transferring completeness in Kripke semantics

Suppose that $\mathcal{K} = \langle W, R \rangle$ and $\mathcal{K}' = \langle W', R' \rangle$ are Kripke frames and that φ is a surjective interior map from $\mathcal{K}'$ to $\mathcal{K}$. Also suppose that $\mathbf{K} = \langle W, R, D \rangle$ is a predicate frame (see Section 3). For each $w \in W'$, define the domain $D'_w = D_{\varphi(w)}$. It is easy to check that $\mathbf{K}' = \langle W', R', D' \rangle$ is a predicate frame, i.e., that the system D' of domains is expanding. Moreover, it is obvious that if $\mathcal{K}'$ and $\mathbf{K}$ are countable, then so is $\mathbf{K}'$. It is routine to prove the following lemma and corollary.

Lemma 4.1. *If a sentence A is valid in $\mathbf{K} = \langle W, R, D \rangle$ and φ is an interior map from $\langle W', R' \rangle$ to $\langle W, R \rangle$ then A is valid in $\mathbf{K}' = \langle W', R', D' \rangle$, where $D'_w = D_{\varphi(w)}$, for every $w \in W'$.*

Corollary 4.2. *Suppose that $\mathcal{K}^* = \langle W^*, R^* \rangle$ is a Kripke frame. Also suppose that* QS4 *is complete for a class $\mathcal{C}$ of predicate frames and that for every $\langle W, R, D \rangle \in \mathcal{C}$, there is an interior map from $\mathcal{K}^*$ onto $\langle W, R \rangle$. Then* QS4 *is complete for $\mathcal{K}^*$. Moreover, if every predicate frame in $\mathcal{C}$ is countable, then Then* QS4 *is complete for $\mathcal{K}^*$ with expanding countable domains.*

We now apply Corollary 4.2 to a particular Kripke frame, the *infinite binary tree*. Let $2^{<\omega}$ be the set of finite binary sequences, ordered as follows: $b \leq b'$ iff b is an initial segment of b'. The *infinite binary tree* is the Kripke frame $\langle 2^{<\omega}, \leq \rangle$. We write $\langle\rangle$ for the empty sequence, which is the root of this Kripke frame. If $b \in 2^{<\omega}$, we write $b0$ [$b1$] for b concatenated with 0 [1]. And we define $\leq(b) =_{\mathrm{df}} \{b' \in 2^{<\omega} : b \leq b'\}$. In the course of the proof of Lemma 6.2 in [9], it is proved that

Lemma 4.3. *Every countable rooted Kripke frame is the image of $2^{<\omega}$ under some interior map (called in* [9] *a* propositional p-morphism).

From Corollary 4.2, Lemma 4.3 and Theorem 3.1, we get a theorem that will be very useful:

Theorem 4.4. QS4 *is complete for* $2^{<\omega}$ *with expanding countable domains.*

Both this conclusion and the argument for it are analogous to Theorem 3.11 in [11], which states that FOS4 is complete for $2^{<\omega}$ with expanding countable domains.

4.2 Transferring completeness in algebraic semantics

First, a standard definition, with a presentation lifted almost verbatim from [11]. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are TBAs. A function $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a *Boolean homomorphism* iff, for all $a, b \in \mathcal{A}_1$,

1. $h(a \vee b) = h(a) \vee h(b)$
2. $h(a \wedge b) = h(a) \wedge h(b)$
3. $h(-a) = -h(a)$

h is a *homomorphism* if h is a Boolean homomorphism and

4. $h(Ia) = Ih(a)$

An injective homomorphism is an *embedding* and a surjective embedding is an *isomorphism*. $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isomorphic* iff there is an isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$. Note that any interior map φ from the topological space X to the topological space Y induces an embedding $h_\varphi : \mathcal{B}(Y) \to \mathcal{B}(X)$ as follows: for each $S \subseteq Y$, $h_\varphi(S) = \varphi^{-1}[S]$, i.e., the preimage of S under φ.

We will make use of the following obvious lemma in Section 5:

Lemma 4.5. *Suppose that* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are isomorphic TBAs. Then* QS4 *is complete for* $\mathcal{A}_1$ [*with a constant countable domain*] *iff* QS4 *is complete for* $\mathcal{A}_2$ [*with a constant countable domain*]*.*

In Section 5, we will also make a slightly different use of homomorphisms. Suppose that h is a Boolean homomorphism from the σ-complete TBA $\mathcal{A}_1$ into the σ-complete TBA $\mathcal{A}_2$. Say that h *preserves countable joins* iff for every countable set $S \subseteq \mathcal{A}_1$,

$$h(\bigvee_{a\in S} a) = \bigvee_{a\in S} h(a)$$

Lemma 4.6. *Suppose that*

1. *$\mathcal{A}$ and $\mathcal{A}'$ are complete TBAs;*
2. *D is nonempty and countable;*
3. *$\mathfrak{A} = \langle \mathcal{A}, D, V \rangle$ is an algebraic model;*
4. *$\mathcal{A}^\sigma$ is a σ-regular subalgebra of $\mathcal{A}$ (see Section 2.2);*
5. *$\mathcal{A}_{val} \sqsubseteq \mathcal{A}^\sigma \sqsubseteq \mathcal{A}$;*
6. *$h : \mathcal{A}^\sigma \to \mathcal{A}'$ is a Boolean homomorphism preserving countable joins;*
7. *$h|_{\mathcal{A}_{val}}$ is a homomorphism; and*
8. *$\mathfrak{A}' = \langle \mathcal{A}', D, V' \rangle$ is an algebraic model such that $V'(\mathsf{c}) = V(\mathsf{c})$ for any $\mathsf{c} \in \mathsf{Names}$, $V'(\mathsf{f}) = V(\mathsf{f})$ for any $\mathsf{f} \in \mathsf{Func}$, and $V'(\mathsf{P})(d_1, \ldots, d_n) = h(V(\mathsf{P})(d_1, \ldots, d_n))$ for any $\mathsf{P} \in \mathsf{Pred}_n$ and $d_1, \ldots, d_n \in D$.*

Then $Val'(A) = h(Val(A))$ for every D-sentence A. Here $Val'(A)$ is the value of A calculated in $\mathfrak{A}'$; and $Val(A)$ is the value of A calculated in $\mathfrak{A}$, which is the same as its value calculated in $\mathfrak{A}_{val}$ and in $\mathfrak{A}^\sigma = \langle \mathcal{A}^\sigma, D, V \rangle$, by Lemma 2.3.

Proof. Assume Items (1)-(8), above. We show that $Val'(A)$ is defined and $= h(Val(A))$ by induction on the construction of A. The atomic case is given by Item (8). The inductive steps for the Boolean connectives and for $\Box$ are given by Item (7) and the inductive step for $\forall$ is given by Items (2), (4), and (6). $\square$

As we will we see in Section 5, every nontheorem A_0 of QS4 is invalidated by the topological model $\mathfrak{A}_0 = \langle X_0, D_0, V_0 \rangle$ – i.e., the algebraic model $\mathfrak{A}_0 = \langle \mathcal{B}(X_0), D_0, V_0 \rangle$ – for a useful subset X_0 of $[0, 1]$ of measure 1, a useful countable domain D_0, and a useful valuation V_0. We will also see that $Val_0(A) \in Borel(X_0)$ for every D_0-sentence A. Thus $\mathcal{B}(X_0)_{val} \sqsubseteq Borel(X_0) \sqsubseteq \mathcal{B}(X_0)$. There is an easy Boolean homomorphism from $Borel(X_0)$ to $\mathcal{M}(X_0)$ that preserves countable joins. This will allow us to make good use of Lemma 4.6.

4.3 p-morphism from topological to frame models

The following definition and two lemmas are from [9], Section 5.

Definition 4.7. Suppose that $\mathbf{K} = \langle W, R, D\rangle$ is a predicate frame, that $\mathbf{X} = \langle X, D^*\rangle$ is a predicate space, and that $\mathfrak{A} = \langle W, R, D, V\rangle$ and $\mathfrak{A}^* = \langle X, D^*, V^*\rangle$ are models based on $\mathbf{K}$ and $\mathbf{X}$, respectively.

(i) A *predicate p-morphism* (henceforth *p-morphism*) *from* $\mathbf{X}$ *to* $\mathbf{K}$ is an ordered pair $\varphi = \langle\varphi_0, \varphi_1\rangle$, such that

1. φ_0 is an interior map from X onto $\langle W, R\rangle$;
2. $\varphi_1 = (\varphi_{1x})_{x\in X}$ is a family of functions indexed by the members of X;
3. every $\varphi_{1x} : D^* \to D_{\varphi_0(x)}$ is a surjective map; and
4. for every $d \in D^*$ and every $x \in X$, there is an open set $O \subseteq X$, such that both $x \in O$ and for every $y \in O$, $\varphi_{1y}(d) = \varphi_{1x}(d)$.

(ii) A *p-morphism from* $\mathfrak{A}^*$ *to* $\mathfrak{A}$ is a p-morphism from $\mathbf{X}$ to $\mathbf{K}$ such that, for every $x \in X$, for every $\mathsf{P} \in \mathsf{Pred}_n$ $(n \geq 1)$, for every $\mathsf{c} \in \mathsf{Names}$, for every $f \in \mathsf{Func}_n$ $(n \geq 1)$, and for every $d_1, \dots, d_n \in D^*$,

5. $x \in V^*(\mathsf{P})(d_1 \dots d_n)$ iff $\varphi_0(x) \in V(\mathsf{P})(\varphi_{1x}(d_1) \dots \varphi_{1x}(d_n))$;
6. $\varphi_{1x}(V^*(\mathsf{c})) = V(\mathsf{c})$; and
7. $\varphi_{1x}(V^*(\mathsf{f})(d_1, \dots, d_n)) = V(\mathsf{f})(\varphi_{1x}(d_1), \dots, \varphi_{1x}(d_n))$.

Lemma 4.8. *If $\varphi = \langle\varphi_0, \varphi_1\rangle$ is a p-morphism from the topological model $\mathfrak{A}^* = \langle X, \tau, D^*, V^*\rangle$ to the frame model $\mathfrak{A} = \langle W, R, D, V\rangle$, then for every D^*-term t,*

$$\text{for every } x \in X,\ \varphi_{1x}(\mathit{Val}(\mathsf{t})) = \mathit{Val}(\varphi_{1x} \cdot \mathsf{t}),$$

where $\varphi_{1x} \cdot \mathsf{t}$ is the $D_{\varphi_0(x)}$-term obtained from the D^*-term t by replacing every occurrence in t of every $d \in D^*$ with $\varphi_{1x}(d)$.

Lemma 4.9. *If $\varphi = \langle\varphi_0, \varphi_1\rangle$ is a p-morphism from the topological model $\mathfrak{A}^* = \langle X, \tau, D^*, V^*\rangle$ to the frame model $\mathfrak{A} = \langle W, R, D, V\rangle$, then for every D^*-sentence B,*

$$\text{for every } x \in X,\ \mathfrak{A}^*, x \vDash B \text{ iff } \mathfrak{A}, \varphi_0(x) \vDash \varphi_{1x} \cdot B,$$

where $\varphi_{1x} \cdot B$ is the $D_{\varphi_0(x)}$-sentence obtained from the D^*-sentence B by replacing every free occurrence in B of every $d \in D^*$ with $\varphi_{1x}(d)$.

5 Completeness proof

In Section 12 of [11], Lando defines a useful Borel subset X_0 of $[0,1]$ of measure 1, and a surjective interior function, say $\psi_0 : X_0 \to 2^{<\omega}$. Recall the general definition in Section 2.1 of $\mathcal{M}(X)$ for any Borel $X \subseteq [0,1]$. Since $\mathcal{M}(X_0)$ is isomorphic to $\mathcal{M}$ it suffices, by Lemma 4.5, for our main result, Theorem 2.2, to show

Lemma 5.1. QS4 *is complete for* $\mathcal{M}(X_0)$ *with a countable domain.*

We will not review the definitions of X_0 and ψ_0 in the current paper: rather, we will rely on a combination of propositions proved in [11] and additional information that can be gleaned without reviewing the construction. Our next lemma notes four nice properties of X_0 and ψ_0: the first and second are established in [11] and we establish the third and fourth here. But first, we reproduce a definition from [11]. Suppose that $X \subseteq \mathbb{R}$, that Y is a topological space, and that $\varphi : X \to Y$. Then φ *has the M-property* if, for any $S \subseteq Y$ and any open $O \subseteq X$,

$$\text{if } \varphi^{-1}[S] \cap O \neq \emptyset \text{ then } \lambda(\varphi^{-1}[S] \cap O) > 0.$$

Lemma 5.2.

1. *For any* $S \subseteq 2^{<\omega}$, ${\psi_0}^{-1}[S]$ *is a Borel subset of* X_0 (Proposition 12.9 in [11]).
2. ψ_0 *has the M-property* (Proposition 12.10 in [11]).
3. X_0 *is* zero dimensional, *i.e., it has a basis of clopen sets.*
4. X_0 *has a countable basis of clopen sets.*

Proof of (3) and (4). For (3), it suffices to show that $[0,1] - X_0$ is dense in $[0,1]$: in this case, the family of nonempty sets of the form $\{x \in X_0 : y < x < z\}$, where $y, z \in [0,1] - X_0$, form a basis for X_0 of sets clopen in X_0. Suppose that $[0,1] - X_0$ is not dense in $[0,1]$. Then there is some nonempty open interval $I \subseteq X_0$. Choose any $x \in I$, and let $b_0 = \psi_0(x)$. Since ψ_0 is continuous and $\leq(b_0)$ is open in $2^{<\omega}$, the set ${\psi_0}^{-1}[\leq(b_0)]$ is open in $[0,1]$. So, since $x \in {\psi_0}^{-1}[\leq(b_0)] \cap I$, there is a nonempty open interval J such that $x \in J \subseteq {\psi_0}^{-1}[\leq(b_0)] \cap I$. Let ψ be ψ_0 restricted to J: Note that $\psi : J \to \leq(b_0)$ is a surjective interior map. It is easy to convert ψ to a surjective interior map $\psi' : J \to 2^{<\omega}$, as follows. For any $b \geq b_0$, note that

b_0 is an initial segment of b: let $b - b_0$ be the result of deleting this initial segment. Thus, for example, $b_0 - b_0 = \langle\rangle$. And let $\psi'(x) = \psi(x) - b_0$. It is easy to show that $\psi' : J \to 2^{<\omega}$ is a surjective interior map. Also note that J is a Baire space.[6] But by [8], Corollary 4.8, there is no interior map from any Baire space onto $2^{<\omega}$. A contradiction.

For (4) first note that $[0, 1]$ is *second countable*, i.e., it has a countable basis. So any subspace of $[0, 1]$, in particular X_0, is also second countable. But any second countable zero-dimensional space has a countable basis of clopen sets. This follows from Lemma 1.1.5 in [3]: if a topological space has a countable basis, then for every basis B for the space, there is a countable basis $B' \subseteq B$ for the space. □

X_0 is zero-dimensional (Lemma 5.2 (3)), and from the existence of ψ_0 and Lemma 4.3, it follows that X_0 is frame-simulating in the sense of [9], i.e., that any countable rooted Kripke frame is an image of it under an interior map. Thus, we can almost take advantage of Theorem 6.1 in [9], which says that QS4 is complete for any frame-simulating zero-dimensional topological space X with a constant domain of the same cardinality as X. Unfortunately, this only delivers the completeness of QS4 for X_0 with a constant uncountable domain: being of measure 1, X_0 is itself uncountable. But we want something stronger: the completeness of QS4 for $\mathcal{M}(X_0)$ with a constant *countable* domain. To get a countable domain rather than an uncountable domain, we will exploit the fact that X_0 has a *countable* basis of clopen sets (Lemma 5.2 (4)). We also have to do a little more work to get completeness for $\mathcal{M}(X_0)$ in addition to completeness for X_0.

At this point, any names and function symbols in $\mathcal{L}$ introduce distracting complications that are best set aside in a first run through the completeness proof. For this reason, we first prove Lemma 5.1 when $\mathcal{L}$ has no names or function symbols, and give the more general proof in Section 6.

Proof of Lemma 5.1, when $\mathcal{L}$ has no names or function symbols. Let A_0 be any nontheorem of QS4 in the language $\mathcal{L}$. Our task is to find a countable nonempty domain D_0 and to show that A_0 is not valid in the predicate TBA

[6] A topological space X is a *Baire space* if the intersection of each countable family of open dense sets in X is dense in X. One version of the Baire Category Theorem says that any complete metric space, such as $\mathbb{R}$, is a Baire space: see, e.g., [2], p. 299, Theorem 4.1. And it is easy to show that any open subset of a Baire space is a Baire space: see, e.g., [2], Exercise 2. So J is a Baire space.

$\langle \mathcal{M}(X_0), D_0 \rangle$. En route, we will see that A_0 is not valid in the predicate space $\langle X_0, D_0 \rangle$ – equivalently, in the predicate TBA $\langle \mathcal{B}(X_0), D_0 \rangle$.

Given that A_0 is a nontheorem of QS4, there is, by Theorem 4.4, a frame model $\mathfrak{A} = \langle 2^{<\omega}, \leq, D, V \rangle$, where D_b is countable for each $b \in 2^{<\omega}$, and where $Val(A_0) \neq 2^{<\omega}$. Let $\mathbf{K} = \langle 2^{<\omega}, \leq, D \rangle$. We define a predicate space $\mathbf{X}_0 = \langle X_0, D_0 \rangle$ and a p-morphism from $\mathbf{X}_0$ to $\mathbf{K}$ as follows. First let $D_0 = \{\langle b, k, d \rangle : b \in 2^{<\omega}, k \in \mathbb{N} \text{ and } d \in D_b\}$. Note that D_0 is countably infinite. Our predicate p-morphism is $\psi = \langle \psi_0, \psi_1 \rangle$, where the surjective interior function $\psi_0 : X_0 \to 2^{<\omega}$ defined in [11] is already assumed, and where the family $\psi_1 = \{\psi_{1x}\}_{x \in X}$ of functions is defined shortly, after some stage-setting.

Note that, for any $b \in 2^{<\omega}$, the set $\leq(b)$ is open in $2^{<\omega}$. So, since ψ_0 is continuous, the set ${\psi_0}^{-1}[\leq(b)]$ is open in X_0. Since X_0 has a countable basis of clopen sets, we can express ${\psi_0}^{-1}[\leq(b)]$ as a countable union of clopen sets O^b_k as follows:

$$ {\psi_0}^{-1}[\leq(b)] = \bigcup_{k \in \mathbb{N}} O^b_k $$

For each $x \in X$, we define $\psi_{1x} : D_0 \to D_{2^{<\omega}}$ as follows:

$$ \psi_{1x}(\langle b, k, d \rangle) \quad = \quad \begin{cases} d \text{ if } x \in O^b_k \\ d_0 \text{ if } x \notin O^b_k \end{cases} $$

We have to check that the ψ_{1x} satisfy Items (3) and (4) in Definition 4.7.

Item (3). First we check that $\psi_{1x} : D_0 \to D_{\psi_0(x)}$. So suppose $\langle b, k, d \rangle \in D_0$. If $x \notin O^b_k$, then $\psi_{1x}(\langle b, k, d \rangle) = d_0 \in D_{\langle\rangle} \subseteq D_{\psi_0(x)}$. On the other hand, suppose that $x \in O^b_k$. Then $\psi_0(x) \in \psi_0[O^b_k] \subseteq \psi_0[\leq(b)]$. So $\psi_0(x) \in \leq(b)$. So $b \leq \psi_0(x)$. Also, $d \in D_b$ since $\langle b, k, d \rangle \in D_0$. So $\psi_{1x}(\langle b, k, d \rangle) = d \in D_{\psi_0(x)}$, as desired. Next we check that $\psi_{1x} : D_0 \to D_{\psi_0(x)}$ is surjective. Suppose that $d \in D_{\psi_0(x)}$. Clearly $\psi_0(x) \in \leq(\psi_0(x))$. So $x \in {\psi_0}^{-1}[\leq(\psi_0(x))]$. So, $x \in O^{\psi_0(x)}_k$ for some $k \in \mathbb{N}$. So $\psi_{1x}(\langle \psi_0(x), k, d \rangle) = d$, which suffices.

Item (4). Choose $\langle b, k, d \rangle \in D_0$ and $x \in X$. We want to show that there is an open set $O \subseteq X$, such that both $x \in O$ and for every $y \in O$, $\psi_{1y}(\langle b, k, d \rangle) = \psi_{1x}(\langle b, k, d \rangle)$. If $x \in O^b_k$, let $O = O^b_k$. Then note that $x \in O$, and for every $y \in O$, we have $\psi_{1y}(\langle b, k, d \rangle) = d = \psi_{1x}(\langle b, k, d \rangle)$. On the other hand, if $x \notin O^b_k$, then let $O = X - O^b_k$. Note that O is open, since O^b_k is clopen. Also note that $x \in O$, and for every $y \in O$, we have $\psi_{1y}(\langle b, k, d \rangle) = d_0 = \psi_{1x}(\langle b, k, d \rangle)$, as desired.

Now define a valuation V_0 on $\mathbf{X}_0$ as follows:

$$V_0(\mathsf{P})(\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle)$$
$$= \{x \in X_0 : \psi_0(x) \in V(\mathsf{P})(\psi_{1x}(\langle b_1, k_1, d_1\rangle), \ldots, \psi_{1x}(\langle b_n, k_n, d_n\rangle))\}.$$

So ψ is a p-morphism from the topological model $\mathfrak{A}_0 = \langle X_0, D_0, V_0\rangle$ to the frame model $\mathfrak{A} = \langle W, R, D, V\rangle$; see Definition 4.7, Item (5).

Let's use $Val_0(B)$ for the value of any D_0-sentence B calculated in $\mathfrak{A}_0$. By Lemma 4.9, $Val_0(A_0) \neq X_0$, so that A_0 is not valid in the predicate frame $\langle X_0, D_0\rangle$, equivalently, in the complete predicate TBA $\langle \mathcal{B}(X_0), D_0\rangle$. We would like to transfer that invalidity to the complete predicate TBA $\langle \mathcal{M}(X_0), D_0\rangle$. To this end, we define the projection map $\pi : Borel(X_0) \to \mathcal{M}(X_0)$ as follows: $\pi(S) = |S|$, for every borel $S \subseteq X_0$. And we define a valuation V' on $\mathcal{M}(X_0)$ as follows:

$$V'(\mathsf{P})(\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle) = \pi(V(\mathsf{P})(\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle))$$

for any $\mathsf{P} \in \mathsf{Pred}_n$ and $\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle \in D_0$.

Note that $\mathfrak{A}' = \langle \mathcal{M}(X_0), D_0, V'\rangle$ is an algebraic model, since $\mathcal{M}(X_0)$ is a complete TBA. The following lemma references Lemma 4.6, above:

Lemma 5.3.

1. *$\mathcal{B}(X_0)$ and $\mathcal{M}(X_0)$ are complete TBAs;*
2. *D_0 is nonempty and countable;*
3. *$\mathfrak{A}_0 = \langle \mathcal{B}(X_0), D_0, V_0\rangle$ is an algebraic model;*
4. *$Borel(X_0)$ is a σ-regular subalgebra of $\mathcal{B}(X_0)$;*
5. *$\mathcal{B}(X_0)_{val} \sqsubseteq Borel(X_0) \sqsubseteq \mathcal{B}(X_0)$;*
6. *$\pi : Borel(X_0) \to \mathcal{M}(X_0)$ is a Boolean homomorphism preserving countable joins;*
7. *$\pi|_{\mathcal{B}(X_0)_{val}}$ is a homomorphism; and*
8. *$\mathfrak{A}' = \langle \mathcal{M}(X_0), D_0, V'\rangle$ is an algebraic model such that*

$$V'(\mathsf{P})(\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle) = h(V(\mathsf{P})(\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle))$$

for any $\mathsf{P} \in \mathsf{Pred}_n$ and $\langle b_1, k_1, d_1\rangle, \ldots, \langle b_n, k_n, d_n\rangle \in D_0$.

Proof. Items (1), (2), (3), (4) and (8) are either obvious, already noted, or true by definition.

Proof of Item (5). Given item (4), it suffice to show that $\mathcal{B}(X_0)_{val} \sqsubseteq Borel(X_0)$. And for this, it suffices to show that $Val_0(B) \in Borel(X_0)$, for any D_0-sentence B. For ease of exposition, we will assume that B has exactly two members of D_0 occurring in it as names, say $\langle b_1, k_1, d_1\rangle$ and $\langle b_2, k_2, d_2\rangle$: the proof clearly generalizes to the case when B has any finite number of members of D_0 occurring in it as names. Note that $Val_0(B) = \{x \in X_0 : \mathfrak{A}_0, x \Vdash B\}$. So by Lemma 4.9, $Val_0(B) = \{x \in X_0 : \mathfrak{A}, \psi_0(x) \Vdash \psi_{1x} \cdot B\}$. Some notation: for any $d, d' \in D$, let $B(d, d')$ be the D-sentence that results from replacing every occurrence of $\langle b_1, k_1, d_1\rangle$ in B by d and every occurrence of $\langle b_2, k_2, d_2\rangle$ in B by d'. Note that, for $i = 1$ or 2, we have $\psi_{1x} \cdot \langle b_i, k_i, d_i\rangle = d_i$ if $x \in O^{b_i}_{k_i}$ and $\psi_{1x} \cdot \langle b_i, k_i, d_i\rangle = d_0$ if $x \notin O^{b_i}_{k_i}$. Thus, for any $x \in X_0$, the D-sentence $\psi_{1x} \cdot B$ is one of $B(d_1, d_2)$, $B(d_0, d_2)$, $B(d_1, d_0)$, or $B(d_0, d_0)$. More particularly, for any $x \in X_0$,

$$\begin{array}{lll} x \in Val_0(B) & \text{iff} & \mathfrak{A}_0, x \Vdash B \\ & \text{iff} & \mathfrak{A}, \psi_0(x) \Vdash \psi_{1x} \cdot B \\ & \text{iff} & x \in O^{b_1}_{n_1} \cap O^{b_2}_{n_2} \text{ and } \mathfrak{A}, \psi_0(x) \Vdash B(d_1, d_2) \quad \text{or} \\ & & x \in O^{b_1}_{n_1} - O^{b_2}_{n_2} \text{ and } \mathfrak{A}, \psi_0(x) \Vdash B(d_1, d_0) \quad \text{or} \\ & & x \in O^{b_2}_{n_2} - O^{b_1}_{n_1} \text{ and } \mathfrak{A}, \psi_0(x) \Vdash B(d_0, d_2) \quad \text{or} \\ & & x \in X_0 - (O^{b_1}_{n_1} \cup O^{b_2}_{n_2}) \text{ and } \mathfrak{A}, \psi_0(x) \Vdash B(d_0, d_0) \end{array}$$

Thus,

$$\begin{array}{rcl} Val_0(B) & = & (O^{b_1}_{n_1} \cap O^{b_2}_{n_2} \cap {\psi_0}^{-1}[Val(B(d_1, d_2))]) \\ & & \cup\ ((O^{b_1}_{n_1} - O^{b_2}_{n_2}) \cap {\psi_0}^{-1}[Val(B(d_1, d_0))]) \\ & & \cup\ ((O^{b_2}_{n_2} - O^{b_1}_{n_1}) \cap {\psi_0}^{-1}[Val(B(d_0, d_2))]) \\ & & \cup\ ((X_0 - (O^{b_1}_{n_1} \cup O^{b_2}_{n_2})) \cap {\psi_0}^{-1}[Val(B(d_0, d_0))]) \end{array}$$

Note that ${\psi_0}^{-1}[Val(B(d, d'))]$ is Borel in X_0 for any $d, d' \in D_W$, by Lemma 5.2 (1). Obviously, $O^{b_1}_{n_1}$ and $O^{b_2}_{n_2}$ are Borel in X_0. So $Val_0(B)$ is also Borel in X_0.

Proof of Item (6). Given the definition of finite joins, finite meets, and complements in the quotient algebra $\mathcal{Q}(X) = Borel(X_0)/Null(X_0)$, π preserves finite joins, finite meets, and complements. To see that π preserves countable joins, it suffices to note Lemma 10.3 in [11], according to which, for any countable collection $\mathcal{C} \subseteq Borel(X_0)$, we have $\bigvee_{S \in \mathcal{C}} |S| = |\bigcup_{S \in \mathcal{C}} S|$. (The proof in [11] is given for $Borel([0, 1])$, but goes through for $Borel(X_0)$ for any X_0 of measure 1.)

Proof of Item (7). Let $h = \pi|_{\mathcal{B}(X_0)_{val}}$. Given Item (6), h preserves finite joins, finite meets, and complements. We want to show that h also preserves interiors. Thus, we want to show that $h(Int(S)) = I(h(S))$ for every $S \in \mathcal{B}(X_0)_{val}$, where $Int(S)$ the the interior of S in the topological space X_0. Thus, we want to show, for every D-sentence B, that

$$|Int(Val_0(B))| = \bigvee\{|O| : |O| \leq |Val_0(B)| \text{ and } O \text{ is open in } X_0\}.$$

Our proof is close to the second part of the proof of Proposition 11.6 in [11].

($\leq$). $|Int(Val_0(B))| \leq |Val_0(B)|$, since $Int(Val_0(B)) \subseteq Val_0(B)$. And $Int(Val_0(B))$ is open in X_0. So $|Int(Val_0(B))| \leq \bigvee\{|O| : |O| \leq |Val_0(B)|$ and O is open in $X_0\}$.

($\geq$). We want to show that $|Int(Val_0(B))|$ is an upper bound on $\{|O| : |O| \leq |Val_0(B)|$ and O is open in $X_0\}$. For ease of exposition, we will assume that B has exactly two members of D_0 occurring in it as names, say $\langle b_1, k_1, d_1\rangle$ and $\langle b_2, k_2, d_2\rangle$: the proof clearly generalizes to the case when B has any finite number of members of D_0 occurring in it as names. So, by the work in the proof of Item (5) applied to $\neg B$ rather than to B, there are open sets $O_1, \ldots, O_4$ and D-sentences $B_1, \ldots, B_4$ such that

$$\begin{aligned} Val_0(\neg B) \quad = \quad & O_1 \cap {\psi_0}^{-1}[Val(\neg B_1)] \\ & \cup (O_2 \cap {\psi_0}^{-1}[Val(\neg B_2)]) \\ & \cup (O_3 \cap {\psi_0}^{-1}[Val(\neg B_3)]) \\ & \cup (O_4 \cap {\psi_0}^{-1}[Val(\neg B_4)]) \end{aligned}$$

The openness of the O_i follows from the clopenness of $O^{b_1}_{n_1}$ and $O^{b_2}_{n_2}$. We will use the above identity shortly.

Choose any open $O \subseteq X_0$ with $|O| \leq |Val_0(B)|$. We want to show that $|O| \leq |Int(Val_0(B))|$. It suffices to show that $O \subseteq Val_0(B)$: if so, then $O \subseteq Int(Val_0(B))$ so that $|O| \leq |Int(Val_0(B))|$, as desired. Suppose that $O \not\subseteq Val_0(B)$. Since $|O| \leq |Val_0(B)|$, we have $O \subseteq Val_0(B) \cup N$, for some $N \in Null(X_0)$ – see Lemma 10.2 in [11]. So, since $O \not\subseteq Val_0(B)$, we have $x \in N \cap (O - Val_0(B))$, for some $x \in X_0$. So $x \in Val_0(\neg B)$. So $x \in O_i \cap {\psi_0}^{-1}[Val(\neg B_i)]$ for some $i = 1, \ldots, 4$.

Let $w = \psi_0(x)$. Since $x \in O$ and $x \in O_i$, we have ${\psi_0}^{-1}[\{w\}] \cap O \cap O_i \neq 0$. So $\lambda({\psi_0}^{-1}[\{w\}] \cap O \cap O_i) > 0$, since ψ_0 has the M-property (Lemma 5.2 (2)). Also, $w \in Val(\neg B_i)$. Now we claim that ${\psi_0}^{-1}[\{w\}] \cap O \cap O_i \subseteq N$: this will give us our contradiction since $\lambda({\psi_0}^{-1}[\{w\}] \cap O \cap O_i) > 0$. To see this claim, suppose that $y \in {\psi_0}^{-1}[\{w\}] \cap O \cap O_i$. Since $y \in {\psi_0}^{-1}[\{w\}]$ and $w \in \mathit{Val}(\neg B_i)$, we have $y \in {\psi_0}^{-1}[\mathit{Val}(\neg B_i)]$. And since $y \in O_i$ we have $y \in \mathit{Val}_0(\neg B)$. So since $y \in O$, we have $y \in O - \mathit{Val}_0(B)$. So $y \in N$, since $O \subseteq \mathit{Val}_0(B) \cup N$. □

Given Lemmas 4.6 and 5.3, $\mathit{Val}'(A_0) = \pi(\mathit{Val}_0(A_0))$. It will suffice for us to show that $\mathit{Val}'(A_0) \neq 1$ in the model $\mathfrak{A}' = \langle \mathcal{M}(X_0), D_0, V' \rangle$, i.e., that $\mathit{Val}'(A_0) \neq |X_0|$. For this, it suffices to show that $\lambda(\mathit{Val}_0(\neg A_0)) > 0$. First, recall that A_0 is a sentence in the language $\mathcal{L}$, i.e., no members of D_0 occur in A_0 as constants. Thus, by Lemma 4.9, $\mathit{Val}_0(\neg A_0) = {\psi_0}^{-1}[\mathit{Val}(\neg A_0)]$. Now $\mathit{Val}(\neg A_0) \neq \emptyset$, since $\mathit{Val}(A_0) \neq 2^{<\omega}$. So $\mathit{Val}_0(\neg A_0) \neq \emptyset$. Note that $\mathit{Val}_0(\neg A_0) = {\psi_0}^{-1}[\mathit{Val}(\neg A_0)] \cap X_0$ and that X_0 is open in X_0: thus $\lambda(\mathit{Val}_0(\neg A_0)) > 0$ by Lemma 5.2 (2). □

6 Names and function symbols

When the language $\mathcal{L}$ has names or functions symbols, the completeness proof breaks down on page 18, when the valuation V_0 is defined. Recall: we have, at this point in the proof, a predicate frame $\mathbf{K} = \langle W, R, D \rangle$, a frame model $\mathfrak{A} = \langle W, R, D, V \rangle$, a predicate space $\mathbf{X}_0 = \langle X_0, D_0 \rangle$, and a predicate p-morphism ψ from $\mathbf{X}_0$ to $\mathbf{K}$. This allowed us to define a valuation V_0 on $\mathbf{X}_0$ so that ψ is also a predicate p-morphism from the topological model $\langle X_0, D_0, V_0 \rangle$ to the frame model $\langle W, R, D, V \rangle$. But when $\mathcal{L}$ contains names and function symbols, we cannot, for all we know, define such a valuation V_0.

The problem is addressed in [9] by constructing a new predicate p-morphism from $\langle X_0, \mathsf{Term}(D_0) \rangle$ to $\mathbf{K} = \langle W, R, D \rangle$ out of the original predicate p-morphism from $\langle X_0, D_0 \rangle$ to $\mathbf{K} = \langle W, R, D \rangle$. Unfortunately, there is a slight but correctable glitch: the syntax of $\mathsf{Term}(D_0)$-terms in the expanded language $\mathcal{L}(\mathsf{Term}(D_0))$ is then ambiguous. Suppose for example that $d \in D_0$ and $\mathsf{f} \in \mathsf{Func}_1$. Then the $\mathsf{Term}(D_0)$-term $\mathsf{f}d$ can be analyzed as a composite term, in particular, as the application of the function symbol f to d, the latter of which is a name in $\mathcal{L}(\mathsf{Term}(D_0))$; or $\mathsf{f}d$ can be analyzed as a name in $\mathcal{L}(\mathsf{Term}(D_0))$ since $\mathsf{f}d \in \mathsf{Term}(D_0)$. In particular, we lose the unique decomposition of terms, which threatens the recursive definition of $\mathit{Val}_0(\mathsf{t})$ for terms t in the language $\mathcal{L}(\mathsf{Term}(D_0))$. (Here I am replacing D^* in [9] with D_0, so that the discussion dovetails with our above proof.) All the proofs in [9] still go through, since the valuations that are relevant all deliver a unique value for $Val_0(\mathsf{t})$, regardless of the analysis of, for example, $\mathsf{f}d$. But here, we will be a little more careful. What we want is a new domain so that every member of the new domain is basically a copy of a member of $\mathsf{Term}(D_0)$, without the syntactic ambiguity just described. Here is one way to implement this idea.

Start the completeness proof as in Section 5, up to and including the definition of the predicate p-morphism ψ from $\mathbf{X}_0$ to $\mathbf{K}$, together with the proof that this is indeed a predicate p-morphism. At this point let $\mathcal{L}^*$ be a language just like $\mathcal{L}$, except that every name c [function symbol f] is replaced by a name c^* [function symbol f^*], making sure that the new names [function symbols] do not occur already in the syntax of $\mathcal{L}(D_0)$. For any nonempty domain D, define $\mathsf{Term}^*(D)$ as the set of terms in the language $\mathcal{L}^*(D)$, i.e, the language $\mathcal{L}^*$ expanded with the members of D as names. In particular, $\mathsf{Term}^*(D_0)$ is the set of terms in the language $\mathcal{L}^*(D_0)$: note that this resolves any ambiguity in the language $\mathcal{L}(\mathsf{Term}^*(D_0))$, since every member of $\mathsf{Term}^*(D_0)$ is simply a name in $\mathcal{L}(\mathsf{Term}^*(D_0))$, and every term in $\mathcal{L}(\mathsf{Term}^*(D_0))$ admits of only one syntactic analysis.

Now define a new predicate p-morphism $\psi^* = \langle {\psi^*}_0, {\psi^*}_1 \rangle$ from $\mathbf{X}_0^* = \langle X_0, \mathsf{Term}^*(D_0)\rangle$ to $\mathbf{K} = \langle W, R, D\rangle$ as follows. First, ${\psi^*}_0 = \psi_0$. Second, ${\psi^*}_{1x}(\mathsf{t})$ is defined, for $x \in X_0$ and $\mathsf{t} \in \mathsf{Term}^*(D_0)$,as follows:

$$\begin{aligned}
{\psi^*}_{1x}(d) &= \psi_{1x}(d), \text{ for } d \in D_0 \\
{\psi^*}_{1x}(\mathsf{c}^*) &= V(\mathsf{c}), \text{ for } \mathsf{c} \in \mathsf{Names} \\
{\psi^*}_{1x}(\mathsf{f}^*\mathsf{t}_1 \dots \mathsf{t}_n) &= V(\mathsf{f})({\psi^*}_{1x}(\mathsf{t}_1) \dots {\psi^*}_{1x}(\mathsf{t}_n)), \\
&\quad \text{for } \mathsf{f} \in \mathsf{Func}_n \text{ and } \mathsf{t}_1, \dots, \mathsf{t}_n \in \mathsf{Term}^*(D_0).
\end{aligned}$$

The proof that ψ^* is indeed a predicate p-morphism from $\mathbf{X}_0^*$ to $\mathbf{K}$ follows the proof of Claim 1 in the proof of Corollary 5.4 in [9].

Now define a valuation V_0 for the predicate space $\mathbf{X}_0^*$.

- If $\mathsf{P} \in \mathsf{Pred}_n$ and $\mathsf{t}_1, \dots, \mathsf{t}_n \in \mathsf{Term}^*(D_0)$, then $V_0(\mathsf{P})(\mathsf{t}_1, \dots, \mathsf{t}_n) = \{x \in X_0 : {\psi^*}_0(x) \in V(\mathsf{P})({\psi^*}_{1x}(t_1), \dots, {\psi^*}_{1x}(t_n))\}$
- if $\mathsf{c} \in \mathsf{Names}$, then $V_0(\mathsf{c}) = \mathsf{c}^*$, and
- if $\mathsf{f} \in \mathsf{Func}_n$ and $\mathsf{t}_1, \dots, \mathsf{t}_n \in \mathsf{Term}^*(D_0)$, then $V_0(\mathsf{f})(\mathsf{t}_1, \dots, \mathsf{t}_n) = \mathsf{f}^*\mathsf{t}_1 \dots \mathsf{t}_n$.

Let $\mathfrak{A}_0 = \langle X_0, \mathsf{Term}^*(D_0), V_0\rangle$. The proof that ψ^* is a predicate p-morphism from the topological model $\mathfrak{A}_0$ to the frame model $\mathfrak{A}$ follows the proof of Claim 2 in the proof of Corollary 5.4 in [9].

At this point, the updated completeness proof resumes after the definition of V_0, now newly defined. We have to redefine the valuation V' on $\mathcal{M}(X_0)$, as follows:

- If $\mathsf{P} \in \mathsf{Pred}_n$ and $\mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}^*(D_0)$, then $V'(\mathsf{P})(\mathsf{t}_1, \ldots, \mathsf{t}_n) = \pi(V_0(\mathsf{P})(\mathsf{t}_1, \ldots, \mathsf{t}_n))$
- $V'(\mathsf{c}) = V_0(\mathsf{c})$ for any $\mathsf{c} \in \mathsf{Names}$, and
- $V'(\mathsf{f}) = V_0(\mathsf{f})$ for any $\mathsf{f} \in \mathsf{Func}$

As in Section 5, $\mathfrak{A}' = \langle \mathcal{M}(X_0), D_0, V' \rangle$ is an algebraic model, since $\mathcal{M}(X_0)$ is a complete TBA.

Now we have to restate and reprove Lemma 5.3. In the statement of the lemma

- Item 1 is left as is;
- Item 2 now reads, $\mathsf{Term}^*(D_0)$ is nonempty and countable;
- Item 3 now reads, $\mathfrak{A}_0 = \langle \mathcal{B}(X_0), \mathsf{Term}^*(D_0), V_0 \rangle$ is an algebraic model;
- Item 4 is left as is;
- Item 5 is left as is;
- Item 6 is left as is;
- Item 7 is left as is; and
- Item 8 now reads, $\mathfrak{A}' = \langle \mathcal{M}(X_0), \mathsf{Term}^*(D_0), V' \rangle$ is an algebraic model such that $V'(\mathsf{c}) = V_0(\mathsf{c})$ for any $\mathsf{c} \in \mathsf{Names}$, $V'(\mathsf{f}) = V_0(\mathsf{f})$ for any $\mathsf{f} \in \mathsf{Func}$, and $V'(\mathsf{P})(\mathsf{t}_1, \ldots, \mathsf{t}_n) = h(V_0(\mathsf{P})(\mathsf{t}_1, \ldots, \mathsf{t}_n))$ for any $\mathsf{P} \in \mathsf{Pred}_n$ and $\mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}^*(D_0)$.

The proof of the newly stated Lemma 5.3 gets tricky, since we now have not only ordered triplets $\langle b, k, d \rangle$ in our quantifier domain in the model $\mathfrak{A}_0$, but also terms in the language $\mathcal{L}^*$ built out of these triplets. We must be careful: these terms are treated as unanalyzable names in the language $\mathcal{L}(\mathsf{Term}^*(D_0))$, the language interpreted by the models in play. For example, if $b \in 2^{<\omega}$, $k \in \mathbb{N}$ and $d \in D_b$, and also if $\mathsf{f} \in \mathsf{Func}_1$, then $\mathsf{f}^*\langle b, k, d \rangle$ is an analyzable expression in the language $\mathcal{L}^*(D_0)$, but is simply a name in the language $\mathcal{L}(\mathsf{Term}^*(D_0))$. Thus, when working in the latter language, we cannot assume that the name $\langle b, k, d\rangle$ occurs in the term $\mathsf{f}^*\langle b, k, d\rangle$, which is just another name. So we have to be careful about using Lemma 4.9, and Lemma 4.8, on which the former depends.

There is a way out. We associate every $\mathsf{t} \in \mathsf{Term}^*(D_0)$ a with new term $\mathsf{t}^\dagger \in \mathsf{Term}(D_0)$. Informally, just remove the *'s. Formally, proceed recursively as follows:

$$\begin{aligned}
\langle b, k, d\rangle^\dagger &= \langle b, k, d\rangle, \text{ if } \langle b, k, d\rangle \in D_0 \\
\mathsf{c}^{*\dagger} &= \mathsf{c}, \text{ if } \mathsf{c} \in \mathsf{Names} \\
(\mathsf{f}^*\mathsf{t}_1 \ldots \mathsf{t}_n)^\dagger &= \mathsf{f}\mathsf{t}_1{}^\dagger \ldots \mathsf{t}_n{}^\dagger, \text{ if } \mathsf{f} \in \mathsf{Func}_n \text{ and } \mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}^*(D_0)
\end{aligned}$$

Next, we associate every $\mathsf{t} \in \mathsf{Term}(\mathsf{Term}^*(D_0))$ with a new term $\mathsf{t}^\dagger \in \mathsf{Term}(D_0) \subseteq \mathsf{Term}(\mathsf{Term}^*(D_0))$. Again, informally, just remove the *'s. Formally, proceed recursively as follows, with $\mathsf{t}^\dagger$ already defined when $\mathsf{t} \in \mathsf{Term}^*(D_0)$:

$$\begin{aligned}
\mathsf{c}^\dagger &= \mathsf{c}, \text{ if } \mathsf{c} \in \mathsf{Names} \\
(\mathsf{f}\mathsf{t}_1 \ldots \mathsf{t}_n)^\dagger &= \mathsf{f}\mathsf{t}_1{}^\dagger \ldots \mathsf{t}_n{}^\dagger, \text{ if } \mathsf{f} \in \mathsf{Func}_n \text{ and } \mathsf{t}_1, \ldots, \mathsf{t}_n \in \mathsf{Term}(\mathsf{Term}^*(D_0))
\end{aligned}$$

Finally, we associate every $\mathsf{Term}^*(D_0)$-sentence B with a D_0-sentence $B^\dagger$: simply replace every occurrence in B of any $t \in \mathsf{Term}^*(D_0)$ with $t^\dagger$.

It is easy to show the following:

- $Val_0(\mathsf{t}^\dagger) = V_0(\mathsf{t})$, for every $\mathsf{t} \in \mathsf{Term}^*(D_0)$;
- $Val_0(\mathsf{t}^\dagger) = Val_0(\mathsf{t})$, for every $\mathsf{t} \in \mathsf{Term}(\mathsf{Term}^*(D_0))$;
- $Val_0(B^\dagger) = Val_0(B)$, for every $(\mathsf{Term}^*(D_0))$-sentence B;
- $Val'(\mathsf{t}^\dagger) = V'(\mathsf{t})$, for every $\mathsf{t} \in \mathsf{Term}^*(D')$;
- $Val'(\mathsf{t}^\dagger) = Val'(\mathsf{t})$, for every $\mathsf{t} \in \mathsf{Term}(\mathsf{Term}^*(D'))$; and
- $Val'(B^\dagger) = Val'(B)$, for every $(\mathsf{Term}^*(D'))$-sentence B.

Now we are ready to prove the updated Lemma 5.3, in particular the hard updated Items (5) and (7): the proofs of the other updated items are exactly the same as in Section 5.

Proof of updated Item (5). As before, given item (4), it suffice to show that $\mathcal{B}(X_0)_{val} \sqsubseteq Borel(X_0)$. And for this, it suffices to show that $Val_0(B) \in Borel(X_0)$, but this time for any $\mathsf{Term}^*(D_0)$-sentence B. But here we recall that $Val_0(B^\dagger) = Val_0(B)$ for any $\mathsf{Term}^*(D_0)$-sentence B and that $B^\dagger$ is a D_0-sentence. So, after all, it suffices to show that $Val_0(B) \in Borel(X_0)$, but this time for any D_0-sentence B. From this point on, the proof of Item (5) proceeds exactly as in Section 5, except that ψ_{1x} is everywhere replaced by ${\psi^*}_{1x}$.

Proof of updated Item (7). The proof proceeds as in Section 5, up until the sentence that begins, "Thus, we want to show, for every D_0-sentence B..." Instead, we want to show, for every $\mathsf{Term}^*(D_0)$-sentence B, that

$$|Int(Val_0(B))| = \bigvee\{|O| : |O| \leq |Val_0(B)| \text{ and } O \text{ is open in } X_0\}. \qquad (*)$$

But, as in the proof of updated Item (5), we recall that $Val_0(B^\dagger) = Val_0(B)$ for any $\mathsf{Term}^*(D_0)$-sentence B and that $B^\dagger$ is a D_0-sentence. So it suffices to show $(*)$ for every D_0-sentence B after all. The rest of the proof proceeds as in Section 5.

Given our updated version of Lemma 5.3, the rest of the completeness proof proceeds exactly as in Section 5.

7 Strong completeness

Completeness is often improved to *strong* completeness: indeed the canonical QS4 completeness proof in Kripke semantics is actually a strong completeness proof. So the question of strong completeness naturally arises in the presence of any completeness theorem.

To define strong completeness in the algebraic semantics, we start with a few preliminaries. A nonempty finite set Γ of sentences of $\mathcal{L}$ is *consistent* iff the negation of their conjunction is not a theorem of QS4. A possibly infinite set Γ of sentences of $\mathcal{L}$ is *consistent* iff every nonempty finite subset is consistent. A set Γ of sentences is *satisfied* in an algebraic model $\mathfrak{A} = \langle \mathcal{A}, D, V \rangle$ iff $Val(\Gamma) =_{\text{df}} \bigwedge_{A \in \Gamma} Val(A)$ exists and $\neq 0$. Γ is *satisfiable* in a predicate TBA $\langle \mathcal{A}, D \rangle$ iff Γ is satisfied in some algebraic model $\langle \mathcal{A}, D, V \rangle$. Γ is *satisfiable* in a TBA $\mathcal{A}$ iff Γ is satisfied in $\langle \mathcal{A}, D \rangle$ for some domain D. Note that QS4 is complete for $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$, $\langle \mathcal{A}, D, V \rangle$] iff every finite consistent set Γ of sentences is satisfiable in $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$, $\langle \mathcal{A}, D, V \rangle$]. We say that QS4 is *strongly complete* for $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$, $\langle \mathcal{A}, D, V \rangle$] iff every consistent set Γ of sentences is satisfiable in $\mathcal{A}$ [$\langle \mathcal{A}, D \rangle$, $\langle \mathcal{A}, D, V \rangle$]. Finally, we say that QS4 is *strongly complete* for $\mathcal{A}$ with a constant countable domain iff there is some countable domain D such that QS4 is strongly complete for $\langle \mathcal{A}, D \rangle$. Analogous definitions can be given for the topological and Kripke semantics.

Completeness is sometimes easy and sometimes hard to improve to strong completeness. The proof in [9] that QS4 is complete for the rational line is already a strong completeness proof. We claim that Lando's result in [10] that S4 is complete for $\mathcal{M}$ can be improved to strong completeness with only a few adjustments; ditto for her result in [11] that FOS4 is complete for $\mathcal{M}$ with expanding countable domains. There is a fairly easy proof in [8] that S4 is strongly complete for the rational line; and a somewhat more difficult proof that S4 is strongly complete for any dense-in-itself metric space. Despite all of these strong completeness claims, we could not see how to improve our completeness result here to strong completeness, so we leave it as an open question:

Open question 7.1. Is QS4 *strongly* complete for $\mathcal{M}$ (with a constant countable domain)?

Dedication. This paper is dedicated to the memory of Grigori Ephraimovich Mints, who taught me the topological semantics of modal and intuitionistic logic.

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