QUANTIFYING OVER PROPOSITIONS IN RELEVANCE LOGIC:
NONAXIOMATISABILITY OF PRIMARY
INTERPRETATIONS OF \( \forall p \) AND \( \exists p \)

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A typical approach to semantics for relevance (and other) logics: specify a class of algebraic structures and take a model to be one of these structures, \( \alpha \), together with some function or relation which associates with every formula \( A \) a subset of \( \alpha \). (This is the approach of, among others, Urquhart, Routley and Meyer and Fine.) In some cases there are restrictions on the class of subsets of \( \alpha \) with which a formula can be associated: for example, in the semantics of Routley and Meyer [1973], a formula can only be associated with subsets which are closed upwards. It is natural to take a proposition of \( \alpha \) to be such a subset of \( \alpha \), and, further, to take the propositional quantifiers to range over these propositions. (Routley and Meyer [1973] explicitly consider this interpretation.) Given such an algebraic semantics, we call (following Routley and Meyer [1973], who follow Henkin [1950]) the above-described interpretation of the quantifiers the primary interpretation associated with the semantics.\(^1\)

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\(^1\) I am much indebted to Nuel Belnap for his constant help and encouragement, and, not least of all, for rather closely inspecting the proofs. I thank Aldo Antonelli for asking me whether the systems studied here fail to be arithmetical (in the recursion theoretic sense of Hinman [1978], Odifreddi [1989] and others) as well as recursively enumerable. Not only are the systems here nonarithmetical, they are recursively isomorphic to full second-order logic. Finally I thank a referee and Richard Shore for pointing in the right direction toward proving this stronger result.

\(^1\) A secondary interpretation of the quantifiers can be developed according to which a secondary model is specified by an algebra, \( \alpha \), a class, \( \pi \), of subsets of \( \alpha \) and a function which assigns to every formula a member of \( \pi \). (\( \pi \) may be a strict subclass of the class of what I have called "propositions.") To get completeness theorems for natural quantified relevance logics, we must restrict our attention to models for which \( \pi \) obeys certain closure conditions.

In addition, a substitutional interpretation of the quantifiers can be developed — Routley and Meyer [1973] consider such an interpretation as well as a primary interpretation. In forthcoming work, I plan to discuss these alternatives.
Section I shows that the primary interpretation of the universal propositional quantifier, $\forall p$, associated with Urquhart's semilattice semantics\(^2\) (Urquhart [1972], [1973], [1992]) is not axiomatisable—indeed the system based on it is recursively isomorphic to full second-order (classical) logic. Section II shows that the primary interpretation of $\forall p$ and $\exists p$, associated with Routley and Meyer's relational semantics (Routley and Meyer [1973]) is not axiomatisable—again, the system based on it is recursively isomorphic to full second-order logic.

These results rely on the following two definitions and theorem from recursion theory. The definitions are given for sets, $A$ and $B$, of natural numbers, but they can be taken to apply to sets of formulas of languages whose syntax can be recursively arithmetised.

**Definition** (Post [1944]; see Odifreddi [1989, p. 324]). $A$ is $1$-**reducible to** $B$ ($A \leq_1 B$) if, for some 1-1 recursive function $f$, $x \in A$ iff $f(x) \in B$. Note: $\leq_1$ is reflexive and transitive.

**Definition** (Post [1944]; see Odifreddi [1989, p. 324]). $A$ is recursively isomorphic to $B$ if, for some 1-1 onto recursive function, $f$, $x \in A$ iff $f(x) \in B$.

**Isomorphism Theorem** (Myhill [1955]; see Odifreddi [1989, p. 325]). $A$ is recursively isomorphic to $B$ iff $A \leq_1 B$ and $B \leq_1 A$.

§1. The primary interpretation of $\forall p$ associated with Urquhart’s semilattice semantics.

1.1. **Formal tools.** The object language has the following vocabulary: $(\cdot)$, $\to$, $\&$, $\forall$, and a countably infinite list of propositional variables, $p_1, p_2, \ldots, p_n, \ldots$.\(^3\) The set of formulas is defined in the usual way. We use $A, B, C, D$ as metavariables ranging over formulas and $p, q, r, \ldots$ as metavariables ranging over propositional variables. Also, we use the following notation:

- for any set, or family of sets, $S$:
  - $\cup S = \{x : (\exists y \in S)(x = y)\}$;
  - $\cap S = \{x : (\forall y \in S)(x = y)\}$;
  - $\mathcal{P}(S) = \{x : (\forall y \in x)(y \in S)\}$;
  - for any two sets, $S$ and $S'$: $S \setminus S' = \{a \in S : a \notin S'\}$.

1.2. The formal semantics. (Except for Theorem 1, Definition 6, and clause (iv) in Definition 4, our presentation is a notational variant of that in Urquhart [1973] and [1992].)

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\(^2\)Here, I investigate the interpretation of $\forall p$ and not of $\exists p$ for two related reasons:

1. Urquhart’s semantics is concerned with the implicational-conjunctive behaviour of relevance logic, so it seems appropriate to leave $\exists$ out of the picture, as least as a primitive.

2. The secondary interpretation motivated by Urquhart’s semantics is complete for the logic which results when we add (with a reasonable set of axioms) $\forall p$ to $R_{\&}$; but not when we add both $\forall p$ and $\exists p$. (Compare the incompleteness of Urquhart’s semantics for $R_{\&}$.)

\(^3\)The language could be supplied with a stock of propositional constants, without significant effect on the current project.
Definition 1. A semilattice is an ordered pair \( L = \langle L, \leq \rangle \) such that \( \leq \) partially orders \( L \), and such that any two points, \( a \) and \( b \), in \( L \) have a least upper bound, \( a \circ b \), in \( L \). A semilattice with 0 is a semilattice with a \( \leq \)-smallest element, 0.

Definition 2. A consequence model or c-model (the terminology is from Urquhart [1992]; or "model") is an ordered pair \( M = \langle L, \phi \rangle \), where \( L = \langle L, \leq \rangle \) is a semilattice with 0, and where \( \phi \) is a function which assigns to each atomic formula a subset of \( L \). \( M \) assigns the set \( \phi(p) \) to the atomic formula \( p \). (Note that for Urquhart's semantics, the class of algebraic structures under consideration is the class of semilattices with 0. Note also that there is no restriction on which subsets of \( L \) can be the semantic value of a propositional variable; so the set of propositions is just \( \mathcal{P}(L) \).)

Definition 3. Given a c-model, \( M = \langle L, \leq, \phi \rangle \), and a subset, \( S \) of \( L \), \( M[S/p] \) is the c-model which is just like \( M \) except that it assigns \( S \) to \( p \).

Definition 4. Given a c-model, \( M = \langle L, \leq, \phi \rangle \), and a formula \( A \), \( M(A) \), the subset of \( L \) assigned to \( A \) by \( M \), is defined thus:

- (i) if \( A \) is atomic, \( M(A) = \phi(A) \);
- (ii) \( M((A \& B)) = M(A) \cap M(B) \);
- (iii) \( M((A \rightarrow B)) = \{ a \in L : (\forall b \in M(A)) (a \circ b \in M(B)) \} \);
- (iv) \( M(\forall pA) = \bigcap \{ M[S/p](A) : S \subseteq L \} \).

Definition 5 (Validity). Given a semilattice with 0, \( L = \langle L, \leq \rangle \), a model \( M \) and a formula \( A \):

- \( M \models A \) iff 0 \( \in M(A) \). (\( M \) validates or satisfies \( A \).)
- \( L \models A \) iff (\( \forall M = \langle L, \phi \rangle \) \( (M \models A) \). (\( L \) validates or satisfies \( A \).)
- \( A \) is c-valid iff (\( \forall M \) \( (M \models A) \).)

For quantifier-free formulas, the above notions correspond exactly to Urquhart's semantic notions.

Urquhart's Soundness and Completeness Theorem:

If \( A \) is quantifier-free, then \( A \) is c-valid iff \( A \) is a theorem of \( \mathbf{R}_{\&} \).

Definition 6. \( \mathbf{R}_{\&} \mathbf{P^+} = \{ A : A \text{ is a c-valid formula} \} \). (\( \mathbf{R}_{\&} \mathbf{P^+} \) is the logic based on the primary interpretation of \( \forall p \) associated with Urquhart's semilattice semantics.)

Theorem 1. \( \mathbf{R}_{\&} \mathbf{P^+} \) is recursively isomorphic to full second-order classical logic.

1.3. Proof of Theorem 1: Preliminaries. Nerode and Shore [1980] show that full second-order classical logic (\( \mathbf{L}^2 \)) is recursively isomorphic to the theory of distributive lattices with second-order monadic quantification over ideals (DLMII). In the appendix, we show that DLMII is 1-reducible to the monadic theory of semilattices with 0 (\( \mathbf{MSL0} \)). And so, \( \mathbf{L}^2 \) is 1-reducible to \( \mathbf{MSL0} \). To show that \( \mathbf{R}_{\&} \mathbf{P^+} \) is recursively isomorphic to \( \mathbf{L}^2 \), we show:

1. \( \mathbf{MSL0} \) is 1-reducible to \( \mathbf{R}_{\&} \mathbf{P^+} \) (§1.4); and
2. \( \mathbf{R}_{\&} \mathbf{P^+} \) is 1-reducible to \( \mathbf{L}^2 \) (§1.5).

To make this precise, we give some definitions.

Definition 7. The full second-order language (SOL) has the following vocabulary: (\( , , \), \( \neg , \), \( \forall \); the equals sign, =; a countably infinite list of individual variables: \( x_1, x_2, \ldots, x_n, \ldots \); and, for each \( m \geq 1 \), a countably infinite list of \( m \)-place relation variables: \( X^{m}_{11}, X^{m}_{22}, \ldots, X^{m}_{n1} \ldots \). We also include a 2-place relation constant, \( \leq \), and an individual constant, 0. (These are not necessary, but they simplify things.)
The set of formulas, and the notions of model and validity are defined in the usual way, as are $\&$, $\lor$, and $\exists$. Second-order logic ($L^2$) is the set of formulas validated by every model.

Definition 8. The monadic second-order language (MSOL) is exactly like SOL except that rather than a countable list of $m$-place relation variables for each $m \geq 1$, it has a countable list of 1-place predicate variables: $X_1, X_2, \ldots, X_n, \ldots$.

Definition 9. We use the following abbreviation in SOL and MSOL, for individual variables, $x$, $y$, and $z$. Here, $w$ is the first in the list of individual variables distinct from $x$, $y$, and $z$:

$$z \approx x \circ y =_{df} (x \leq z \land y \leq z \land (\forall w)((x \leq w \land y \leq w) \supset z \leq w)).$$

Definition 10. *semilat* is the following formula of SOL and MSOL:

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 [x_1 \leq x_1 \land ((x_1 \leq x_2 \land x_2 \leq x_1) \supset x_1 = x_2)$$

$$\land ((x_1 \leq x_2 \land x_2 \leq x_3) \supset x_1 \leq x_3) \land \forall x_1 \forall x_2 \exists x_3 (x_3 \approx x_1 \circ x_2).$$

*semilat* 0 is the following formula: ($\forall x_1 (0 \leq x_1)$).

Definition 11. The monadic second-order theory of semilattices with 0 (MSL0) is the set of MSOL formulas which are validated by every model which validates *semilat* 0.

Theorem 2. $L^2$ is 1-reducible to MSL0.

(See the appendix for the proof.)

1.4. Proof of Theorem 1: MSL0 is 1-reducible to $R_\& P_+$. In Tables 1 and 2, $L = \langle L, \leq \rangle$ and $M = \langle L, \phi \rangle$.

Note for Table 1. Given a c-model $M$ and a formula $A$, $A$ can be thought of as playing two roles:

1. $A$ names a subset of the lattice, namely, $M(A)$; and
2. $A$ makes a claim about the c-model.

For example $(p \rightarrow q)$ names the set $M(p \rightarrow q)$ and says that $M(p) \subseteq M(q)$, since $M \models (p \rightarrow q)$ iff $M(p) \subseteq M(q)$. Table 1 lists formulas constructed with the primitive object language connectives and indicates what the formulas say and what they name. The blank entries are those of no particular interest.

<table>
<thead>
<tr>
<th>Formula</th>
<th>What the formula names: $M(\text{Formula})=$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A &amp; B$</td>
<td>$M \models A$ and $M \models B$</td>
</tr>
<tr>
<td>$A \rightarrow B$</td>
<td>$M(A) \subseteq M(B)$</td>
</tr>
<tr>
<td>$\forall p A$</td>
<td>$(\forall S \subseteq L)(M[S/p] \models A)$</td>
</tr>
</tbody>
</table>
Note 1 for Table 2. The last two columns of Table 2 play the same role as the last two columns of Table 1: they tell us how we can interpret the object language formulas. The entries in these last two columns represent more or less substantial claims, which are left to the reader to prove.

Note 2 for Table 2. Quantification over propositions amounts, under the primary interpretation under consideration, to quantification over the subsets of whatever semilattice we are considering. The effect of quantifying over elements of the semilattice can be had by restricting a particular quantifier to atomic subsets of the lattice and by informally identifying an element \(a \in L\) with the atomic set \(\{a\} \subseteq L\). This remark should explain the importance of the last row.

<table>
<thead>
<tr>
<th>Definiendum</th>
<th>Definiens</th>
<th>What the definiendum says: (M \models \text{Definiendum iff})</th>
<th>What the definiendum names: (M(\text{Definiendum})=)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((A \lor B))</td>
<td>(\forall p(((A \rightarrow p) &amp; (B \rightarrow p)) \rightarrow p))</td>
<td>(M \models A) or (M \models B)</td>
<td>(M(A) \cup M(B))</td>
</tr>
<tr>
<td>((A \leftrightarrow B))</td>
<td>((A \rightarrow B) &amp; (B \rightarrow A))</td>
<td>(M(A) = M(B))</td>
<td></td>
</tr>
<tr>
<td>(F)</td>
<td>(\forall p p)</td>
<td></td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(t)</td>
<td>(\forall p (p \rightarrow p))</td>
<td></td>
<td>({0})</td>
</tr>
<tr>
<td>((A \supset B))</td>
<td>((A &amp; t) \rightarrow B)</td>
<td>(M \models A \Rightarrow M \models B)</td>
<td></td>
</tr>
<tr>
<td>(\neg A)</td>
<td>(A \supset F)</td>
<td></td>
<td>(M \not\models A)</td>
</tr>
<tr>
<td>((A \circ B))</td>
<td>(\forall p ((A \rightarrow (B \rightarrow p)) \rightarrow p))</td>
<td></td>
<td>({a \circ b: a \in M(A) &amp; b \in M(B)})</td>
</tr>
<tr>
<td>(A \in L)</td>
<td>(\neg (A \rightarrow F) &amp; \forall p (((A &amp; p) \rightarrow F) \lor (A \rightarrow p)))</td>
<td></td>
<td>(M(A) = {a} \text{ for some } a \text{ in } L)</td>
</tr>
</tbody>
</table>

Postscript to Table 2. With "\(\supset\)" and "\(\neg\)" we can reflect metalinguistic (classical) implication and negation in the object language—see the entries in the third column. "\(t\)" is also useful: it names \(\{0\}\); so under the heuristic identification (explained above) of atomic subsets of semilattices with their elements, "\(t\)" names 0.

Definition 12. Define a recursive 1-1 function, \(f\), from MSOL formulas to MSOL formulas as follows. Here, for \(i \geq 1\), \(x_i(p_i)\) is the \(i\)th individual (propositional) variable in the list of individual (propositional) variables in MSOL (the object language). \(x_0\) is the MSOL constant 0, and \(p_0\) is the defined object language constant, \(t\).
f(x_i = x_j) = (p_{2i} \leftrightarrow p_{2j});
f(x_i \leq x_j) = (p_{2i} \circ p_{2j} \rightarrow p_{2j});
f(x_i x_j) = (p_{2j} \rightarrow p_{2i+1});
f((A \supset B)) = (f(A) \supset f(B));
f(\neg A) = \neg f(A);
f(\forall x_i A) = \forall p_{2i}.(p_{2i} \in L \supset f(A));
f(\forall X_i A) = \forall p_{2i+1} f(A).

**Definition 13.** Let \( g \) be the following recursive 1-1 function from MSOL formulas to MSOL formulas. Suppose \( A \) is an MSOL formula and \( n \) is the greatest number such that \( x_n \) or \( X_n \) appears in \( A \). Let \( g(A) = \forall x_1 \cdots \forall x_n \forall X_1 \cdots \forall X_n A \).

**Lemma 1.** For any MSOL formula, \( A, A \in \text{MSOL0} \iff g(A) \in \text{MSOL0}. \)

**Lemma 2.** For any closed MSOL formula, \( A, A \in \text{MSOL0} \iff f(A) \in \text{R}_{\&} P +. \)

**Corollary 1.** For any MSOL formula, \( A, A \in \text{MSOL0} \iff g(A) \in \text{R}_{\&} P +. \)

**Corollary 2.** \( \text{MSOL0} \) is 1-reducible to \( \text{R}_{\&} P +. \)

**I.5. Proof of Theorem 1: \( \text{R}_{\&} P + \) is 1-reducible to \( L^2. \)

**Definition 14.** Define a 1-1 recursive function \( f \) from object language formulas to SOL formulas as follows. Note: given an object language formula \( A: X_i \) will be free in \( f(A) \) just in case \( p_i \) is free in \( A \); the only individual variable free in \( f(A) \) will be \( x_i \); and \( f(A) \) will contain no \( m \)-ary second-order variables for \( m \geq 2. \)

\[
f(p_i) = X_i x_i;
f((A \& B)) = (f(A) \& f(B));
f((A \rightarrow B)) = \forall x_3 (f(A)[x_2/x_1] \supset \forall x_3(x_3 \approx x_2 \circ x_1 \supset f(B)[x_3/x_1]));
f(\forall p_i A) = \forall X_i f(A).
\]

**Definition 15.** We define another 1-1 recursive function, \( g \), which assigns to every formula of the object language a formula in SOL:

\[
g(A) = (\text{semilat} 0 \supset f(A)[0/x_1]).
\]

**Lemma 3.** For any object language formula, \( A, A \in \text{R}_{\&} P + \iff g(A) \in L^2. \)

**Corollary 3.** \( \text{R}_{\&} P + \) is 1-reducible to \( L^2. \)

Theorem 1 is a corollary to Corollaries 2 and 3 and Theorem 2.

**§II. The primary interpretation of \( \forall p \) and \( \exists p \) associated with Routley and Meyer’s relational semantics.** (The formal tools are the same as for §I, except that the object language has the following additional vocabulary: \( \lor, \neg, \exists \). Routley and Meyer also add the two-place connective \( \circ \), but this can be defined, in any extension of \( R \) by \( (A \circ B) = \neg (A \rightarrow \neg B). \)

**II.1. The formal semantics.** (Except for clauses (vi) and (vii) of Definition 20, Definition 22 and Theorem 3, our presentation is a notational variant of Routley and Meyer’s. Most of the terminology is theirs.)

**Definition 16.** A relevant model structure (rms) is a quadruple \( K = \langle 0, K, R, * \rangle \) where \( K \) is a set, \( 0 \in K \), \( R \) is a ternary relation on \( K \) and \( * \) is a unary function on \( K \) satisfying postulates p1–p6 to follow (for all \( a, b, c, d \in K \)). Before we state the
postulates, define (for \(a, b, c, d \in K\)):

(d1) \(a < b\) iff \(R0ab\);
(d2) \(R^2abcd\) iff \((\exists x \in K)(Rabx \text{ and } Rxcd)\).

The postulates are

(p1) \(R0aa\);
(p2) \(Raab\);
(p3) \(R^2abcd \Rightarrow R^2acbd\);
(p4) \(R^20abc \Rightarrow Rabc\);
(p5) \(Rabc \Rightarrow Rac*b*\);
(p6) \(a** = a\).

(For Routley and Meyer’s semantics the class of algebraic structures under consideration is the class of rms’s.)

Before continuing, we state a useful Lemma:

**Lemma 4.** (1) \(Rabc \Rightarrow Rbac\).
(2) \((\forall a, b \in K)(\exists c \in K)(Rabc)\).

**Proof.** (1) Suppose \(Rabc\). Since \(R0aa\) (p1), \(R^20abc\) (d2). So \(R^20bac\) (p3). So \(Rbac\) (p4).

(2) Choose \(a, b \in K\). Now, \(R0*0*0*\) (p2). So \(R0*00\) (p5 and p6). So \(R00*0\) (1). So, since \(R0bb\) (p1), \(R^2*0*bb\) (d2). So \(R0*bb\) (p4). Also, \(R0aa\) (p1). So \(Ra0a\) (1). So \(Raa*0*\) (p5). So \(R^2aa*bb\) (d2). So \(R^2aba*b\) (p3). So \((\exists c)(Rabc \& Rca*b)\) (d2). So \((\exists c \in K)(Rabc)\).

**Definition 17.** Given an rms \(K = \langle 0, K, R, * \rangle\), a \(K\)-proposition is any subset, \(P\), of \(K\) such that

\[
[(b \in P \& b < a) \Rightarrow a \in P].
\]

\(\pi(K)\) (we often just write “\(\pi\)” is the set of \(K\)-propositions.\(^4\)

**Definition 18.** A relational model (or “model”) is an ordered pair \(M = \langle K, \phi \rangle\) where \(K\) is an rms and \(\phi\) is a function which assigns to each atomic formula a member of \(\pi\). \(M\) assigns the set \(\phi(p)\) to the atomic formula \(p\).

**Definition 19.** Given a model, \(M = \langle K, \phi \rangle\), and a \(K\)-proposition \(P, M[P/p]\) is the model which is just like \(M\) except that it assigns \(P\) to \(p\).

**Definition 20.** Given a model, \(M = \langle 0, K, R, *, \phi \rangle\), and a formula \(A, M(A)\), the subset of \(K\) assigned to \(A\) by \(M\), is defined by:

(i) if \(A\) is atomic, \(M(A) = \phi(A)\);
(ii) \(M((A \& B)) = M(A) \cap M(B)\);
(iii) \(M((A \lor B)) = M(A) \cup M(B)\);
(iv) \(M((A \rightarrow B)) = \{ c \in K : (\forall a \in M(A))(\forall b \in K)(Rcab \Rightarrow b \in M(B))\}\);
(v) \(M(\neg A) = \{ a \in K : a* \notin M(A)\}\);
(vi) \(M(\forall pA) = \bigcap \{M[P/p](A) : P \in \pi\}\);
(vii) \(M(\exists pA) = \bigcup \{M[P/p](A) : P \in \pi\}\).

Note: for every formula \(A, M(A) \in \pi\).

\(^4\)Routley and Meyer [1973] define \(\pi(K)\) as the algebra of propositions rather than simply the set, but our definition suffices for the present purposes.
Definition 21 (Validity). Given an rms \( K \), a model \( M \) and a formula \( A \): 
\[ M \models A \text{ iff } 0 \in M(A), \text{ (} M \text{ validates or satisfies } A \text{).} \]
\[ K \models A \text{ iff } (\forall M = \langle K, \phi \rangle)(M \models A). \text{ (} K \text{ validates or satisfies } A \text{).} \]
\[ A \text{ is valid iff } (\forall M)(M \models A). \]

For quantifier-free formulas, the above notions correspond exactly to Routley and Meyer's semantic notions.

Routley and Meyer's Soundness and Completeness Theorem. If \( A \) is quantifier-free, then \( A \) is valid iff \( A \) is a theorem of \( R \).

Definition 22. \( \text{RP}^+ = \{ A : A \text{ is a valid formula} \} \). (\( \text{RP}^+ \) is the logic based on the primary interpretation of \( \forall p \) and \( \exists p \) associated with Routley and Meyer’s relational semantics.)

Theorem 3. \( \text{RP}^+ \) is recursively isomorphic to full second-order logic.

II.2. Proof of Theorem 3: Preliminaries. As in §1, we encode MSL0 in the object language and encode \( \text{RP}^+ \) in SOL. The latter is similar to §1.5. To accomplish the former, we associate with every semilattice with 0, the rms defined presently.

Definition 23. If \( L = \langle L, \leq \rangle \) is a semilattice with 0, let \( K_L = \langle 0, L, R, * \rangle \) where 0 is the \( \leq \)-smallest element of \( L \), and for \( a, b, c \) in \( L \):
if \( 0, a, b, c \) are distinct then \( Rabc \); if and only if \( a \leq b \), then \( Rabb \) and \( Rbab \) and \( Rba \); and \( a^* = a \).

Theorem 4. \( K_L \) is an rms.

Proof. \( K_L \) must satisfy p1—p6 of Definition 16 (§1.1). p1, p2, p5 and p6 are straightforward. To see that \( K_L \) satisfies p4, suppose that \( R^20abc \). Then \( (\exists x \in L) \text{ (} R0ax \text{ and } Rxbc \text{).} \) By the definition of \( R, x = a \). So \( Rabc \). Before we show that \( K_L \) satisfies p3, note that \( R \) is symmetric—i.e., the following are equivalent: \( Rabc; Rbab; Rcab; Rcba; Rabc \). As a result, the following are equivalent: \( R^2abcd; R^2badc; R^2baced; R^2badc; R^2cdab; R^2dcab; R^2dcba; R^2dcba. \) Now, suppose that \( R^2abcd \). To show that \( R^2acbd \), we consider cases and subcases.

Case 1. \( a = 0 \). \( R0bcd \). So, by p4, \( Rbdc \). So \( Rcbd \). So, since \( R0cc, R^20cbd \), as desired.

Case 2. \( b, c, \) or \( d = 0 \). Using the above equivalences, reduce this to Case 1.

Case 3. \( b = c \). \( R^2acbd \) follows immediately from \( R^2abcd \).

Case 4. \( a = d \). \( R^2abca \). So, by the above equivalences, \( R^2acba \), as desired.

Case 5. \( a, b, c, d \neq 0 \) and \( b \neq c \) and \( a \neq d \).

Case 5.1. \( a \neq c \) and \( b \neq d \).
Case 5.11. $L$ is finite. Then $L$ has a $\leq$-greatest member, say $x$. Whether $x = a$
or $x = c$ or neither, $Racx$. Similarly $Rcbd$. So $R^2acbd$ as desired.

Case 5.12. $L$ is infinite. Then choose any $x$ distinct from $0$, $a$, $b$, $c$ and $d$. By the definition of $R$, $Racx$ and $Rbcd$. So $R^2acbd$ as desired.

Case 5.2. $a = c$. $Raca$. Also, since $b \neq c$, $a \neq b$. So, since $a \neq d$, $Rabd$. So $R^2acbd$, as desired.

Case 5.3. $b = d$. Using the above equivalences, reduce this to Case 5.2.

Q.E.D.

(We note that $K_4$ is singular as defined in §II.3, Definition 24, below, and superclassical as defined in §II.4, Definition 25, below.)

Not every rms is such an rms. The trick is to find an object language formula $SEMILAT$ which is validated by an rms just in case the rms is related to some semilattice with 0 as in Definition 23. Then, after we have defined functions $f$ and $g$ similar to those defined in Definitions 12 and 13 (§I.4, above), our main theorem should look something like this: For any MSOL formula $A$, $A \in MSL0$ iff $(SEMILAT \supset f_0(A)) \in RP^+$.

There is a special difficulty for the Routley-Meyer semantics which is not present for the Urquhart semantics. In the context of Urquhart's semantics, we were able to reflect metalinguistic implication in the object language by defining a two-place object language connective, $\supset$, with the following property: for every $M$, $M \models (A \supset B)$ iff $(M \models A \Rightarrow M \models B)$. This ability to reflect metalinguistic implication in the object language figured prominently in the translation from the MSOL to the object language in two ways (see Definition 12, §I.4):

(1) it allowed the restriction of propositional quantification to "elements" of $L$; and

(2) it provided a convenient translation of $(A \supset B)$.

Unfortunately, we have not been able to define such a connective in the context of Routley and Meyer's semantics for $RP^+$. (Conjecture: no such connective is definable.) For this reason, we take a detour through a restricted class of rms's, the superclassical rms's, discussed in §II.4, below. But first we restrict attention to singular rms's.

II.3. Singular relevant model structures. It would be very convenient if, for every rms $K = \langle 0, K, R, * \rangle$, and for every $a, b \in K$, $[a < b$ and $b < a] \Rightarrow a = b]$. This is not in general the case, but the following definition, theorem and corollary allow us to restrict our attention to rms's in which it is the case.

Definition 24. $K = \langle 0, K, R, \ast \rangle$ is a singular relevant model structure (srms) iff $K$ is an rms and $(\forall a, b \in K)((R0ab$ and $R0ba) \Rightarrow a = b)$. A model $M$ is a singular model (s-model) iff $M = \langle K, \phi \rangle$ for some singular $K$.

Theorem 5. For every model $M = \langle 0, K, R, *, \phi \rangle$ there is an s-model $M'$ such that $M'$ validates exactly the same formulas which $M$ validates.

Proof. Consider some model $M = \langle 0, K, R, *, \phi \rangle$. For $a, b \in K$, define $a \approx b$ as $(R0ab$ and $R0ba)$. $\approx$ is an equivalence relation. For any $a \in K$ let $a' = \{b \in K : b \approx a\}$. Let $K' = \{a' : a \in K\}$. Let $R'$ be the following relation on $K'$: $\langle a, b', c' \rangle : Rabc$. Define $(a')^\ast$ as $(a^\ast)$, and $\phi'(A)$ as $(\phi(A))'$. $(\ast$ and $\phi'$ are well defined.) Let $M' = \langle 0', K', R', \ast', \phi' \rangle$. Then $M'$ is an s-model which validates exactly the same formulas as $M$.

Q.E.D.
COROLLARY 4. \( \mathbf{RP^+} = \{A: A \text{ is validated by every s-model}\} \).
Henceforth, we restrict attention to srms's.

II.4. Superclassical relevant model structures.

DEFINITION 25. An rms (srms) \( K = \langle 0, K, R, * \rangle \) is superclassical iff \((\forall a \in K) (a^* = a)\). We use the abbreviation "Krms" ("Ksrms") for "superclassical rms" ("superclassical srms"). (The terminology and abbreviations are adapted from Routley, Meyer, Plumwood and Brady [1982], where such and similar rms's are studied.) A model (s-model) \( M = \langle K, \phi \rangle \) is superclassical iff \( K \) is. We use the term "K-model" ("Ksrms-model") for "superclassical model" ("superclassical s-model").

Note 1. If \( K \) is a Ksrms, then \( \pi(K) = \mathcal{P}(K) \).

Note 2. If \( L \) is a semilattice with 0, then \( K_L \) is a Ksrms.

DEFINITION 26. The logic \( \mathbf{KR} \) is the unquantified propositional logic which results by adding the following axiom scheme to \( \mathbf{R} \): \((A \& \sim A) \to B\). (See Routley, Meyer, Plumwood and Brady [1982].)

THEOREM 6 (Routley, Meyer, Plumwood and Brady [1982]). If \( A \) contains no propositional quantifiers, then \( A \) is a theorem of \( \mathbf{KR} \) iff \( A \) is validated by every \( K \)-model.

DEFINITION 27. \( \mathbf{KRP^+} = \{A: A \text{ is validated by every K-model}\} \).

THEOREM 7. \( \mathbf{KRP^+} = \{A: A \text{ is validated by every Ks-model}\} \). (See Theorem 4, §II.3, above.)

THEOREM 8. \( \mathbf{KRP^+} \) is recursively isomorphic to full second-order logic.

Given Theorem 2 (§I.3), Theorems 3 and 8 follow from these results: §II.5: \( \mathbf{MSL0} \) is 1-reducible to \( \mathbf{KRP^+} \); §II.6: \( \mathbf{KRP^+} \) is 1-reducible to \( \mathbf{RP^+} \); §II.7: \( \mathbf{RP^+} \) is 1-reducible to \( \mathbf{L}^2 \).

II.5. Proof of Theorems 3 and 8: \( \mathbf{MSL0} \) is 1-reducible to \( \mathbf{KRP^+} \).

For Table 3, let \( K = \langle 0, K, R, * \rangle \) be an srms; let \( a, b, c; \) etc., range over the elements of \( K \); and let \( S \) and \( S' \) range over the subsets of \( K \).

Note for Table 3. The definition of "semilattice-like" is independent of the choice of \( L \).

<table>
<thead>
<tr>
<th>Definiendum</th>
<th>Definien</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a])</td>
<td>({b: a &lt; b}). Note: ([a] \in \pi), and, if ( K ) is a Ksrms, then ([a] = {a}).</td>
</tr>
<tr>
<td>(a \preceq b)</td>
<td>(Rbba) Note: (\preceq) is a partial ordering for an interesting class of srms's (including semilattice-like srms's, as defined in this table, below). (\preceq) is, in general, quite different from (&lt;).</td>
</tr>
<tr>
<td>(S \circ S')</td>
<td>({c: (\exists a \in S)(\exists b \in S')(Rabc)})</td>
</tr>
<tr>
<td>(K) is semilattice-like</td>
<td>(K) is isomorphic to (K_L) for some semilattice with 0, (L) (where &quot;isomorphic&quot; is defined for rms's in the obvious way).</td>
</tr>
<tr>
<td>(\langle K, \phi \rangle) is semilattice-like</td>
<td>(K) is semilattice-like.</td>
</tr>
</tbody>
</table>
The notes and postscripts to Tables 1 and 2 in §I.4 are of substantial importance in interpreting Tables 4 and through 6 in the present section.

For Tables 4 through 6, assume that $K = \langle 0, K, R, \ast \rangle$ is a Ks-model and that $M = \langle K, \phi \rangle$ is a Ks-model.

Note for Table 6. In the last row of the second column in Table 6 we use the abbreviations $(\forall p \in K)A$ and $(\exists p \in K)A$. These stand for $\forall p(p \in K \Rightarrow A)$ and $\exists p(p \in K \& A)$, respectively.

Now, to show that $\text{MSL}_0$ is 1-reducible to $\text{KRP}^+$, define functions $f$ and $g$ as in Definitions 12 and 13 (in §I.4, above) with the following exception: in the definition of the recursive function, $f$, (Definition 12) replace the clauses for $f(x_i \leq x_j)$, $f(\neg A)$, and $f((\forall x_i)A)$ with (respectively):

\[
f(x_i \leq x_j) = (p_{2i} \subseteq p_{2j});
f(\neg A) = \neg f(A); \text{ and}
f((\forall x_i)A) = (\forall p_{2i})(p_{2i} \in K \Rightarrow f(A)).
\]

**Table 4. Object language connectives and their two meta-linguistic interpretation.**

<table>
<thead>
<tr>
<th>Formula</th>
<th>What the formula says: $M \models$ Formula iff</th>
<th>What the formula names: $M(\text{Formula})$ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(A &amp; B)$</td>
<td>$M \models A$ and $M \models B$</td>
<td>$M(A) \cap M(B)$</td>
</tr>
<tr>
<td>$(A \lor B)$</td>
<td>$M \models A$ or $M \models B$</td>
<td>$M(A) \cup M(B)$</td>
</tr>
<tr>
<td>$(A \rightarrow B)$</td>
<td>$M(A) \subseteq M(B)$</td>
<td></td>
</tr>
<tr>
<td>$\sim A$</td>
<td>$M \not\models A$</td>
<td>$K \setminus M(A)$</td>
</tr>
<tr>
<td>$\exists p A$</td>
<td>$(\exists S \subseteq K)(M[S/p] \models A)$</td>
<td></td>
</tr>
<tr>
<td>$\forall p A$</td>
<td>$(\forall S \subseteq K)(M[S/p] \models A)$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5. Preliminary object language definitions.**

<table>
<thead>
<tr>
<th>Definiendum</th>
<th>Definiens</th>
<th>What the definiendum says: $M \models$ Definiendum iff</th>
<th>What the definiendum names: $M(\text{Definiendum})$ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$\forall pp$</td>
<td></td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\forall p(p \rightarrow p)$</td>
<td></td>
<td>${0}$</td>
</tr>
<tr>
<td>$(A \supset B)$</td>
<td>$(\sim A \lor B)$</td>
<td>$M \models A \rightarrow M \models B$</td>
<td>$K \setminus M(A) \cup M(B)$</td>
</tr>
<tr>
<td>$(A \leftrightarrow B)$</td>
<td>$((A \rightarrow B) &amp; (B \rightarrow A))$</td>
<td>$M(A) = M(B)$</td>
<td></td>
</tr>
<tr>
<td>$(A \circ B)$</td>
<td>$\sim (A \rightarrow \sim B)$</td>
<td></td>
<td>$M(A) \circ M(B)$</td>
</tr>
</tbody>
</table>
### Table 6. Further object language definitions.

<table>
<thead>
<tr>
<th>Definiendum</th>
<th>Definiens</th>
<th>What the definiendum says: ( M \models ) Definiendum iff</th>
</tr>
</thead>
<tbody>
<tr>
<td>((A \in K))</td>
<td>(~(A \to F) &amp; \forall p(((A &amp; p) \to F) \vee (A \to p)))</td>
<td>( M(A) = {a} ) for some ( a ) in ( K )</td>
</tr>
<tr>
<td>(R(A, B, C))</td>
<td>(A \in K &amp; B \in K &amp; C \in K &amp; (C \to A \circ B))</td>
<td>( M(A) = {a}, M(B) = {b}, ) and ( M(C) = {c}, ) where ( R_{abc} )</td>
</tr>
<tr>
<td>((A \leq B))</td>
<td>(R(B, B, A))</td>
<td>( M(A) = {a} ) and ( M(B) = {b} ), where ( a \leq b )</td>
</tr>
<tr>
<td>(SEMILAT)</td>
<td>((\forall p \in K)(\forall q \in K)(\forall r \in K)[(p \leq q &amp; p \leq r \Rightarrow (p \leftrightarrow q)) &amp; &amp; ((p \leq q &amp; q \leq r) \Rightarrow r \leq p)] &amp; (\forall p \in K) (\exists r \in K)[(p \leq r &amp; q \leq r &amp; (\forall s \in K)((p \leq s &amp; q \leq s) \Rightarrow r \leq s)] &amp; (\forall p \in K)(t \leq p))</td>
<td>( M ) is semilattice-like</td>
</tr>
</tbody>
</table>

Then, we have the following lemmas and corollaries. (Compare with Lemmas 1 and 2 and Corollaries 1 and 2 in §1.4.)

**Lemma 5.** For any MSOL formula, \( A, A \in MSOL \) iff \( g(A) \in MSOL \).

**Lemma 6.** For any closed MSOL formula, \( A, A \in MSOL \) iff \((SEMILAT \supset f(A)) \in KRP^+.\)

**Corollary 5.** For any MSOL formula, \( A, A \in MSOL \) iff \((SEMILAT \supset f g(A)) \in KRP^+.\)

**Corollary 6.** \( MSOL \) is 1-reducible to \( KRP^+.\)

**II.6. Proof of Theorems 3 and 8: KRP^+ is 1-reducible to RP^+.** Theorem 9, below, suffices.

**Lemma 7.** If \( M = \langle 0, K, R, *, \phi \rangle \) is an s-model then \( M \) is a Ks-model iff \( M \models \forall p \forall q((p \& \sim p) \rightarrow q) \).

**Proof.** (\( \Rightarrow \)) is straightforward.

(\( \Leftarrow \)). Suppose \( M \models \forall p \forall q((p \& \sim p) \rightarrow q) \). Choose any \( a \in K \), and let \( M' = M[[a]/p][\emptyset/q] \). Then \( M' \models (p \& \sim p) \rightarrow q \). So \([a] \cap M'(\sim p) \subseteq \emptyset \). So \( a \notin M'(\sim p) \), So \( a^* \in [a] \). Similarly \( a^{**} \in [a^*] \). So \( a^* < a^{**} = a < a^* \). So, since \( M \) is an s-model, \( a^* = a \).

**Theorem 9.** \( A \in KRP^+ \) iff \([\forall p \forall q((p \& \sim p) \rightarrow q) \& t] \rightarrow A \in RP^+.\)

**Proof.** (\( \Leftarrow \)). Suppose \([\forall p \forall q((p \& \sim p) \rightarrow q) \& t] \rightarrow A \notin RP^+.\) Then there is an s-model \( M = \langle 0, K, R, *, \phi \rangle \) such that \( M \equiv [\forall p \forall q((p \& \sim p) \rightarrow q) \& t] \rightarrow A \). So there exists

\[ M \models [\forall p \forall q((p \& \sim p) \rightarrow q) \& t] \rightarrow A. \]

So, \( M \models A \). So \( A \in KRP^+.\)

(\( \Rightarrow \)). Suppose \([\forall p \forall q((p \& \sim p) \rightarrow q) \& t] \rightarrow A \notin RP^+.\) Then there is an s-model \( M = \langle 0, K, R, *, \phi \rangle \) such that \( M \equiv [\forall p \forall q((p \& \sim p) \rightarrow q) \& t] \rightarrow A \). So there exists
$b \in K$ such that

1. $b \in M(\forall p \forall q ((p \& \sim p) \rightarrow q))$;
2. $b \in M(t)$; and
3. $b \notin M(A)$.

**Claim.** $M$ is a $K$s-model.

**Proof of Claim.** Choose any $a \in K$, and let $M' = M[[a]/p][\emptyset/q]$. Then, by
(1), $b \in M'(p \& \sim p) \rightarrow q$. So, $(\forall c, d \in K)(c \in M'(p \& \sim p)$ and $Rbcd) \Rightarrow d \in \emptyset]$. So, $(\forall c, d \in K)(c \notin M'(p \& \sim p)$ or not $Rbcd)$. Now, by Lemma 4 (§II.1, above),
$(\forall c \in K)(\exists d \in K)(Rbcd)$. Therefore, $(\forall c \in K)(c \notin M'(p \& \sim p))$. So $a \notin M'(p \& \sim p)$.
Now, $a \in [a] = M'(p)$. So, $a \notin M'(\sim p)$. So, $a^* \in [a]$. So $a < a^*$. This argument is
completely general wrt $a$; so $a^* < a^{**}$. So $a^* < a$. So, since $M$ is an s-model, $a^* = a$.
So $M$ is a $K$s-model.

By (2), $b \in M(t)$. So $0 < b$. So $R0b0$. So $R0b*0*$. So $R0b0$. So $b < 0$. So, since $M$
is an s-model, $b = 0$. So, since $b \notin M(A)$, $0 \notin M(A)$. So $M$ is a $K$s-model such that
$M \not\models A$. So $A \notin \text{KRP} +$. Q.E.D.

**II.7.** **Proof of Theorems 3 and 8:** $\text{RP} +$ is 1-reducible to $L^2$. To show this, encode $\text{RP} +$
into $L^2$. Proceed as with the encoding of $\text{R}_2 \text{P} +$ into $L^2$ (in §I.6, above),
with the following three exceptions:

1. take the language of $L^2$ to contain both a 3-place relation constant, $R$, and
   a one place function constant, $*$, rather than the 2-place relation constant
   $\leq$ (see §I.3, Definition 7);
2. replace the defined formula *semilat* 0 (defined in §I.3, Definition 10, and
   used in §I.5 Definition 15) with the following formula, *rms*, in the
   language of $L^2$: $\forall x_1 \forall x_2 \forall x_3 \forall x_4 (*((x_1)) = x_1 \& R0x_1x_1 \& Rx_1x_1x_1 \&
   [\exists x_5 (Rx_1x_2x_3 \& Rx_3x_1x_5 \& Rx_5x_3x_4) \supset \exists x_6 (Rx_1x_3x_5 \& Rx_5x_3x_4) \&
   [\exists x_4 (R0x_1x_5 \& Rx_5x_2x_3) \supset Rx_1x_2x_3] \& [Rx_1x_2x_3 \supset Rx_1*((x_3)))]]$;
3. in the definition of the recursive function, $f$ (§I.5, Definition 14), replace
   the clause for $f((A \rightarrow B))$, and add a clause for $f(\sim A)$ as follows:

   \[
   f((A \rightarrow B)) = \forall x_2 \forall x_3 ((f(A)[x_2/x_1] \& Rx_1x_2x_3) \supset f(B)[x_3/x_1]);
   \]

   \[
   f(\sim A) = \neg f(A)[*(x_1)/x_1].
   \]

**§III.** **Technical and logico-philosophical remarks.**

**III.1.** **Technical remarks.** We have used a quite general method for proving that
systems resulting from an algebraically-motivated primary interpretation of
propositional quantification are recursively isomorphic to full second-order classical
logic. Fine [1970] defines modal analogues to $\text{R}_2 \text{P} +$ and $\text{RP} +$. Assuming a
possible world semantics with an accessibility relation, he defines $\text{K} \pi +$, $\text{T} \pi +$,
$\text{K4} \pi +$, $\text{S4} \pi +$, $\text{S4.2} \pi +$, and $\text{Bn} +$, and he discusses proofs that second-order arithmetic can be encoded into these. One can use methods similar to those in the
present paper to show that these modal systems are recursively isomorphic to second-order logic. A similar project would be to get a grip on the complexity of the primary interpretation of the propositional quantification associated with Kripke's
semantics for intuitionism (see Kripke [1963]). (I do not know what the result would be; the problem is with reflecting metalinguistic (classical) implication in the object
language. See the discussion in §II.6, above.)
We know that some primary interpretations are axiomatisable: Kaplan [1970] and Fine [1970] axiomatise the primary interpretation of the quantifiers over $S5$ associated with the Kripke’s possible world semantics—indeed, Fine [1970] shows that $S5n+$ is decidable. Such results suggest that some algebras (for example, sets of possible worlds with no accessibility relation) do not have enough structure for our methods. Some algebras might have too much. Perhaps some very general results are in the neighborhood. Perhaps there are interesting sufficient and/or necessary conditions on an algebraic semantics for the resulting primary interpretations of the propositional quantifiers to be axiomatisable, or arithmetical, or recursively isomorphic to second-order logic.

III.2. Logico-philosophical remarks. One might wonder whether these results bode well or ill for relevance logic. They seem to bode ill: after all, such a natural interpretation of the propositional quantifiers is not axiomatisable within a logical context which prides itself on its proof-theoretic motivation. But this misses the point of the proof-theoretic motivation of the relevance project. The primary interpretations are semantically motivated, and their nonaxiomatisability just goes to show that you should not put too much trust in your (formal) semantic intuitions when approaching $R$ and its cousins. There is an interpretation of the propositional quantifiers which readily avails itself to a natural axiomatisation and a natural deduction system (see note 1). The mild semantic awkwardness of this secondary interpretation should not be too hard to take for anyone who countenances the relevant model structures of Routley and Meyer [1973] and the frames of Fine [1974]; like the Routley-Meyer and the Fine semantics, the secondary semantics for propositional quantifiers is motivated by proof-theoretic rather than semantic considerations. Indeed, this discussion of the primary and secondary interpretations of the propositional quantifiers may be a testament to the degree to which, in relevance logic at least, semantics is abstract proof theory.

§IV. Appendix Proof of Theorem 2, §I.3. Recall that Theorem 2, in §I.3 states that $L^2$ is 1-reducible to MSL0. First we give some definitions.

Definition 28. A lattice is an ordered pair $L = \langle L, \leq \rangle$ where

(1) $\leq$ partially orders $L$; and

(2) each pair, $\{a, b\}$ of points in $L$ has a least upper bound, $a \circ b$, and a greatest lower bound, $a \land b$, in $L$.

A distributive lattice is a lattice such that, for points $a, b,$ and $c$:

$a \circ (b \land c) = (a \circ b) \land (a \circ b)$ and $a \land (b \circ c) = (a \land b) \circ (a \land c)$.

Definition 29. Given a lattice $L = \langle L, \leq \rangle$ an ideal is a nonempty subset $I$ of $L$ which is such that $(\forall a, b \in I)(a \circ b \in I)$ and $(\forall a \in I)(\forall b \in L)(if b \leq a then b \in I)$.

Definition 30. The theory of distributive lattices with second-order monadic quantification over ideals (DLMI) is the set of sentences in MSOL which are validated by every model (1) whose underlying structure is a distributive lattice; (2) which assigns ideals to predicate variables; and (3) for which we interpret monadic second-order quantification as ranging over ideals.

Theorem 2 of §I.3 follows from Theorems 10 and 11 below and the following:

Theorem 2.2 (Nerode and Shore [1980]). DLMI is recursively isomorphic to $L^2$. 

We continue with some definitions.

Note. Just as we defined the MSOL formula \( z \approx x \circ y \) (Definition 9, §1.3), we can define \( z \approx x \land y \). Using these, we can define the formula (with one free predicate variable): \( X \) is an ideal. Also, just as we defined (Definition 10) the formula semilat (which “says that” the model is a semilattice), we can define a formula, distlat, which “says that” the model is a distributive lattice.

**Definition 33.** The monadic theory of distributive lattices (MDL) is the set of MSOL formulas validated by all models which validate distlat. The monadic theory of semilattices (MSL) is the set of MSOL formulas which are validated by all models which validate semilat.

**Theorem 10.** DLMI is 1-reducible to MSL.

**Proof.** MDL is 1-reducible to MSL, since \( A \in \text{MDL} \iff (\text{distlat} \Rightarrow A) \in \text{MSL} \). To show that DLMI is 1-reducible to MDL, first let \( f \) be the recursive 1-1 function which maps an MSOL formula \( A \) to an MSOL formula just like \( A \), except that subformulas of the form \( \forall X B \) are replaced by \( \forall X (X \text{ is an ideal} \Rightarrow f(B)) \). Let \( g \) be as in Definition 13 (§1.4). Then, for all MSOL formulas \( A, A \in \text{DLMI} \iff fg(A) \in \text{MDL} \). Q.E.D.

**Theorem 11.** MSL is 1-reducible to MSL0.

**Proof.** First let \( h \) be the following 1-1 recursive function from MSOL formulas to MSOL formulas: \( h(A) \) results by replacing every occurrence of \( x_i \) by \( x_{i+1} \) and by replacing every occurrence of 0 by \( x_1 \). Next, let \( g \) be as in Definition 13 in §1.4. Finally let the 1-1 recursive function \( f \) map a formula \( A \) to a formula just like \( A \) except that occurrences of subformulas of the form \( \forall X B \) are replaced by \( \forall X (\neg x = 0 \Rightarrow f(B)) \). Note: \( A \in \text{MSL} \iff fhg(A) \in \text{MSL0} \). Q.E.D.

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