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QUANTIFYING OVER PROPOSITIONS IN RELEVANCE LOGIC: NONAXIOMATISABILITY OF PRIMARY INTERPRETATIONS OF $\forall p$ AND $\exists p$*

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A typical approach to semantics for relevance (and other) logics: specify a class of algebraic structures and take a *model* to be one of these structures, α, together with some function or relation which associates with every formula A a subset of α. (This is the approach of, among others, Urquhart, Routley and Meyer and Fine.) In some cases there are restrictions on the class of subsets of α with which a formula can be associated: for example, in the semantics of Routley and Meyer [1973], a formula can only be associated with subsets which are closed upwards. It is natural to take a *proposition* of α to be such a subset of α, and, further, to take the propositional quantifiers to range over these propositions. (Routley and Meyer [1973] explicitly consider this interpretation.) Given such an algebraic semantics, we call (following Routley and Meyer [1973], who follow Henkin [1950]) the above-described interpretation of the quantifiers the *primary* interpretation associated with the semantics.[1]

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[1]A *secondary* interpretation of the quantifiers can be developed according to which a *secondary* model is specified by an algebra, α, a class, π, of subsets of α and a function which assigns to every formula a member of π. (π may be a strict subclass of the class of what I have called "propositions.") To get completeness theorems for natural quantified relevance logics, we must restrict our attention to models for which π obeys certain closure conditions.

In addition, a *substitutional* interpretation of the quantifiers can be developed—Routley and Meyer [1973] consider such an interpretation as well as a primary interpretation. In forthcoming work, I plan to discuss these alternatives.

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Section I shows that the primary interpretation of the universal propositional quantifier, $\forall p$, associated with Urquhart's semilattice semantics[2] (Urquhart [1972], [1973], [1992]) is not axiomatisable—indeed the system based on it is recursively isomorphic to full second-order (classical) logic. Section II shows that the primary interpretation of $\forall p$ and $\exists p$, associated with Routley and Meyer's relational semantics (Routley and Meyer [1973]) is not axiomatisable—again, the system based on it is recursively isomorphic to full second-order logic.

These results rely on the following two definitions and theorem from recursion theory. The definitions are given for sets, A and B, of natural numbers, but they can be taken to apply to sets of formulas of languages whose syntax can be recursively arithmetised.

DEFINITION (Post [1944]; see Odifreddi [1989, p. 324]). A is *1-reducible* to B ($A \leq_1 B$) if, for some 1-1 recursive function f, $x \in A$ iff $f(x) \in B$. Note: $\leq_1$ is reflexive and transitive.

DEFINITION (Post [1944]; see Odifreddi [1989, p. 324]). A is *recursively isomorphic* to B if, for some 1-1 onto recursive function, f, $x \in A$ iff $f(x) \in B$.

ISOMORPHISM THEOREM (Myhill [1955]; see Odifreddi [1989, p. 325]). A is recursively isomorphic to B iff $A \leq_1 B$ and $B \leq_1 A$.

§I. The primary interpretation of $\forall p$ associated with Urquhart's semilattice semantics.

I.1. Formal tools. The object language has the following vocabulary: (,), $\rightarrow$, &, $\forall$, and a countably infinite list of propositional variables, $p_1, p_2, \ldots, p_n, \ldots$.[3] The set of formulas is defined in the usual way. We use A, B, C, D as metavariables ranging over formulas and $p, q, r, \ldots$ as metavariables ranging over propositional variables. Also, we use the following notation:

for any set, or family of sets, S:

$\cup S$ $\quad \{x: (\exists y \in S)(x \in y)\}$;

$\cap S$ $\quad \{x: (\forall y \in S)(x \in y)\}$;

$\mathscr{P}(S)$ $\quad \{x: (\forall y \in x)(y \in S)\}$;

for any two sets, S and S': $S \backslash S' = \{a \in S: a \notin S'\}$.

I.2. The formal semantics. (Except for Theorem 1, Definition 6, and clause (iv) in Definition 4, our presentation is a notational variant of that in Urquhart [1973] and [1992].)

[2] Here, I investigate the interpretation of $\forall p$ and not of $\exists p$ for two related reasons:

(1) Urquhart's semantics is concerned with the implicational-conjunctional behaviour of relevance logic, so it seems appropriate to leave $\exists$ out of the picture, as least as a primitive.

(2) The *secondary* interpretation motivated by Urquhart's semantics is complete for the logic which results when we add (with a reasonable set of axioms) $\forall p$ to $\mathbf{R}_{\rightarrow\&}$; but not when we add both $\forall p$ and $\exists p$. (Compare the incompleteness of Urquhart's semantics for $\mathbf{R}_{\rightarrow\&\vee}$.)

[3] The language could be supplied with a stock of propositional constants, without significant effect on the current project.

DEFINITION 1. A *semilattice* is an ordered pair $\mathbf{L} = \langle L, \leq \rangle$ such that $\leq$ partially orders L, and such that any two points, a and b, in L have a least upper bound, $a \circ b$, in L. A *semilattice with* 0 is a semilattice with a $\leq$-smallest element, 0.

DEFINITION 2. A *consequence model* or c-*model* (the terminology is from Urquhart [1992]; or "model") is an ordered pair $M = \langle \mathbf{L}, \phi \rangle$, where $\mathbf{L} = \langle L, \leq \rangle$ is a semilattice with 0, and where ϕ is a function which assigns to each atomic formula a subset of L. M *assigns* the set $\phi(p)$ to the atomic formula p. (Note that for Urquhart's semantics, the class of algebraic structures under consideration is the class of semilattices with 0. Note also that there is no restriction on which subsets of L can be the semantic value of a propositional variable; so the set of *propositions* is just $\mathscr{P}(L)$.)

DEFINITION 3. Given a c-model, $M = \langle L, \leq, \phi \rangle$, and a subset, S of L, $M[S/p]$ is the c-model which is just like M except that it assigns S to p.

DEFINITION 4. Given a c-model, $M = \langle L, \leq, \phi \rangle$, and a formula A, $M(A)$, the subset of L assigned to A by M, is defined thus:

(i) if A is atomic, $M(A) = \phi(A)$;
(ii) $M((A \mathbin{\&} B)) = M(A) \cap M(B)$;
(iii) $M((A \to B)) = \{a \in L: (\forall b \in M(A))(a \circ b \in M(B))\}$;
(iv) $M(\forall pA) = \bigcap \{M[S/p](A): S \subseteq L\}$.

DEFINITION 5 (Validity). Given a semilattice with 0, $\mathbf{L} = \langle L, \leq \rangle$, a model M and a formula A:

$M \models A$ iff $0 \in M(A)$. (*M validates* or *satisfies A*.)
$\mathbf{L} \models A$ iff $(\forall M = \langle \mathbf{L}, \phi \rangle)\,(M \models A)$. (**L** *validates* or *satisfies A*.)
A is c-*valid* iff $(\forall M)\,(M \models A)$.

For quantifier-free formulas, the above notions correspond exactly to Urquhart's semantic notions.

URQUHART'S SOUNDNESS AND COMPLETENESS THEOREM:

If A is quantifier-free, then A is c-valid iff A is a theorem of $\mathbf{R}_{\to\&}$.

DEFINITION 6. $\mathbf{R}_{\to\&}\mathbf{P+} = \{A: A$ is a c-valid formula$\}$. ($\mathbf{R}_{\to\&}\mathbf{P+}$ is the logic based on the primary interpretation of $\forall p$ associated with Urquhart's semilattice semantics.)

THEOREM 1. *$\mathbf{R}_{\to\&}\mathbf{P+}$ is recursively isomorphic to full second-order classical logic.*

I.3. Proof of Theorem 1: Preliminaries. Nerode and Shore [1980] show that full second-order classical logic ($\mathbf{L}^2$) is recursively isomorphic to the theory of distributive lattices with second-order monadic quantification over ideals (**DLMI**). In the appendix, we show that **DLMI** is 1-reducible to the monadic theory of semilattices with 0 (**MSL0**). And so, $\mathbf{L}^2$ is 1-reducible to **MSL0**. To show that $\mathbf{R}_{\to\&}\mathbf{P+}$ is recursively isomorphic to $\mathbf{L}^2$, we show:

(1) **MSL0** is 1-reducible to $\mathbf{R}_{\to\&}\mathbf{P+}$ (§1.4); and
(2) $\mathbf{R}_{\to\&}\mathbf{P+}$ is 1-reducible to $\mathbf{L}^2$ (§I.5).

To make this precise, we give some definitions.

DEFINITION 7. The *full second-order language* (SOL) has the following vocabulary: (,), $\supset, \neg, \forall$; the equals sign, $=$; a countably infinite list of individual variables: $x_1, x_2, \ldots, x_n, \ldots$; and, for each $m \geq 1$, a countably infinite list of m-place relation variables: $X_1^m, X_2^m, \ldots, X_n^m, \ldots$. We also include a 2-place relation constant, $\leq$, and an individual constant, 0. (These are not necessary, but they simplify things.)

The set of *formulas*, and the notions of *model* and *validity* are defined in the usual way, as are &, $\vee$, and $\exists$. *Second-order logic* ($\mathbf{L}^2$) is the set of formulas validated by every model.

DEFINITION 8. The *monadic second-order language* (MSOL) is exactly like SOL except that rather than a countable list of m-place relation variables for each $m \geq 1$, it has a countable list of 1-place predicate variables: $X_1, X_2, \ldots, X_n, \ldots$.

DEFINITION 9. We use the following abbreviation in SOL and MSOL, for individual variables, x, y, and z. Here, w is the first in the list of individual variables distinct from x, y, and z:

$$z \approx x \circ y =_{\mathrm{df}} (x \leq z \;\&\; y \leq z \;\&\; (\forall w)((x \leq w \;\&\; y \leq w) \supset z \leq w)).$$

DEFINITION 10. *semilat* is the following formula of SOL and MSOL:

$$\forall x_1 \forall x_2 \forall x_3 [x_1 \leq x_1 \;\&\; ((x_1 \leq x_2 \;\&\; x_2 \leq x_1) \supset x_1 = x_2) \\ \;\&\; ((x_1 \leq x_2 \;\&\; x_2 \leq x_3) \supset x_1 \leq x_3)] \;\&\; \forall x_1 \forall x_2 \exists x_3 (x_3 \approx x_1 \circ x_2).$$

semilat 0 is the following formula: (*semilat* & $(\forall x_1)(0 \leq x_1)$).

DEFINITION 11. The *monadic second-order theory of semilattices with* 0 (**MSL0**) is the set of MSOL formulas which are validated by every model which validates *semilat* 0.

THEOREM 2. $\mathbf{L}^2$ is 1-reducible to **MSL0**.
(See the appendix for the proof.)

I.4. Proof of Theorem 1: MSL0 is 1-reducible to $\mathbf{R}_{\rightarrow\&}\mathbf{P}+$. In Tables 1 and 2, $\mathbf{L} = \langle L, \leq \rangle$ and $M = \langle \mathbf{L}, \phi \rangle$.

Note for Table 1. Given a c-model M and a formula A, A can be thought of as playing two roles:

(1) *A names* a subset of the lattice, namely, $M(A)$; and
(2) *A makes a claim* about the c-model.

For example $(p \rightarrow q)$ *names* the set $M(p \rightarrow q)$ and *says that* $M(p) \subseteq M(q)$, since $M \models (p \rightarrow q)$ iff $M(p) \subseteq M(q)$. Table 1 lists formulas constructed with the primitive object language connectives and indicates what the formulas *say* and what they *name*. The blank entries are those of no particular interest.

TABLE 1. Object language connectives and their two meta-linguistic interpretations.

Formula	What the formula says: $M \models$ Formula iff	What the formula names: M(Formula) =
$(A \;\&\; B)$	$M \models A$ and $M \models B$	$M(A) \cap M(B)$
$(A \rightarrow B)$	$M(A) \subseteq M(B)$	
$\forall pA$	$(\forall S \subseteq L)(M[S/p] \models A)$	

Note 1 *for Table* 2. The last two columns of Table 2 play the same role as the last two columns of Table 1: they tell us how we can interpret the object language formulas. The entries in these last two columns represent more or less substantial claims, which are left to the reader to prove.

Note 2 *for Table* 2. Quantification over propositions amounts, under the primary interpretation under consideration, to quantification over the *subsets* of whatever semilattice we are considering. The effect of quantifying over *elements* of the semilattice can be had by restricting a particular quantifier to *atomic* subsets of the lattice and by informally identifying an element $a \in L$ with the atomic set $\{a\} \subseteq L$. This remark should explain the importance of the last row.

TABLE 2. Object language definitions.

Definiendum	Definiens	What the definiendum says: $M \models$ Definiendum iff	What the definiendum names: M(Definiendum) =
$(A \vee B)$	$\forall p(((A \to p) \& (B \to p)) \to p)$	$M \models A$ or $M \models B$	$M(A) \cup M(B)$
$(A \leftrightarrow B)$	$(A \to B) \& (B \to A)$	$M(A) = M(B)$	
F	$\forall pp$		$\varnothing$
t	$\forall p(p \to p)$		$\{0\}$
$(A \supset B)$	$(A \& t) \to B$	$(M \models A \Rightarrow M \models B)$	
$\neg A$	$A \supset F$	$M \not\models A$	
$(A \circ B)$	$\forall p((A \to (B \to p)) \to p)$		$\{a \circ b\colon a \in M(A) \& b \in M(B)\}$
$A \in L$	$\neg(A \to F) \& \forall p(((A \& p) \to F) \vee (A \to p))$	$M(A) = \{a\}$ for some a in L	

Postscript to Table 2. With "$\supset$" and "$\neg$" we can reflect metalinguistic (classical) implication and negation in the object language—see the entries in the third column. "t" is also useful: it names $\{0\}$; so under the heuristic identification (explained above) of atomic subsets of semilattices with their elements, "t" names 0.

DEFINITION 12. Define a recursive 1-1 function, f, from MSOL formulas to MSOL formulas as follows. Here, for $i \geq 1$, $x_i(p_i)$ is the ith individual (propositional) variable in the list of individual (propositional) variables in MSOL (the object language). x_0 is the MSOL constant 0, and p_0 is the defined object language constant, t.

$$f(x_i = x_j) = (p_{2i} \leftrightarrow p_{2j});$$
$$f(x_i \leq x_j) = (p_{2i} \circ p_{2j} \rightarrow p_{2j});$$
$$f(X_i x_j) = (p_{2j} \rightarrow p_{2i+1});$$
$$f((A \supset B)) = (f(A) \supset f(B));$$
$$f(\neg A) = \neg f(A);$$
$$f(\forall x_i A) = \forall p_{2i}(p_{2i} \in L \supset f(A));$$
$$f(\forall X_i A) = \forall p_{2i+1} f(A).$$

DEFINITION 13. Let g be the following recursive 1-1 function from MSOL formulas to MSOL formulas. Suppose A is an MSOL formula and n is the greatest number such that x_n or X_n appears in A. Let $g(A) = \forall x_1 \cdots \forall x_n \forall X_1 \cdots \forall X_n A$.

LEMMA 1. *For any* MSOL *formula, A, $A \in$* **MSL0** *iff $g(A) \in$* **MSL0**.

LEMMA 2. *For any closed* MSOL *formula, A, $A \in$* **MSL0** *iff $f(A) \in$* $\mathbf{R}_{\rightarrow\&}\mathbf{P+}$.

COROLLARY 1. *For any* MSOL *formula, A, $A \in$* **MSL0** *iff $fg(A) \in$* $\mathbf{R}_{\rightarrow\&}\mathbf{P+}$.

COROLLARY 2. **MSL0** *is* 1*-reducible to* $\mathbf{R}_{\rightarrow\&}\mathbf{P+}$.

I.5. Proof of Theorem 1: $\mathbf{R}_{\rightarrow\&}\mathbf{P+}$ is 1-reducible to $\mathbf{L}^2$.

DEFINITION 14. Define a 1-1 recursive function f from object language formulas to SOL formulas as follows. Note: given an object language formula A: X_i^1 will be free in $f(A)$ just in case p_i is free in A; the only individual variable free in $f(A)$ will be x_1; and $f(A)$ will contain no m-ary second-order variables for $m \geq 2$.

$$f(p_i) = X_i^1 x_1;$$
$$f((A \,\&\, B)) = (f(A) \,\&\, f(B));$$
$$f((A \rightarrow B)) = \forall x_2(f(A)[x_2/x_1] \supset \forall x_3(x_3 \approx x_2 \circ x_1 \supset f(B)[x_3/x_1]));$$
$$f(\forall p_i A) = \forall X_i^1 f(A).$$

DEFINITION 15. We define another 1-1 recursive function, g, which assigns to every formula of the object language a formula in SOL:

$$g(A) = (\mathit{semilat}\, 0 \supset f(A)[0/x_1]).$$

LEMMA 3. *For any object language formula, A, $A \in$* $\mathbf{R}_{\rightarrow\&}\mathbf{P+}$ *iff* $g(A) \in \mathbf{L}^2$.

COROLLARY 3. $\mathbf{R}_{\rightarrow\&}\mathbf{P+}$ *is* 1*-reducible to* $\mathbf{L}^2$.

Theorem 1 is a corollary to Corollaries 2 and 3 and Theorem 2.

§II. The primary interpretation of $\forall p$ and $\exists p$ associated with Routley and Meyer's relational semantics. (The formal tools are the same as for §I, except that the object language has the following additional vocabulary: $\vee, \sim, \exists$. Routley and Meyer also add the two-place connective $\circ$, but this can be defined, in any extension of **R** by $(A \circ B) = \sim(A \rightarrow \sim B)$.)

II.1. The formal semantics. (Except for clauses (vi) and (vii) of Definition 20, Definition 22 and Theorem 3, our presentation is a notational variant of Routley and Meyer's. Most of the terminology is theirs.)

DEFINITION 16. A *relevant model structure* (rms) is a quadruple $\mathbf{K} = \langle 0, K, R, * \rangle$ where K is a set, $0 \in K$, R is a ternary relation on K and $*$ is a unary function on K satisfying postulates p1–p6 to follow (for all $a, b, c, d \in K$). Before we state the postulates, define (for $a, b, c, d \in K$):

(d1) $a < b$ iff $R0ab$;

(d2) R^2abcd iff $(\exists x \in K)(Rabx$ and $Rxcd)$.

The postulates are

(p1) $R0aa$;

(p2) $Raaa$;

(p3) $R^2abcd \Rightarrow R^2acbd$;

(p4) $R^20abc \Rightarrow Rabc$;

(p5) $Rabc \Rightarrow Rac^*b^*$;

(p6) $a^{**} = a$.

(For Routley and Meyer's semantics the class of algebraic structures under consideration is the class of rms's.)

Before continuing, we state a useful Lemma:

LEMMA 4. (1) $Rabc \Rightarrow Rbac$.

(2) $(\forall a, b \in K)(\exists c \in K)(Rabc)$.

PROOF. (1) Suppose $Rabc$. Since $R0aa$ (p1), R^20abc (d2). So R^20bac (p3). So $Rbac$ (p4).

(2) Choose $a, b \in K$. Now, $R0^*0^*0^*$ (p2). So $R0^*00$ (p5 and p6). So $R00^*0$ (1). So, since $R0bb$ (p1), R^200^*bb (d2). So $R0^*bb$ (p4). Also, $R0aa$ (p1). So $Ra0a$ (1). So Raa^*0^* (p5). So R^2aa^*bb (d2). So R^2aba^*b (p3). So $(\exists c)(Rabc$ & $Rca^*b)$ (d2). So $(\exists c \in K)(Rabc)$. Q.E.D.

DEFINITION 17. Given an rms $\mathbf{K} = \langle 0, K, R, * \rangle$, a $\mathbf{K}$-*proposition* is any subset, P, of K such that

$$[(b \in P \ \&\ b < a) \Rightarrow a \in P].$$

$\pi(\mathbf{K})$ (we often just write "π") is the set of $\mathbf{K}$-propositions.[4]

DEFINITION 18. A *relational model* (or "model") is an ordered pair $M = \langle \mathbf{K}, \phi \rangle$ where $\mathbf{K}$ is an rms and ϕ is a function which assigns to each atomic formula a member of π. *M assigns* the set $\phi(p)$ to the atomic formula p.

DEFINITION 19. Given a model, $M = \langle \mathbf{K}, \phi \rangle$, and a $\mathbf{K}$-proposition P, $M[P/p]$ is the model which is just like M except that it assigns P to p.

DEFINITION 20. Given a model, $M = \langle 0, K, R, *, \phi \rangle$, and a formula A, $M(A)$, the subset of K assigned to A by M, is defined by:

(i) if A is atomic, $M(A) = \phi(A)$;

(ii) $M((A \ \&\ B)) = M(A) \cap M(B)$;

(iii) $M((A \vee B)) = M(A) \cup M(B)$;

(iv) $M((A \to B)) = \{c \in K: (\forall a \in M(A))(\forall b \in K)(Rcab \Rightarrow b \in M(B))\}$;

(v) $M(\sim A) = \{a \in K: a^* \notin M(A)\}$;

(vi) $M(\forall pA) = \bigcap\{M[P/p](A): P \in \pi\}$;

(vii) $M(\exists pA) = \bigcup\{M[P/p](A): P \in \pi\}$.

Note: for every formula A, $M(A) \in \pi$.

[4]Routley and Meyer [1973] define $\pi(\mathbf{K})$ as the *algebra* of propositions rather than simply the set, but our definition suffices for the present purposes.

DEFINITION 21 (Validity). Given an rms $\mathbf{K}$, a model M and a formula A:
$M \models A$ iff $0 \in M(A)$. (M *validates* or *satisfies* A.)
$\mathbf{K} \models A$ iff $(\forall M = \langle \mathbf{K}, \phi \rangle)(M \models A)$. ($\mathbf{K}$ *validates* or *satisfies* A.)
A is **valid** iff $(\forall M)(M \models A)$.[5]

For quantifier-free formulas, the above notions correspond exactly to Routley and Meyer's semantic notions.

ROUTLEY AND MEYER'S SOUNDNESS AND COMPLETENESS THEOREM. If A is quantifier-free, then A is valid iff A is a theorem of $\mathbf{R}$.

DEFINITION 22. $\mathbf{RP+} = \{A\colon A$ is a valid formula$\}$. ($\mathbf{RP+}$ is the logic based on the primary interpretation of $\forall p$ and $\exists p$ associated with Routley and Meyer's relational semantics.)

THEOREM 3. $\mathbf{RP+}$ is recursively isomorphic to full second-order logic.

II.2. Proof of Theorem 3: Preliminaries. As in §I, we encode **MSL0** in the object language and encode $\mathbf{RP+}$ in SOL. The latter is similar to §I.5. To accomplish the former, we associate with every semilattice with 0, the rms defined presently.

DEFINITION 23. If $\mathbf{L} = \langle L, \leq \rangle$ is a semilattice with 0, let $\mathbf{K_L} = \langle 0, L, R, * \rangle$ where 0 is the $\leq$-smallest element of L, and for a, b, c in L:
if $0, a, b, c$ are distinct then $Rabc$; and if $a \leq b$, then $Rabb$ and $Rbab$ and $Rbba$; and $a^* = a$.

THEOREM 4. $\mathbf{K_L}$ *is an* rms.

PROOF. $\mathbf{K_L}$ must satisfy p1–p6 of Definition 16 (§II.1). p1, p2, p5 and p6 are straightforward. To see that $\mathbf{K_L}$ satisfies p4, suppose that $R^2 0abc$. Then $(\exists x \in L)$ ($R0ax$ and $Rxbc$). By the definition of R, $x = a$. So $Rabc$. Before we show that $\mathbf{K_L}$ satisfies p3, note that R is symmetric—i.e., the following are equivalent: $Rabc$; $Rbac$; $Racb$; $Rbca$; $Rcab$; $Rcba$. As a result, the following are equivalent: R^2abcd; R^2abdc; R^2bacd; R^2badc; R^2cdab; R^2dcab; R^2cdba; R^2dcba. Now, suppose that R^2abcd. To show that R^2acbd, we consider cases and subcases.

Case 1. $a = 0$. $R0bcd$. So, by p4, $Rbcd$. So $Rcbd$. So, since $R0cc$, $R^2 0cbd$, as desired.

Case 2. b, c, or $d = 0$. Using the above equivalences, reduce this to Case 1.

Case 3. $b = c$. R^2acbd follows immediately from R^2abcd.

Case 4. $a = d$. R^2abca. So, by the above equivalences, R^2acba, as desired.

Case 5. $a, b, c, d \neq 0$ and $b \neq c$ and $a \neq d$.

Case 5.1. $a \neq c$ and $b \neq d$.

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[5] Among formulas constructed with only the connectives &, →, and only the quantifier ∀:
(1) there are formulas valid in the Urquhart-inspired sense (of §I), which are not valid in the Routley-Meyer-inspired sense (of §II); and
(2) there are formulas valid in the Routley-Meyer-inspired sense which are not valid in the Urquhart-inspired sense.

A formula of the first kind is the formula which says that every proposition has a classical negation: $\forall p \exists q[((p \,\&\, q) \rightarrow \forall rr) \,\&\, (\exists rr \rightarrow (p \vee q))]$. A formula of the second kind is the formula which says that every proposition has a relevant negation. To define this formula, first define $A \circ B$ as $\forall p(A \rightarrow ((B \rightarrow p) \rightarrow p))$. (Note: the definition conforms to the requirement that the formula of interest be constructed using only &, →, and ∀. This is why we do not here define $A \circ B$ in the usual way.) Then define f as $\exists q(q \circ \forall p(((p \rightarrow q) \rightarrow q) \rightarrow p))$. (The definitions of $A \circ B$ and f are from Anderson, Belnap and Dunn [1991].) The desired formula is: $\forall p \exists q[(p \rightarrow (q \rightarrow f)) \,\&\, ((q \rightarrow f) \rightarrow p)]$.

Case 5.11. L is finite. Then L has a $\leq$-greatest member, say x. Whether $x = a$ or $x = c$ or neither, $Racx$. Similarly $Rxbd$. So R^2acbd as desired.

Case 5.12. L is infinite. Then choose any x distinct from 0, a, b, c and d. By the definition of R, $Racx$ and $Rxbd$. So R^2acbd as desired.

Case 5.2. $a = c$. $Raca$. Also, since $b \neq c$, $a \neq b$. So, since $a \neq d$, $Rabd$. So R^2acbd, as desired.

Case 5.3. $b = d$. Using the above equivalences, reduce this to Case 5.2.

Q.E.D.

(We note that $\mathbf{K_L}$ is *singular* as defined in §II.3, Definition 24, below, and *superclassical* as defined in §II.4, Definition 25, below.)

Not every rms is such an rms. The trick is to find an object language formula *SEMILAT* which is validated by an rms just in case the rms is related to some semilattice with 0 as in Definition 23. Then, after we have defined functions f and g similar to those defined in Definitions 12 and 13 (§I.4, above), our main theorem should look something like this: For any MSOL formula A, $A \in \mathbf{MSL0}$ iff $(SEMILAT \supset fg(A)) \in \mathbf{RP+}$.

There is a special difficulty for the Routley-Meyer semantics which is not present for the Urquhart semantics. In the context of Urquhart's semantics, we were able to reflect metalinguistic implication in the object language by defining a two-place object language connective, $\supset$, with the following property: for every M, $M \models (A \supset B)$ iff $(M \models A \Rightarrow M \models B)$. This ability to reflect metalinguistic implication in the object language figured prominently in the translation from the MSOL to the object language in two ways (see Definition 12, §I.4):

(1) it allowed the restriction of propositional quantification to "elements" of L; and

(2) it provided a convenient translation of $(A \supset B)$.

Unfortunately, we have not been able to define such a connective in the context of Routley and Meyer's semantics for **RP+**. (Conjecture: no such connective is definable.) For this reason, we take a detour through a restricted class of rms's, the *superclassical* rms's, discussed in §II.4, below. But first we restrict attention to *singular* rms's.

II.3. Singular relevant model structures. It would be very convenient if, for every rms $\mathbf{K} = \langle 0, K, R, * \rangle$, and for every $a, b \in K$, $[(a < b \text{ and } b < a) \Rightarrow a = b]$. This is not in general the case, but the following definition, theorem and corollary allow us to restrict our attention to rms's in which it is the case.

DEFINITION 24. $\mathbf{K} = \langle 0, K, R, * \rangle$ is a *singular* relevant model structure (srms) iff $\mathbf{K}$ is an rms and $(\forall a, b \in K)((R0ab \text{ and } R0ba) \Rightarrow a = b)$. A model M is a *singular* model (s-model) iff $M = \langle \mathbf{K}, \phi \rangle$ for some singular $\mathbf{K}$.

THEOREM 5. *For every model $M = \langle 0, K, R, *, \phi \rangle$ there is an s-model M' such that M' validates exactly the same formulas which M validates.*

PROOF. Consider some model $M = \langle 0, K, R, *, \phi \rangle$. For $a, b \in K$, define $a \approx b$ as ($R0ab$ and $R0ba$). $\approx$ is an equivalence relation. For any $a \in K$ let $a' = \{b \in K : b \approx a\}$. Let $K' = \{a' : a \in K\}$. Let R' be the following relation on K': $\{\langle a', b', c' \rangle : Rabc\}$. Define $(a')^{*'}$ as $(a^*)'$, and $\phi'(A)$ as $(\phi(A))'$. ($*'$ and ϕ' are well defined.) Let $M' = \langle 0', K', R', *', \phi' \rangle$. Then M' is an s-model which validates exactly the same formulas as M. Q.E.D.

COROLLARY 4. **RP+** $= \{A$: *A is validated by every s-model*$\}$.

Henceforth, we restrict attention to srms's.

II.4. Superclassical relevant model structures.

DEFINITION 25. An rms (srms) $\mathbf{K} = \langle 0, K, R, * \rangle$ is *superclassical* iff $(\forall a \in K)(a^* = a)$. We use the abbreviation "Krms" ("Ksrms") for "superclassical rms" ("superclassical srms"). (The terminology and abbreviations are adapted from Routley, Meyer, Plumwood and Brady [1982], where such and similar rms's are studied.) A model (s-model) $M = \langle \mathbf{K}, \phi \rangle$ is *superclassical* iff **K** is. We use the term "K-model" ("Ks-model") for "superclassical model" ("superclassical s-model").

Note 1. If **K** is a Ksrms, then $\pi(\mathbf{K}) = \mathscr{P}(\mathbf{K})$.

Note 2. If **L** is a semilattice with 0, then $\mathbf{K_L}$ is a Ksrms.

DEFINITION 26. The logic **KR** is the unquantified propositional logic which results by adding the following axiom scheme to **R**: $((A \ \& \sim A) \rightarrow B)$. (See Routley, Meyer, Plumwood and Brady [1982].)

THEOREM 6 (Routley, Meyer, Plumwood and Brady [1982]). *If A contains no propositional quantifiers, then A is a theorem of* **KR** *iff A is validated by every K-model.*

DEFINITION 27. **KRP+** $= \{A$: A is validated by every K-model$\}$.

THEOREM 7. **KRP+** $= \{A$: *A is validated by every Ks-model*$\}$. (*See Theorem* 4, §II.3, *above.*)

THEOREM 8. **KRP+** *is recursively isomorphic to full second-order logic.*

Given Theorem 2 (§I.3), Theorems 3 and 8 follow from these results: §II.5: **MSL0** is 1-reducible to **KRP+**; §II.6: **KRP+** is 1-reducible to **RP+**; §II.7: **RP+** is 1-reducible to $\mathbf{L}^2$.

II.5. Proof of Theorems 3 and 8: MSL0 is 1-reducible to KRP+.

For Table 3, let $\mathbf{K} = \langle 0, K, R, * \rangle$ be an srms; let a, b, c, etc., range over the elements of K; and let S and S' range over the subsets of K.

Note for Table 3. The definition of "semilattice-like" is independent of the choice of **L**.

TABLE 3. Useful meta-linguistic definitions.

Definiendum	Definiens
$[a]$	$\{b$: $a < b\}$. Note: $[a] \in \pi$, and, if **K** is a Ksrms, then $[a] = \{a\}$.
$a \preceq b$	*Rbba* Note: $\preceq$ is a partial ordering for an interesting class of srms's (including *semilattice-like* srms's, as defined in this table, below). $\preceq$ is, in general, quite different from $<$.
$S \circ S'$	$\{c$: $(\exists a \in S)(\exists b \in S')(Rabc)\}$
K is *semilattice-like*	**K** is isomorphic to $\mathbf{K_L}$ for some semilattice with 0, **L** (where "isomorphic" is defined for rms's in the obvious way).
$\langle \mathbf{K}, \phi \rangle$ is *semilattice-like*	**K** is semilattice-like.

The notes and postscripts to Tables 1 and 2 in §I.4 are of substantial importance in interpreting Tables 4 and through 6 in the present section.

For Tables 4 through 6, assume that $\mathbf{K} = \langle 0, K, R, * \rangle$ is a Ksrms and that $M = \langle \mathbf{K}, \phi \rangle$ is a Ks-model.

Note for Table 6. In the last row of the second column in Table 6 we use the abbreviations $(\forall p \in K)A$ and $(\exists p \in K)A$. These stand for $\forall p(p \in K \supset A)$ and $\exists p(p \in K \ \&\ A)$, respectively.

Now, to show that **MSL0** is 1-reducible to **KRP+**, define functions f and g as in Definitions 12 and 13 (in §I.4, above) with the following exception: in the definition of the recursive function, f, (Definition 12) replace the clauses for $f(x_i \leq x_j)$, $f(\neg A)$, and $f((\forall x_i)A)$ with (respectively):

$$f(x_i \leq x_j) = (p_{2i} \precsim p_{2j});$$
$$f(\neg A) = {\sim} f(A); \text{ and}$$
$$f((\forall x_i)A) = (\forall p_{2i})(p_{2i} \in K \supset f(A)).$$

TABLE 4. Object language connectives and their two meta-linguistic interpretation.

Formula	What the formula says: $M \models$ Formula iff	What the formula names: M(Formula) =
$(A \ \&\ B)$	$M \models A$ and $M \models B$	$M(A) \cap M(B)$
$(A \vee B)$	$M \models A$ or $M \models B$	$M(A) \cup M(B)$
$(A \rightarrow B)$	$M(A) \subseteq M(B)$	
${\sim} A$	$M \not\models A$	$K \backslash M(A)$
$\exists pA$	$(\exists S \subseteq K)(M[S/p] \models A)$	
$\forall pA$	$(\forall S \subseteq K)(M[S/p] \models A)$	

TABLE 5. Preliminary object language definitions.

Definiendum	Definiens	What the definiendum says: $M \models$ Definiendum iff	What the definiendum names: M(Definiendum) =
F	$\forall pp$		$\varnothing$
t	$\forall p(p \rightarrow p)$		$\{0\}$
$(A \supset B)$	$({\sim} A \vee B)$	$M \models A \Rightarrow M \models B$	$K \backslash M(A) \cup M(B)$
$(A \leftrightarrow B)$	$((A \rightarrow B) \ \&\ (B \rightarrow A))$	$M(A) = M(B)$	
$(A \circ B)$	${\sim}(A \rightarrow {\sim} B)$		$M(A) \circ M(B)$

TABLE 6. Further object language definitions.

Definiendum	Definiens	What the definiendum says: $M \models$ Definiendum iff
$(A \in K)$	$\sim(A \to F) \mathbin{\&} \forall p(((A \mathbin{\&} p) \to F) \vee (A \to p))$	$M(A) = \{a\}$ for some a in K
$R(A, B, C)$	$A \in K \mathbin{\&} B \in K \mathbin{\&} C \in K \mathbin{\&} (C \to A \circ B)$	$M(A) = \{a\}$, $M(B) = \{b\}$, and $M(C) = \{c\}$, where $Rabc$
$(A \preceq B)$	$R(B, B, A)$	$M(A) = \{a\}$ and $M(B) = \{b\}$, where $a \preceq b$
$SEMILAT$	$(\forall p \in K)(\forall q \in K)(\forall r \in K)[p \preceq p \mathbin{\&} ((p \preceq q \mathbin{\&} q \preceq p) \supset (p \leftrightarrow q)) \mathbin{\&} ((p \preceq q \mathbin{\&} q \preceq r) \supset p \preceq r)] \mathbin{\&} (\forall p \in K) (\forall q \in K)(\exists r \in K)[p \preceq r \mathbin{\&} q \preceq r \mathbin{\&} (\forall s \in K)((p \preceq s \mathbin{\&} q \preceq s) \supset r \preceq s)] \mathbin{\&} (\forall p \in K)(t \preceq p)$	M is semilattice-like

Then, we have the following lemmas and corollaries. (Compare with Lemmas 1 and 2 and Corollaries 1 and 2 in §I.4.)

LEMMA 5. *For any* MSOL *formula*, A, $A \in$ **MSL0** *iff* $g(A) \in$ **MSL0**.

LEMMA 6. *For any closed* MSOL *formula*, A, $A \in$ **MSL0** *iff* $(SEMILAT \supset f(A)) \in$ **KRP+**.

COROLLARY 5. *For any* MSOL *formula*, A, $A \in$ **MSL0** *iff* $(SEMILAT \supset fg(A)) \in$ **KRP+**.

COROLLARY 6. **MSL0** *is 1-reducible to* **KRP+**.

II.6. Proof of Theorems 3 and 8: KRP+ is 1-reducible to RP+. Theorem 9, below, suffices.

LEMMA 7. *If* $M = \langle 0, K, R, *, \phi\rangle$ *is an s-model then* M *is a Ks-model iff* $M \models \forall p \forall q((p \mathbin{\&} \sim p) \to q)$.

PROOF. $(\Rightarrow)$ is straightforward.

$(\Leftarrow)$. Suppose $M \models \forall p \forall q((p \mathbin{\&} \sim p) \to q)$. Choose any $a \in K$, and let $M' = M[[a]/p][\varnothing/q]$. Then $M' \models (p \mathbin{\&} \sim p) \to q$. So $[a] \cap M'(\sim p) \subseteq \varnothing$. So $a \notin M'(\sim p)$. So $a^* \in [a]$. Similarly $a^{**} \in [a^*]$. So $a^* < a^{**} = a < a^*$. So, since M is an s-model, $a^* = a$. Q.E.D.

THEOREM 9. $A \in$ **KRP+** *iff* $[\forall p \forall q((p \mathbin{\&} \sim p) \to q) \mathbin{\&} t] \to A \in$ **RP+**.

PROOF. $(\Leftarrow)$. Suppose $[\forall p \forall q((p \mathbin{\&} \sim p) \to q) \mathbin{\&} t] \to A \in$ **RP+**, and suppose that M is a Ks-model. By Lemma 7, $M \models \forall p \forall q((p \mathbin{\&} \sim p) \to q)$, and, since $M \models t$, $M \models \forall p \forall q((p \mathbin{\&} \sim p) \to q) \mathbin{\&} t$. Also, since M is an rms,

$$M \models [\forall p \forall q((p \mathbin{\&} \sim p) \to q) \mathbin{\&} t] \to A.$$

So, $M \models A$. So $A \in$ **KRP+**.

$(\Rightarrow)$. Suppose $[\forall p \forall q((p \mathbin{\&} \sim p) \to q) \mathbin{\&} t] \to A \notin$ **RP+**. Then there is an s-model $M = \langle 0, K, R, *, \phi\rangle$ such that $M \not\models [\forall p \forall q((p \mathbin{\&} \sim p) \to q) \mathbin{\&} t] \to A$. So there exists

$b \in K$ such that

(1) $b \in M(\forall p \forall q((p \,\&\, {\sim} p) \rightarrow q))$;

(2) $b \in M(t)$; and

(3) $b \notin M(A)$.

Claim. M is a Ks-model.

Proof of Claim. Choose any $a \in K$, and let $M' = M[[a]/p][\varnothing/q]$. Then, by (1), $b \in M'((p \,\&\, {\sim} p) \rightarrow q)$. So, $(\forall c, d \in K)[(c \in M'(p \,\&\, {\sim} p)$ and $Rbcd) \Rightarrow d \in \varnothing]$. So, $(\forall c, d \in K)(c \notin M'(p \,\&\, {\sim} p)$ or not $Rbcd)$. Now, by Lemma 4 (§II.1, above), $(\forall c \in K)(\exists d \in K)(Rbcd)$. Therefore, $(\forall c \in K)(c \notin M'(p \,\&\, {\sim} p))$. So $a \notin M'(p \,\&\, {\sim} p)$. Now, $a \in [a] = M'(p)$. So, $a \notin M'({\sim} p)$. So, $a^* \in [a]$. So $a < a^*$. This argument is completely general wrt a; so $a^* < a^{**}$. So $a^* < a$. So, since M is an s-model, $a^* = a$. So M is a Ks-model.

By (2), $b \in M(t)$. So $0 < b$. So $R00b$. So $R0b^*0^*$. So $R0b0$. So $b < 0$. So, since M *is an s-model*, $b = 0$. So, since $b \notin M(A)$, $0 \notin M(A)$. So M is a Ks-model such that $M \not\models A$. So $A \notin$ **KRP+**. Q.E.D.

II.7. Proof of Theorems 3 and 8: RP+ is 1-reducible to $\mathbf{L}^2$. To show this, encode **RP+** int $\mathbf{L}^2$. Proceed as with the encoding of $\mathbf{R}_{\rightarrow \&}\mathbf{P+}$ into $\mathbf{L}^2$ (in §1.6, above), with the following three exceptions:

(1) take the language of $\mathbf{L}^2$ to contain both a 3-place relation constant, R, and a one place function constant, $*$, rather than the 2-place relation constant $\leq$ (see §1.3, Definition 7);

(2) replace the defined formula *semilat* 0 (defined in §1.3, Definition 10, and used in §1.5 Definition 15) with the following formula, *rms*, in the language of $\mathbf{L}^2$: $\forall x_1 \forall x_2 \forall x_3 \forall x_4 \{*(*(x_1)) = x_1 \,\&\, R0x_1x_1 \,\&\, Rx_1x_1x_1 \,\&\, [\exists x_5(Rx_1x_2x_5 \,\&\, Rx_5x_3x_4) \supset \exists x_5(Rx_1x_3x_5 \,\&\, Rx_5x_2x_4)] \,\&\, [\exists x_5(R0x_1x_5 \,\&\, Rx_5x_2x_3) \supset Rx_1x_2x_3] \,\&\, [Rx_1x_2x_3 \supset Rx_1{*}(x_3){*}(x_2)]\}$;

(3) in the definition of the recursive function, f (§I.5, Definition 14), replace the clause for $f((A \rightarrow B))$, and add a clause for $f({\sim} A)$ as follows:

$$f((A \rightarrow B)) = \forall x_2 \forall x_3((f(A)[x_2/x_1] \,\&\, Rx_1x_2x_3) \supset f(B)[x_3/x_1]);$$
$$f({\sim} A) = \neg f(A)[*(x_1)/x_1].$$

§III. Technical and logico-philosophical remarks.

III.1. Technical remarks. We have used a quite general method for proving that systems resulting from an algebraically-motivated primary interpretation of propositional quantification are recursively isomorphic to full second-order classical logic. Fine [1970] defines modal analogues to $\mathbf{R}_{\rightarrow \&}\mathbf{P+}$ and **RP+**. Assuming a possible world semantics with an accessibility relation, he defines **Kπ+**, **Tπ+**, **K4π+**, **S4π+**, **S4.2π+**, and **Bπ+**, and he discusses proofs that second-order arithmetic can be encoded into these. One can use methods similar to those in the present paper to show that these modal systems are recursively isomorphic to second-order logic. A similar project would be to get a grip on the complexity of the primary interpretation of the propositional quantification associated with Kripke's semantics for intuitionism (see Kripke [1963]). (I do not know what the result would be; the problem is with reflecting metalinguistic (classical) implication in the object language. See the discussion in §II.6, above.)

We know that some primary interpretations *are* axiomatisable: Kaplan [1970] and Fine [1970] axiomatise the primary interpretation of the quantifiers over **S5** associated with the Kripke's possible world semantics—indeed, Fine [1970] shows that **S5**π**+** is decidable. Such results suggest that some algebras (for example, sets of possible worlds with no accessibility relation) do not have enough structure for our methods. Some algebras might have too much. Perhaps some very general results are in the neighborhood. Perhaps there are interesting sufficient and/or necessary conditions on an algebraic semantics for the resulting primary interpretations of the propositional quantifiers to be axiomatisable, or arithmetical, or recursively isomorphic to second-order logic.

III.2. Logico-philosophical remarks. One might wonder whether these results bode well or ill for relevance logic. They seem to bode ill: after all, such a natural interpretation of the propositional quantifiers is not axiomatisable within a logical context which prides itself on its proof-theoretic motivation. But this misses the point of the proof-theoretic motivation of the relevance project. The primary interpretations are *semantically* motivated, and their nonaxiomatisability just goes to show that you should not put too much trust in your (formal) semantic intuitions when approaching **R** and its cousins. There *is* an interpretation of the propositional quantifiers which readily avails itself to a natural axiomatisation and a natural deduction system (see note 1). The mild semantic awkwardness of this secondary interpretation should not be too hard to take for anyone who countenances the relevant model structures of Routley and Meyer [1973] and the frames of Fine [1974]; like the Routley-Meyer and the Fine semantics, the secondary semantics for propositional quantifiers is motivated by proof-theoretic rather than semantic considerations. Indeed, this discussion of the primary and secondary interpretations of the propositional quantifiers may be a testament to the degree to which, in relevance logic at least, semantics is abstract proof theory.

§IV. Appendix Proof of Theorem 2, §I.3. Recall that Theorem 2, in §I.3 states that $\mathbf{L}^2$ is 1-reducible to **MSL0**. First we give some definitions.

DEFINITION 28. A *lattice* is an ordered pair $\mathbf{L} = \langle L, \leq \rangle$ where

(1) $\leq$ partially orders L; and

(2) each pair, $\{a, b\}$ of points in L has a least upper bound, $a \circ b$, and a greatest lower bound, $a \wedge b$, in L.

A *distributive lattice* is a lattice such that, for points a, b, and c:

$$a \circ (b \wedge c) = (a \circ b) \wedge (a \circ b) \quad \text{and} \quad a \wedge (b \circ c) = (a \wedge b) \circ (a \wedge c).$$

DEFINITION 29. Given a lattice $\mathbf{L} = \langle L, \leq \rangle$ an *ideal* is a nonempty subset I of L which is such that $(\forall a, b \in I)(a \circ b \in I)$ and $(\forall a \in I)(\forall b \in L)(\text{if } b \leq a \text{ then } b \in I)$.

DEFINITION 30. The *theory of distributive lattices with second-order monadic quantification over ideals* (**DLMI**) is the set of sentences in MSOL which are validated by every model (1) whose underlying structure is a distributive lattice; (2) which assigns ideals to predicate variables; and (3) for which we interpret monadic second-order quantification as ranging over ideals.

Theorem 2 of §I.3 follows from Theorems 10 and 11 below and the following:

THEOREM 2.2 (Nerode and Shore [1980]). **DLMI** *is recursively isomorphic to* $\mathbf{L}^2$.

We continue with some definitions.

Note. Just as we defined the MSOL formula $z \approx x \circ y$ (Definition 9, §I.3), we can define $z \approx x \wedge y$. Using these, we can define the formula (with one free predicate variable): X is an ideal. Also, just as we defined (Definition 10) the formula *semilat* (which "says that" the model is a semilattice), we can define a formula, *distlat*, which "says that" the model is a distributive lattice.

DEFINITION 33. The *monadic theory of distributive lattices* (**MDL**) is the set of MSOL formulas validated by all models which validate *distlat*. The *monadic theory of semilattices* (**MSL**) is the set of MSOL formulas which are validated by all models which validate *semilat*.

THEOREM 10. **DLMI** *is* 1-*reducible to* **MSL**.

PROOF. **MDL** is 1-reducible to **MSL**, since $A \in$ **MDL** iff $(distlat \supset A) \in$ **MSL**. To show that **DLMI** is 1-reducible to **MDL**, first let f be the recursive 1-1 function which maps an MSOL formula A to an MSOL formula just like A, except that subformulas of the form $(\forall XB)$ are replaced by $\forall X(X$ is an ideal $\supset f(B))$. Let g be as in Definition 13 (§I.4). Then, for all MSOL formulas A, $A \in$ **DLMI** iff $fg(A) \in$ **MDL**. Q.E.D.

THEOREM 11. **MSL** *is* 1-*reducible to* **MSL0**.

PROOF. First let h be the following 1-1 recursive function from MSOL formulas to MSOL formulas: $h(A)$ results by replacing every occurrence of x_i by x_{i+1} and by replacing every occurrence of 0 by x_1. Next, let g be as in Definition 13 in §I.4. Finally let the 1-1 recursive function f map a formula A to a formula just like A except that occurrences of subformulas of the form $\forall xB$ are replaced by $\forall x(\neg x = 0 \supset f(B))$. Note: $A \in$ **MSL** iff $fhg(A) \in$ **MSL0**. Q.E.D.

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