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RELEVANT IDENTITY

ABSTRACT. We begin to fill a lacuna in the relevance logic enterprise by providing a foundational analysis of identity in relevance logic. We consider rival interpretations of identity in this context, settling on the *relevant indiscernibility* interpretation, an interpretation related to Dunn's *relevant predication* project. We propose a general test for the *stability* of an axiomatisation of identity, relative to this interpretation, and we put various axiomatisations to this test. We fill our discussion out with both formal and philosophical remarks on identity in relevance logic.

KEY WORDS: relevance logic, relevant logic, identity, predication

Anderson and Belnap's relevance logic has been an ongoing concern for over thirty-five years. Propositional relevance logics have been extensively studied, and some of these studies have been extended to the first order. Despite this work there is an important lacuna in the relevance logic enterprise: there is little consensus concerning the proper interpretation and axiomatisation of *identity* within relevance logics, even within a logic as well understood as **R**. Rival axiomatisations have been proposed, but we have had little to go on in assessing these, apart from naïve relevance intuitions. The jury is still out on how to express transitivity (see below) and on what to say about substitution, $((x = y \,\&\, A[y/u]) \to A[x/u])$. The prevalent attitude in the literature is one of tolerance towards any plausible set of axioms, pending further research.

Here, we address this gap by providing a foundational methodological analysis of relevant identity. We consider rival interpretations of identity in first order **R**. We motivate the *relevant indiscernibility* interpretation, and propose a general test for the *stability* of an axiomatisation relative to this interpretation. We put various axiomatisations to this test, ultimately opting for a logic we call $\mathbf{R}^{\forall\exists x=}$. (One striking feature of $\mathbf{R}^{\forall\exists x=}$ is the failure of substitution.)

There are at least two reasons to engage in this project. Firstly, any first order logic worth its salt should incorporate a principled notion of identity. Secondly, our project is interdependent with another project, one of general philosophical interest: Dunn's project of developing and clarifying a notion of *relevant predication* (Dunn (1987, 1990, 1990a, 1990b, 199+ and 199+ a); see also Kremer (1997)). Dunn proposes a way of expressing,

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in $\mathbf{R}^{\forall\exists x}$ with identity, the claim that F is *genuinely a property* of x. Kremer (1997) argues that Dunn's proposal relies on an implicit interpretation of identity, the interpretation developed here.

Mares's (1992) contribution to our understanding of relevant identity should be noted. He adds a new semantic primitive to Fine's (1988) semantics for first order relevance logics, and shows that various conditions on this primitive correspond to various sets of identity axioms. In the course of his mainly technical investigation, he provides some considerations for and against particular identity axioms. For example, he recognises that expressing transitivity as $(x = y \rightarrow (y = z \rightarrow x = z))$ might be a problem in the logic **E**, since in **E** this implies that all identities are necessary. Further, Mares is motivated by Kremer 1989 to reconsider the axiom of substitution. For the most part, however, Mares's aim is not to present reasons for or against one axiom or another, but to develop a neutral semantic framework in which to model various axiomatisations. So there is still much room for the kind of foundational analysis that we are after: an analysis aimed at giving insight into *which* axioms to accept, or at least into the philosophical consequences of accepting one set of axioms rather than another.

1. Axiomatising Relevant Identity

In axiomatising identity in $\mathbf{R}^{\forall\exists x}$, we want *some* versions of reflexivity, symmetry and transitivity. The only version of reflexivity is

(Refl) $x = x$.

But the principle of symmetry has, in the relevance context, non-equivalent forms, for example,

Relevant Symmetry	$(x = y \rightarrow y = x)$, and
Truth-functional Symmetry	$(\sim x = y \vee y = x)$.

With transitivity, we have even more choices, for example,

Nested Transitivity	$x = y \rightarrow (y = z \rightarrow x = z)$,
Conjoined Transitivity	$(x = y \,\&\, y = z) \rightarrow x = z$, and
Truth-functional Transitivity	$(\sim x = y \vee \sim y = z \vee x = z)$.

Nobody has seriously considered the truth-functional versions of symmetry and transitivity, but both conjoined and nested transitivity have been proposed. (See Dunn 1987 and 1990a.)

In addition to obeying reflexivity, symmetry and transitivity, we expect "=" to interact in distinctive ways with the non-logical vocabulary. In classical logic, this interaction is often expressed with the axiom of the indiscernibility of identicals:

$$(x = y \to (A[x/u] \to A[y/u])).$$

As Dunn 1987 points out, this axiom in its unrestricted form commits us to seeming irrelevancies like

$$(x = y \to (p \to p)).$$

So we might consider weakening indiscernibility. Two proposals can be found in the literature. One is to weaken indiscernibility by positing it only for certain formulas A. The other is to replace indiscernibility with the weaker axiom of substitution:

$$((x = y \,\&\, A[x/u]) \to A[y/u]).$$

Even with substitution we might put restrictions on the formula A: Mares 1992 considers requiring that A contain no occurrences of $\to$.

Given these options, we can make *some* decisions by relying on naïve relevance intuitions. *Re* symmetry: $x = y$ and $y = x$ should be typographical variants of the same claim. So we postulate between them as close a connection as the logic allows, opting for relevant symmetry. *Re* transitivity: we expect *some* relevant connections among $x = y$, $y = z$ and $x = z$. So we restrict our options to nested and conjoined transitivity. Thus we have so far agreed on the following:

(Refl) $x = x$,
(Sym) $x = y \to y = x$,

and either one of

(N. Trans) $x = y \to (y = z \to x = z)$, or
(C. Trans) $(x = y \,\&\, y = z) \to x = z$.

Regarding the various forms of indiscernibility and substitution, our naïve intuitions are of less help. So we consider more substantial interpretations of identity in the relevance context.

2. The traditional interpretation of identity

The traditional interpretation of identity takes the following principle to be fundamental: if "s" and "t" are terms, then "$s = t$" is true iff "s" and "t" refer to the same individual.[1] This principle is *semantic*, articulated in terms of *truth* and *reference*. So we consider Fine's 1988 Kripke-style semantics for first order relevance logic.

In this semantic context, the traditional interpretation of identity has at least two counter-intuitive consequences. First, we must either develop a sense in which a variable can refer to different individuals in different worlds, or accept $(p \rightarrow (x = y \rightarrow x = y))$ as a theorem of relevance logic.[2] Second, and worse, we must either develop a sense in which a single variable "x" can refer to two distinct entities *in the same world*, or accept $(p \rightarrow x = x)$ as a theorem of relevance logic.[3] (Mares 1992 makes similar points.)

These consequences are not definitive strikes against the traditional interpretation. Perhaps we should reconsider our relevance intuitions in the presence of such a special relation as identity. On the other hand, we are sufficiently concerned to search for interpretations of identity that fit more squarely with our naïve relevance intuitions.

3. The indiscernibility interpretation of identity

Another historically important interpretation of identity finds its expression in the following metalinguistic principle: if "s" and "t" are terms then "$s = t$" is true iff "s" and "t" are interchangeable, *salve veritate*, in all contexts. Another expression this interpretation: "$s = t$" is true iff s and t are *indiscernible*: anything true of one is true of the other. On this *indiscernibility* interpretation of identity, "$s = t$" is roughly an infinite conjunction of biconditionals $(B[s/x] \leftrightarrow B[t/x])\,\&\,(C[s/x] \leftrightarrow C[t/x])\,\&\,\ldots$, where $B, C, \ldots$ run through all the formulas of the language. Since p is a formula of the language, this interpretation of identity leads to at least one counter-intuitive principle: $(x = y \rightarrow (p \leftrightarrow p))$.

So we have traded one kind of counter-intuitive consequence, $(p \rightarrow x = x)$, for another, $(x = y \rightarrow (p \leftrightarrow p))$. But the indiscernibility interpretation has an advantage over the traditional interpretation: on the former, $x = y$ can be given an approximate object-language reading. So when challenged by counter-intuitive consequences, a defender of the indiscernibility interpretation has access to a glass not available with the traditional interpretation: "'$x = y$' is an abbreviation of an infinite conjunction, among whose conjuncts is $(p \leftrightarrow p)$. So $(x = y \rightarrow (p \leftrightarrow p))$ is relevantly acceptable, even though it does not seem so at first glance."

Unfortunately, even if we can explain away the seeming irrelevancies produced by the indiscernibility interpretation, it wreaks havoc on an interesting project: Dunn's *relevant predication* project. To Dunn's project we now turn.

4. DUNN'S RELEVANT PREDICATION

Dunn proposes a notion of *relevant predication*, as a formalisation of the intuitive notion of a *real* property. Dunn's specific suggestion is embodied in the following two definitions, where "→" is the contingent relevant implication of **R**:

(1) "c relevantly has the property of being (an x) such that Ax" $=_{\mathrm{df}} \forall x(x = c \rightarrow Ax)$.

(2) "Ax is of a kind to determine relevant properties (with respect to x)" $=_{\mathrm{df}} \forall x(Ax \rightarrow \forall y(y = x \rightarrow Ay))$, where y is not free in A.

Dunn (1987) motivates (1) by contrasting the sentences "Socrates is such that he is wise" and "Reagan is such that Socrates is wise". x's being identical to Socrates is *relevant* to x's being wise: $\forall x(x =$ Socrates $\rightarrow$ x is wise$)$. But x's being identical to Reagan is not relevant to Socrates' being wise: $\sim\forall x(x =$ Reagan $\rightarrow$ Socrates is wise$)$. *Re* (2): the formula Ax generally determines relevant properties iff having the property is relevant to having it relevantly.

For an application of relevant predication, consider Geach's (1969) distinction between real and mere Cambridge change. Geach is unsatisfied with the "Cambridge" criterion for change: the thing, c, has changed if, for some formula Ax, Ac is true at time t, and Ac is false at time $t' > t$. Suppose, for example, that (i) Gx stands for 'x is tall'; (ii) p stands for 'it is raining in Moscow'; (iii) Tracy is tall; and (iv) it is presently raining in Moscow. If it stops raining in Moscow then the formula $(Gx \mathbin{\&} p)$ stops being true of Tracy. Tracy undergoes a Cambridge change, even though, intuitively, she has not changed at all.[4] Geach presents other examples of mere Cambridge change: Socrates becomes shorter than Theaetetus, as Theaetetus grows; Socrates becomes admired by a schoolboy; five ceases to be the number of someone's children. On place to locate the unrealness of these changes in is the unrealness of the corresponding properties: $(Gx \mathbin{\&} p)$; x is shorter than Theaetetus; x is admired by Johnny; and x is the number of Geach's children. Kremer (1997) argues for Dunn's relevant predication as a formalisation of real – in contrast to mere Cambridge – properties. (See also Dunn (199+).)

In $\mathbf{R}^{\forall\exists x}$, Dunn's definition (2) of "Ax determines relevant properties" is equivalent to a particular instance of indiscernibility (assuming symmetry for "="):

$$\forall x \forall y(x = y \to (Ax \to Ay)).$$

So the formulas that are "of a kind to determine relevant properties" are precisely those formulas of which indiscernibility holds. Now we can see why the indiscernibility interpretation of identity would wreak havoc on Dunn's project: *all* formulas would be of a kind to determine relevant properties. So Dunn's definition (1) and (2) would pick no formulas out as distinctive.

5. The relevant indiscernibility interpretation of identity

A close cousin of the indiscernibility interpretation of identity is suggested by the link between Dunn's definition (2), above, and the principle of indiscernibility. According to the *relevant indiscernibility* interpretation of identity (r.i. interpretation) "$s = t$" is true iff "s" and "t" are interchangeable, *salve veritate*, in all *relevant* contexts. A *relevant* context is one expressed by a formula *relevant* in the variable x. On this new interpretation, "$s = t$" is true iff s and t are *relevantly indiscernible*: anything *relevantly* true of the other. "$s = t$" can then be interpreted, intuitively, as an infinite conjunction of biconditionals $(B[s/x] \leftrightarrow B[t/s])$ & $(C[s/x] \leftrightarrow C[t/x])$ & ..., where Bx, Cx, etc., run through *only those formulas that express relevant properties*. A related understanding of "$s = t$" is as the second order formula $\forall G(Gs \leftrightarrow Gt)$, where G is ranges over relevant properties.

The r.i. interpretation is of immediate help in choosing our axiom of transitivity. Identity satisfies nested transitivity, since identity claims are being treated as conjunctions of biconditionals or as universals closures of biconditionals, and *these* satisfy nested transitivity. This makes a relevant property out of "being identical to z", since $\forall x(x = z \to \forall y(x = y \to y = z))$.

The logic of identity is otherwise very weak. For example, if F is a non-logical predicate constant, then we should not expect $(x = y \to (Fx \leftrightarrow Fy))$ to be a theorem. Independently of any interpretation of the non-logical constants, we have no reason to expect that F expresses a relevant property.[5]

The rejection of indiscernibility even for *atomic* formulas suggests that we might want to bring the axiom of substitution

$$\text{(Sub)} \quad (x = y \,\&\, A[y/u]) \rightarrow A[x/u]$$

to the rescue, so as to provide *some* logical content to the identity symbol, "$=$". Otherwise, "$=$" would not interact in any interesting way with the language's non-logical vocabulary.

Unfortunately, substitution in its broadest form is at odds with the r.i. interpretation of identity. Consider some non-logical predicate constant, F. Recall that we cannot assume, as a matter of logic, that F expresses a relevant property. One instance of the axiom of substitution is $((x = y \,\&\, Fx) \rightarrow Fy)$. Given our interpretation of identity, this can be interpreted either as

$$(1) \quad ((Bx \leftrightarrow By) \,\&\, (Cx \leftrightarrow Cy) \,\&\, \cdots \,\&\, Fx) \rightarrow Fy,$$

where $B,\ C$, etc. run through the formulas that express relevant properties; or as

$$(2) \quad (\forall G(Gx \leftrightarrow Gy) \,\&\, Fx) \rightarrow Fy,$$

where G ranges over the relevant properties. The r.i. interpretation of identity makes us suspicious of (1), since we have no reason to believe that the formulas $B,\ C$, etc. contain F in their vocabulary, or that F is expressible in the vocabulary contained in these formulas. We are equally suspicious of (2), since we have no reason to believe that what might be a hokey property expressed by F has any close relationship to the relevant properties over which G ranges. Finally, if F does express a hokey property, then we might not want to allow substitution into the context "Fx" under *any* interpretation of identity. (§15, below, touches on the issue of the substitution of identicals into *transparent* and *opaque* contexts.)

So "$=$" seems to be a symbol with very little logical content. This need not be an unwelcome consequence of the r.i. interpretation. The claim that a particular piece of vocabulary interacts with "$=$" is turned into a substantial non-logical claim. Such a claim might be advanced in the context of a particular *theory*, but can hardly be taken to be a theorem of *logic*.[6] And "$=$" is provided non-logical content in the context of the extra-logical stipulations of theories.

Still, there is a kind of language in which we can develop an interesting notion of identity within a first order relevance *logic*, in the absence of any particular theory. Consider first order relevance languages that have two sorts of first order constants: non-relevant predicate or relational constants, say $F_1, F_2, \ldots, F_n, \ldots$; and relevant predicate constants, say

$G_1, G_2, \ldots, G_n, \ldots,$ that are reserved for relevant properties. This allows us the following theorems: $x = y \rightarrow (G_k x \rightarrow G_k y)$. And this allows us a more interesting logic.

6. First order relevance logics with identity

The r.i. interpretation of identity puts some constraints on our axiomatisation of identity, but *prima facie* allows us room to manoeuvre. This section considers some or our possibilities. Until further notice, we will work in a restricted first order language: a language with propositional constants and unary *relevant* predicate constants, expressing relevant properties, but without any other non-logical vocabulary. We return to broader languages below.

So far our axioms are as follows:

(Refl) $x = x$;
(Sym) $(x = y \rightarrow y = x)$;
(Trans) $(x = y \rightarrow (y = z \rightarrow x = z))$.

Furthermore, we must have the following form of restricted indiscernibility:

(Ind) $(x = y \rightarrow (Gx \rightarrow Gy))$, where G is a relevant predicate constant.

The reason for (Ind) is that we are taking the G's to express relevant properties. In this case $(Gx \leftrightarrow Gy)$, which is a conjunction of $(Gx \rightarrow Gy)$ and $(Gy \rightarrow Gx)$, is among the conjuncts of the infinite conjunction that we are abbreviating by "$x = y$". Note that the postulation of (Ind) is equivalent to the stipulation that Gx expresses a relevant property in Dunn's sense.

So far, we have postulated indiscernibility for atomic formulas. The r.i. interpretation of identity suggests, however, that we should postulate indiscernibility for the non-atomic formula Ax whenever Ax expresses a relevant property with respect to x. Unfortunately, we do not have a clear idea which non-atomic formulas express relevant properties with respect to x. At this point we hold off on extending indiscernibility. Depending on our other axioms, the postulation of indiscernibility for atomic formulas produces indiscernibility for non-atomic formulas. We can think of ourselves as taking up a suggestion made by Urquhart and presented by Dunn (1987): postulate indiscernibility for atomic formulas, 'letting induction on formulas take us where it may with respect to Indiscernibility for compound formulas' (p. 452).

What about substitution,

$$\text{(Sub)} \quad ((x = y \mathbin{\&} A[x/u]) \to A[y/u])?$$

Our above argument against substitution (§5) depended on the existence, in the language, of non-relevant relational constants. In the present context, however, we have no such constants. So in the present context, if the formula A contains free occurrences of some individual variables, then A contains either "=" or some relevant predicate constant. So there might at least be *some* relationship between A and either the infinite conjunction of biconditionals, which is one intuitive reading of "$x = y$"; or the second order universal closure of a biconditional, which is our other intuitive reading of "$x = y$". So stipulating $((x = y \mathbin{\&} A[x/u]) \to A[y/u])$ does not put us in the same danger as it did in §5 of stipulating an irrelevant connection.

We have two further reasons for tolerance towards substitution, at least for the moment. First, though we have rejected unrestricted indiscernibility, $(x = y \to (A[x/u] \to A[y/u]))$, the introduction of unrestricted substitution, $((x = y \mathbin{\&} A[x/u]) \to A[y/u])$, does not introduce the same irrelevancies: $((x = y \mathbin{\&} p) \to p)$ is harmless, in contrast to $(x = y \to (p \to p))$. Second, as far as we know, when we draw up an appropriate list of formulas, Bx, Cx, etc., relevant with respect to x, it might just turn out that $((Bx \leftrightarrow By) \mathbin{\&} (Cx \leftrightarrow Cy) \mathbin{\&} \cdots \mathbin{\&} A[x/u]) \to A[y/u]$, especially if A shares some vocabulary with B, C, etc.

So we are now considering five identity axioms: (Refl), (Sym), (Trans), (Ind) and (Sub). Given our uncertainty about (Sub), we define two logics:

$$\mathbf{R}^{\forall\exists x=}: \quad \mathbf{R}^{\forall\exists x} + \text{(Refl)} + \text{(Sym)} + \text{(Trans)} + \text{(Ind)}$$
$$\mathbf{R}^{\forall\exists x=\mathrm{S}}: \quad \mathbf{R}^{\forall\exists x} + \text{(Refl)} + \text{(Sym)} + \text{(Trans)} + \text{(Ind)} + \text{(Sub)}.$$

7. THE GAME PLAN

So far we have expressed tolerance towards (Sub). Our plan is to argue against it, by arguing against $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ and its fragments, which we call the *substitution logics*. We begin by arguing, in §8 and §9, that the substitution logics are inelegant compared to the substitution-free logics.

Despite this formal inelegance, we might still want (Sub) for *philosophical* reasons. Some of the most interesting *philosophical* work done in first order relevance logic with identity is Dunn's work on relevant predication. And we have suggested that the interpretation of identity that is best suited to Dunn's project is the relevant indiscernibility interpretation. So whatever axiomatisation we choose for identity, we want it to jibe with the r.i.

interpretation. In §10 we develop a notion of a logic's *stability* with respect to the r.i. interpretation. $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ will be seen to be unstable in this sense, and so not to jibe with the r.i. interpretation.

8. PROOF THEORY: CONSECUTION CALCULUSES

One piece of evidence in favour of a proposed logic is its amenability to proof theoretic analyses. Important among proof theoretic tools are consecution calculuses, originally developed for classical and intuitionistic logic by Gentzen (1934). Anderson and Belnap (1975) present a consecution calculus, due to Dunn, for $\mathbf{R}_+ = \mathbf{R}_{\to\,\&\,\vee}$, the positive fragment of $\mathbf{R}$.

Define $\mathbf{R}^x_+$ to be the logic axiomatised by the axioms of $\mathbf{R}_+$, stated in a quantifier and negation free first order language. $\mathbf{R}^{\forall\exists x}$ is a conservative extension of $\mathbf{R}^x_+$.[7] Furthermore, Dunn's consecution calculus for $\mathbf{R}_+$ can equally be treated as a consecution calculus for $\mathbf{R}^x_+$. Finally, $\mathbf{R}^x_+$ can be enriched with identity axioms generating the substitution-free logic $\mathbf{R}^{x=}_+$ and the substitution logic $\mathbf{R}^{x=\mathrm{S}}_+$.

Dunn's calculus for $\mathbf{R}^x_+$ can elegantly be extended to a calculus for $\mathbf{R}^{x=}_+$. To the axioms add

$$\vdash x = x, \text{ and}$$
$$x = y \vdash y = x.$$

To the rules add two forms of $=\vdash$.

$$=\vdash \quad \frac{\Gamma_1 Gx\Gamma_2 \vdash B}{\Gamma_1\mathrm{I}(x = y, Gy)\Gamma_2 \vdash B} \quad \text{where } G \text{ is a relevant predicate constant}$$

$$=\vdash \quad \frac{\Gamma_1 Gx\Gamma_2 \vdash B}{\Gamma_1\mathrm{I}(y = x, Gy)\Gamma_2 \vdash B} \quad \text{where } G \text{ is a relevant predicate constant.}$$

The point against substitution: there seems to be no similarly perspicuous extension of the consecution calculus for $\mathbf{R}^x_+$ to a calculus for $\mathbf{R}^{x=\mathrm{S}}_+$. In fact, we do not even know of an inelegant way to handle substitution.

9. SEMANTICS

Mares (1992) provides semantics for a wide range of relevance logics with identity, including $\mathbf{R}^{\forall\exists x=}$ and $\mathbf{R}^{\forall\exists x=\mathrm{S}}$. Mares begins with Fine's (1988) semantics for first order relevance logics, and introduces a new semantic primitive to reflect the behaviour of identity. He shows that various conditions on this semantic primitive correspond to various axiomatisations of identity.

Here we note that Fine's semantics for $\mathbf{R}^{\forall\exists x}$ can be extended to a semantics for $\mathbf{R}^{\forall\exists x=}$ *without the addition of any new semantic primitives*. We simply add the following four conditions to the conditions on Fine's valuation relation, ϕ (see Fine (1988) for the details of the semantics):

for every ℓ, $\phi(\ell, =, i, i)$;
if $\phi(u, =, i, j)$ then $\phi(u, =, j, i)$;
if $\phi(u, =, i, j)$ and $\phi(v, =, j, k)$ then $\phi(uv, =, i, k)$; and
if $\phi(u, =, i, j)$ and $\phi(v, G, i)$ then $\phi(uv, G, j)$.

The point against substitution: the semantics for $\mathbf{R}^{\forall\exists x=}$ is simpler than the semantics for $\mathbf{R}^{\forall\exists x=\mathrm{S}}$, requiring fewer semantic primitives.

10. Stability, for Relevance Logics with Identity

Our considerations so far suggest that the substitution logics are not as elegant as the substitution-free logics: they do not admit of equally simple proof theoretic or semantic analyses. But formal elegance and philosophical coherence are different, though related, matters. In this section, we begin to develop a sense in which $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ is philosophically unstable, as least with respect to the relevant indiscernibility interpretation of identity.

Recall that, according to the r.i. interpretation, $x = y$ is thought of as an infinite conjunction of biconditionals, $(Bx \leftrightarrow By) \,\&\, (Cx \leftrightarrow Cy) \,\&\, \ldots,$ where B, C, etc., run through the formulas expressing relevant properties. Since $x = y$ is a conjunction of biconditionals, part of the conceptual content of $x = y$ is expressed in our finitary language by (Refl), (Sym) and (Trans). The remainder of its content is expressed by the infinite list,

$$x = y \;\rightarrow\; (B[x/u] \leftrightarrow B[y/u]),$$
$$x = y \;\rightarrow\; (C[x/u] \leftrightarrow C[y/u]),$$
$$\ldots$$

where Bu, Cu run through the formulas relevant in the variable u. To obtain each member of this list as an axiom, it suffices to stipulate

(Relevant Indiscernibility) $x = y \rightarrow (A[x/u] \rightarrow A[y/u])$,
where Au expresses a relevant property with respect to the variable u.

Thus put, this axiom is problematic: it depends on some antecedent answer to the question of which formulas express relevant properties in u. And there is no obvious answer: even if we allow that all the atomic formulas of the form Gu express relevant properties, we do not know which non-atomic formulas express relevant properties.[8]

We have a “chicken and egg” problem here. On the one hand we want to rely on the r.i. interpretation of identity to decide how to introduce identity to $\mathbf{R}^{\forall\exists x}$. The precise meaning of this interpretation of identity depends on which formulas are relevant in which variables. This in turn depends on which axioms we use to introduce identity to $\mathbf{R}^{\forall\exists x}$. Even if the r.i. interpretation of identity is right, we still do not know how to introduce identity to $\mathbf{R}^{\forall\exists x}$ until we have decided how to introduce identity to $\mathbf{R}^{\forall\exists x}$.

Is there a way out of this circle? At the very least, the r.i. interpretation of identity suggests that, whichever formulas express relevant properties, $\mathbf{R}^{\forall\exists x}+$ identity should be axiomatisable as $\mathbf{R}^{\forall\exists x}$ + (Refl) + (Sym) + (Trans) +

(R. Ind) $(x = y \to (A[x/u] \to A[y/u]))$,
where Au is *any* formula, atomic or not, relevant with respect to the variable u (however, this notion of relevance is made precise).

Neither $\mathbf{R}^{\forall\exists x=}$ nor $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ has been given such an axiomatisation. We might ask, however, whether $\mathbf{R}^{\forall\exists x=}$ or $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ *admits* of such an axiomatisation, i.e. a axiomatisation that makes explicit the relevant indiscernibility interpretation of identity.

We develop our key idea here more broadly. Suppose that the logic **L** is either $\mathbf{R}^{\forall\exists x}$ or some fragment or conservative extension of $\mathbf{R}^{\forall\exists x}$, and that the logic **LI** results from adding identity axioms to **L**. **LI** might be $\mathbf{R}^{\forall\exists x=}$ or $\mathbf{R}^{\forall\exists x=\mathrm{S}}$. Here we suppose that the language of **LI** has at least the connective $\to$, individual variables, relational constants and “$=$”. We also suppose that if the language contains $\exists$ then it contains $\forall$. **LI** provides us with a notion of a *relevant* formula:

DEFINITION 1. (i) If the language of **LI** contains $\forall$, then a formula A is **LI**-*relevant in the variable* x iff $\vdash_{\mathbf{LI}} \forall x(A \to \forall y(x = y \to A[y/x]))$, where y is not free in A.

(ii) If the language of **LI** does not contain $\forall$, then a formula A id **LI**-*relevant in the variable* x iff $\vdash_{\mathbf{LI}} A[u/x] \to (u = v \to A[v/x])$, where u and v are not free in A.

If the relevant indiscernibility interpretation of identity works for **LI**, then “$=$” ought to be axiomatisable with (Refl), (Sym), (Trans) and (R. Ind), where the precise form of (R. Ind) depends on the theory of relevant properties provided by **LI**.

DEFINITION 2. **LI** is *stable with respect to the relevant indiscernibility interpretation of identity* iff **LI** can be re-axiomatised by the theorems of **L** together with (Refl), (Sym), (Trans), and the **LI**-*relevant indiscernibility axiom*

$$x = y \to (A[x/u] \to A[y/u]), \quad \text{where } Au \text{ is } \mathbf{LI}\text{-relevant in } u,$$

and the rules of Modus Ponens, Adjunction (if the language of **LI** contains &) and universal generalisation (if the language of **LI** contains ∀). We will call this the *r.i.* axiomatisation of **LI**. We often just say that **LI** is *stable*.

So, according to Definition 2, a fragment, **LI**, of $\mathbf{R}^{\forall\exists x}$ is stable iff it admits of the r.i. axiomatisation of identity, where the precise form of that axiomatisation is given by **LI**'s own theory of relevant predication. Notice that the kind of stability we have defined is a kind of *internal* stability: we judge **LI** by **LI**'s own theory of relevant predication.

All the obvious substitution-free logics are well-behaved in this sense.

THEOREM 3. $\mathbf{R}^{\forall\exists x=}$, $\mathbf{R}^{x=}$, $\mathbf{R}_{+}^{x=}$ *and* $\mathbf{R}_{+}^{\forall\exists x=}$ *are stable.*

Proof. Every axiom of the r.i. axiomatisation of $\mathbf{R}^{\forall\exists x=}$ is a theorem of $\mathbf{R}^{\forall\exists x=}$, and every axiom of the original axiomatisation of $\mathbf{R}^{\forall\exists x=}$ is in the r.i. axiomatisation. Similarly for the other substitution free logics. □

So, in $\mathbf{R}^{\forall\exists x=}$ and its fragments, "=" means what it ought to mean. Things do not go so well for the substitution logics.

THEOREM 4. $\mathbf{R}_{\to\&}^{x=\mathrm{S}}$ *and* $\mathbf{R}_{\to\&}^{\forall x=\mathrm{S}}$ *are not stable.*

(To define $\mathbf{R}_{\to\&}^{x=\mathrm{S}}$, work in a quantifier free first order language with only two connectives, → and & . First define $\mathbf{R}_{\to\&}^{x}$ with the axioms and rules of $\mathbf{R}_{\to\&}$. Then add (Refl), (Sym), (Ind) and (Sub). To get $\mathbf{R}_{\to\&}^{\forall x=\mathrm{S}}$, add quantificational axioms and rules to $\mathbf{R}_{\to\&}^{x=\mathrm{S}}$.)

Proof. The proof of Theorem 4 is long, and the techniques involved are distant from the our main thread. We refer the reader to Kremer (1994). The strategy is to show that $(x = y \,\&\, ((Gy \,\&\, p) \to q)) \to ((Gx \,\&\, p) \to q)$ cannot be proved from the r.i. axiomatisation of $\mathbf{R}_{\to\&}^{\forall x=\mathrm{S}}$. □

We are really interested in the full logic $\mathbf{R}^{\forall\exists x=\mathrm{S}}$. We do not, however, have a proof of

CONJECTURE 5. $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ *is not stable.*

Theorem 4 renders this conjecture likely. At any rate, if the conjecture were false then $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ would be problematic in other interesting senses: the r.i. axiomatisation of $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ would not be a conservative extension of the r.i. axiomatisation of $\mathbf{R}^{x=\mathrm{S}}_{\rightarrow\,\&}$. So there would be $\forall\exists \vee \sim$-free theorems of $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ whose only proofs from the r.i. axioms take a detour through axioms that are built up with $\forall$, $\exists$, $\vee$ or $\sim$. In the remainder of this paper, we will assume that Conjecture 5 is true.

11. Other stable logics

$\mathbf{R}^{\forall\exists x=}$ is not the only stable extension of $\mathbf{R}^{\forall\exists x}$. Define the logic $\mathbf{R}^{\forall\exists x=\mathrm{F}}$ as $\mathbf{R}^{\forall\exists x}$ + (Refl) + (Sym) + (Trans)+ the axiom of *full* indiscernibility:

$$\text{(F. Ind)} \quad x = y \rightarrow (A[x/u] \rightarrow A[y/u]), \quad \text{for any formula } A.$$

Even $\mathbf{R}^{\forall\exists x=\mathrm{F}}$ is stable. What does this say about $\mathbf{R}^{\forall\exists x=\mathrm{F}}$? Though motivated by the non-relevant indiscernibility interpretation of identity (§3), this logic is stable with respect to the *relevant* indiscernibility interpretation in the following sense: we can axiomatise this logic's theory of identity by (Refl), (Sym), (Trans) and *the form of relevant indiscernibility motivated by its theory of relevant predication*. The trouble with $\mathbf{R}^{\forall\exists x=\mathrm{F}}$ is not that it is lacking the kind of internal stability that we have defined, but that its theory of relevant predication is deficient on other grounds.

Another stable logic is the logic generated by the r.i. axiomatisation of $\mathbf{R}^{\forall\exists x=\mathrm{S}}$. Our objection to *this* logic is not that it is unstable, but that is its motivation is to be found in the unstable logic $\mathbf{R}^{\forall\exists x=\mathrm{S}}$. This indicates that our notion of stability is a necessary but not sufficient condition for the coherence of a relevance logic with identity.

We further note that *any* extension of $\mathbf{R}^{\forall\exists x}$ with the axioms (Refl), (Sym), (Trans) and some form of indiscernibility is stable in the present sense. A particular form of indiscernibility might be called for by independently motivated grammatical considerations. In such cases, we have provided no further test for the coherence of the logic.

12. Weak substitution

When Dunn (1987) rejects unlimited indiscernibility, he comforts his readers with the thought that “the traditional principle of reasoning, known as ‘substitution of identicals’ ” need not fail in relevance logic: after all, "we can always have in place of Indiscernibility the weaker [Substitution]" (p. 452).[9] But we have been mounting a sustained argument against substitution. What then of our "traditional principles" concerning identity?

We take heart from the similarity between the failure of

$$(1) \quad [x = y \,\&\, ((Gx \,\&\, p) \to q)] \to ((Gy \,\&\, p) \to q)$$

in $\mathbf{R}^{\forall\exists x=}$ (without substitution), and the failure of

$$(2) \quad [(r \leftrightarrow s) \,\&\, ((r \,\&\, p) \to q)] \to ((s \,\&\, p) \to q)$$

in **R**. Despite any principle of "substitution of equivalents", relevance logicians are comfortable enough with the failure of (2). The standard gloss on (2) is that the logical truths $(p \to p)$ and $(q \to q)$ has been "suppressed": if they are added to the antecedent thus,

$$(2') \quad [(r \leftrightarrow s) \,\&\, (p \to p) \,\&\, (q \to q) \,\&\, ((r \,\&\, p) \to q)] \\ \to ((s \,\&\, p) \to q),$$

then we have a theorem on our hands. Classical logic allows such "suppression" and relevance logic does not. The same gloss can be given on (1) or on any other instance of (Sub), since

$$(1') \quad [x = y \,\&\, (p \to p) \,\&\, (q \to q) \,\&\, ((Gx \,\&\, p) \to q)] \\ \to ((Gy \,\&\, p) \to q)$$

is a theorem of $\mathbf{R}^{\forall\exists x=}$.

If we add the logical constant $\boldsymbol{t}$ to the language, then we can state a general principle of weak substitution. $\boldsymbol{t}$ is standardly interpreted as the conjunction of all theorems, and is added to **R**, or to $\mathbf{R}^{\forall\exists x=}$, via two axioms: $\boldsymbol{t}$; and $\boldsymbol{t} \to (A \to A)$. The principle of *weak substitution* is

$$(x = y \,\&\, A[x/u] \,\&\, \boldsymbol{t}) \to A[y/u].$$

Every instance of weak substitution is a theorem of $\mathbf{R}^{\forall\exists x=}$. So our rejection of substitution is not an outright rejection of the 'substitution of identicals'.

We are still working in a language whose only non-logical constants are propositional constants and *relevant* predicate constants. And we are only claiming the principle of weak substitution for such a language. §14, below, considers the appropriate logic for first order languages with all manner of relational constants. And §15 considers the significance of weak substitution in the context of a broader language.

13. $\mathbf{R}^{\forall\exists x=}$

So far the thrust of our argument has been negative: we have been arguing against $\mathbf{R}^{\forall\exists x=\mathrm{S}}$, and, by association, against the axiom of substitution. Though our discussion has included some considerations in favour of $\mathbf{R}^{\forall\exists x=}$ we have not presented them in one place. In this section, we gather these considerations together.

First $\mathbf{R}^{\forall\exists x=}$ is stable. Though it might not be the only stable logic, it is the simplest for the narrow first order language we are so far considering.

Against $\mathbf{R}^{\forall\exists x=}$, it might be argued that is motivated by too sparse a notion of a relevant predicate since indiscernibility is restricted to *atomic* formulas. In reply we note that Kremer 1989 shows that $\mathbf{R}^{\forall\exists x=}$ allows for a rich and independently motivated notion of relevant predication. This is, in itself, another consideration in favour of $\mathbf{R}^{\forall\exists x=}$.

Furthermore, the substitution-free fragment $\mathbf{R}_{+}^{x=}$ admits of a simple proof theoretic analysis, and $\mathbf{R}^{\forall\exists x=}$ admits of a relatively simple semantic analysis. We suspect that any formal analysis of $\mathbf{R}^{\forall\exists x}$ or its fragments can be extended in similarly simple ways to an analysis of $\mathbf{R}^{\forall\exists x=}$ or the corresponding fragments, but not to other versions of $\mathbf{R}^{\forall\exists x}+$ identity.

Also, though the principle of the substitution fails in $\mathbf{R}^{\forall\exists x=}$, a weaker version of substitution holds: the principle of weak substitution. And the extent to which weak substitution will be seen *not* to hold in broader languages (see §14 and §15), is the extent to which those languages permit the expression of *opaque* contexts for which we do not expect *any* form of substitution to hold.

Finally, $\mathbf{R}^{\forall\exists x=}$ is not beset by the seeming irrelevancies that beset the logics which emerge from the traditional and the indiscernibility interpretations of identity (§2 and §3). And so $\mathbf{R}^{\forall\exists x=}$ harmonises with our naïve relevance intuitions.

14. IDENTITY FOR BROADER FIRST ORDER LANGUAGES

§5 suggested that an appropriate first order language for the logic of relevant identity should have a set of non-relevant relational constants, F_1, F_2, $\ldots$, F_n, $\ldots$; and a set of relevant predicate constants, $G_1, G_2, \ldots, G_n, \ldots$, expressing, as a matter of logic, relevant properties. So far, we have restricted our investigation to a narrow language, whose only relational constants are propositional constants and *relevant* predicate constants. And in this context we have argued against $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ and in favour of $\mathbf{R}^{\forall\exists x=}$. Suppose that we are now working with some broader first order language, with a non-empty stock of non-relevant relational constants. What is the appropriate logic of identity for this broader language?

First, we want this logic to be a conservative extension of the logic for the narrower language. So we want all of the axioms of $\mathbf{R}^{\forall\exists x=}$, and we do *not* want the axiom of substitution in its general form. We *especially* do not want full substitution for arbitrary formulas involving the new vocabulary, since this new vocabulary is less likely to interact with "$=$" than is the old vocabulary. Furthermore, we do not want any new axiom of indiscernibility for the new vocabulary, since we are supposing that atomic formulas formed with the new vocabulary are not such as to express relevant properties as a matter of logic.

One might stipulate that the new vocabulary interacts in *some* interesting way with identity: (1) We could posit restricted forms of substitution: for example, $(x = y \,\&\, A[x/u]) \rightarrow A[y/u]$, when A is *atomic*, or, as suggested by Mares (1992), when A is contains no occurrences of $\rightarrow$. This would allow $(x = y \,\&\, Fx) \rightarrow Fy$ as an axiom, for example, even when F does not express a relevant property. (2) Motivated by the discussion of §12, we could posit a general form of *weak* substitution for languages enriched by $\boldsymbol{t}$: $((x = y \,\&\, A[x/u] \,\&\, \boldsymbol{t}) \rightarrow A[y/u])$. This would allow $((x = y \,\&\, Fx \,\&\, \boldsymbol{t}) \rightarrow Fy)$ as an axiom, even when F does not express a relevant property.

We will comment on these two suggestions in turn. Regarding suggestion (1): §5 already provided one argument against a general axiom of atomic substitution in the presence of non-relevant relational constants. Here we consider two related arguments. First, the resulting logic is not stable in the sense of §10: its r.i. interpretation does not allow us to prove, for example, $((x = y \,\&\, Fx) \rightarrow Fy)$, if F is a non-relevant predicate constant. This drives home our point in §5 that, if Fx does not express a relevant property, then we have no reason to believe that the hokey property expressed by F has *any* close relationship to any relevant properties. Second, suppose that, on some interpretation or in some theory, F does have a relationship to some relevant properties. For example, suppose that in some theory Fx is equivalent to $((Gx \,\&\, p) \rightarrow q)$, where G is a relevant predicate constant. Even in such circumstances we must recognise the possibility that $((x = y \,\&\, Fx) \rightarrow Fy)$ fails. And this is because $[x = y \,\&\, ((Gx \,\&\, p) \rightarrow q)] \rightarrow ((Gy \,\&\, p) \rightarrow q)$ is not a theorem of $\mathbf{R}^{\forall\exists x=}$. So $(x = y \,\&\, Fx) \rightarrow Fy$ should not be a principle of the theory $\mathbf{R}^{\forall\exists x=} + \forall x(Fx \leftrightarrow ((Gx \,\&\, p) \rightarrow q))$. These considerations tell even more strongly against Mares's version of restricted substitution than against "atomic" substitution, since atomic substitution follows from Mares's version.

Regarding suggestion (2): the resulting logic looks unstable, with either the unrestricted or the restricted form of weak substitution. So even weak substitution is an illegitimate new axiom. But weak substitution is still an interesting principle, as we argue in §15.

15. TRANSPARENT AND OPAQUE CONTEXTS

Quine (1953) characterises *opaque*–as opposed to *transparent*–contexts as ones in which we cannot intersubstitute co-referential singular terms *salve veritate*.[10] A canonical example is the formula Ax = "Tracy believe that x denounced Catiline". From A["Cicero"/x] and "Cicero = Tully" we cannot, on most accounts, infer A["Tully"/x]. Note that Ax expresses a Cambridge property (see §4, above); A["Cicero"/x] could go from true to false without a corresponding change in Cicero. Such examples suggest an identification of opaque contexts with Cambridge predicates.

We should resist this. Among other Cambridge predicates are Cx = "x is shorter than Theaetetus" and Bx = "$(Gx \,\&\, p)$", where Gx is "x is tall" and p is "it is raining in Moscow" (see §4, above). Despite the failure of $(x = y \rightarrow (Bx \rightarrow By))$, there is *some* sense in which Bx is transparent, since $((x = y \,\&\, Bx) \rightarrow By)$. And it is not completely implausible that $((x = y \,\&\, Cx) \rightarrow Cy)$. This tempts us to say that a formula Au is a *transparent context in u* iff $((x = y \,\&\, A[x/u]) \rightarrow A[y/u])$. Such a characterisation allows transparent contexts that are not relevant predicates.

Is this characterisation of transparent contexts broad enough? Consider the formula $Au = ((Gu \,\&\, p) \rightarrow q)$, where Gu is relevant (in u). Notice that $((x = y \,\&\, Ax \,\&\, \boldsymbol{t}) \rightarrow Ay)$ is a theorem of $\mathbf{R}^{t\forall\exists x=}$, even though $((x = y \,\&\, Ax) \rightarrow Ay)$ is not. So weak substitution holds for Au although substitution fails. Indeed, suppose that $G_1, G_2, \ldots, G_n$ are relevant predicate constants. And suppose that Au is a formula in which all the free occurrences of u occur in subformulas of the form $G_i u$. Then $((x = y \,\&\, Ax \,\&\, \boldsymbol{t}) \rightarrow Ay)$ is a theorem of $\mathbf{R}^{t\forall\exists x=}$.

This fact suggests two things: first, a broader characterisation of transparent contexts:

"A is a transparent context with respect to u"
$=_{\mathrm{df}} \forall x \forall y((x = y \,\&\, A[x/u] \,\&\, \boldsymbol{t}) \rightarrow A[y/u])$;

and second, a sufficient if not a necessary condition for A to be a transparent context:

whatever is expressed by A is in some sense ultimately reducible to relevant properties, even if we have not yet formulated a language in which relevant properties can be expressed by atomic formulas.

Now there are two very general methodological principles one might urge in the development of a first order language with identity:

(1) in a well behaved formal language, there are no opaque contexts; and
(2) if the logic underlying a particular theory is $\mathbf{R}^{\forall\exists x=}$, then there is some, possibly undiscovered, ideal language with which we can formalise the theory as an $\mathbf{R}^{\forall\exists x=}$-theory, so that each atomic formula $Hx_1 \ldots x_n$ expresses a relevant property in $x_i,\ i = 1, \ldots, n$.[11]

Marcus (1975) expresses sympathy with principle (1), hinting at a link to a principle like (2): "A belief in the principle of substitutivity is grounded in the belief that the pursuit of logical form is not futile." (p. 109) A belief in the principle of substitutivity can be seen as a belief in principle (1). And a belief that "the pursuit of logical form is not futile" can be seen as a belief in the existence of some logically ideal language in which the logical form of every claim is made explicit in its grammatical form.

Notice that, in the context of $\mathbf{R}^{\forall\exists x=}$, a belief in such a logically ideal language is not sufficient to ground the principle of substitutivity, i.e. principle (1). If we want to formulate a *stable* version of $\mathbf{R}^{\forall\exists x=}$ that *also* allows substitutivity, we must not only have a logically ideal language; we must have a language in which each atomic formula $Hx_1 \ldots x_n$ expresses a relevant property in x_i, $i = 1, \ldots, n$. This suggests a *transparent* formulation of $\mathbf{R}^{\forall\exists x=}$, i.e. an extension of $\mathbf{R}^{\forall\exists x}$ with (Refl), (Sym), (Trans) and the following axiom of indiscernibility:

$$(x = y \to (A[x/u] \to A[y/u])),\quad \text{where } A \text{ is atomic and where } u \text{ is free in } A.$$

Every instance of weak substitution is a theorem of such a formulation of $\mathbf{R}^{\forall\exists x=}$: all contexts are transparent.

From a point of view that takes the weaker formulation of $\mathbf{R}^{\forall\exists x=}$ to be fundamental, the transparent formulation of $\mathbf{R}^{\forall\exists x=}$ is not a *logic*, but a *theory*, that depends on the postulation of certain *extra*-logical axioms. On this line, principles (1) and (2) are not logical, but metaphysical in nature. We believe that not much hangs on whether to call the transparent system a *logic* or an extra-logical *theory.* The important point is to see what motivates such a formulation, and what its underlying assumptions are.

16. CONCLUDING REMARKS

We have been considering exactly how to extend $\mathbf{R}^{\forall\exists x}$ to $\mathbf{R}^{\forall\exists x}$ with identity. We are presently considering languages which have a stock of non-relevant relational constants from among $F_1, F_2, \ldots, F_n, \ldots$, and a stock of relevant predicate constants from among $G_1, G_2, \ldots, G_n, \ldots$, which are reserved for the expression of relevant properties.

We proceeded by first urging a relevant indiscernibility interpretation of identity. Given this interpretation we presented a sustained argument against using the axiom of substitution in extending $\mathbf{R}^{\forall\exists x}$. $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ turns out to be unwieldy both proof theoretically and semantically. Further, it is unstable by the lights of its own theory of relevant predication, in the context of the relevant indiscernibility interpretation of identity. So we propose the logic $\mathbf{R}^{\forall\exists x=}$, defined presently, as the appropriate extension of $\mathbf{R}^{\forall\exists x}$. $\mathbf{R}^{\forall\exists x=} = \mathbf{R}^{\forall\exists x}+$ the following axioms:

(Refl) $x = x$;
(Sym) $(x = y \to y = x)$;
(Trans) $(x = y \to (y = z \to x = z))$; and
(Ind) $x = y \to (Gx \to Gy)$,
where G is a relevant predicate constant.

$\mathbf{R}^{\forall\exists x=}$ is motivated by the relevant indiscernibility interpretation, which is in turn motivated by Dunn's notion of relevant predication. Though $\mathbf{R}^{\forall\exists x=}$ is ultimately motivated by the notion of relevant predication, we believe that it can be seen as part of the motivation *for* the notion of relevant predication. The notion of relevant predication has been shown to be *fruitful*: it has helped us work our way through the issue of identity in relevance logic. In particular it has motivated a treatment of identity which goes beyond the "let's see what happens if we postulate such and such axioms" treatments to which we might otherwise be tempted. Furthermore, the logic motivated by the notion of relevant predication harmonises with our other relevance intuitions.

When we first considered how to add identity to $\mathbf{R}^{\forall\exists x}$, we considered a "traditional" interpretation, whose underlying principle is that "$s = t$" is true iff "s" and "t" refer to the same entity. Though this interpretation did not provide good guidance in our choice of axioms, there is still a sense in which its underlying principle is *true*.

The first part of making sense of this is to understand

(1) "s" and "t" refer to the same individual

as

(2) the referent of "s" is identical to the referent of "t".

Given our relevant indiscernibility of identity, (2) can be understood as

(3) the referent of "s" and the referent of "t" share all relevant properties.

And so the metalinguistic principle motivating the traditional interpretation, i.e. the principle

(4) "$s = t$" is true iff "s" and "t" refer to the same individual,

is to be understood as the claim that

(5) "$s = t$" is true iff the referent of "s" and the referent of "t" share all relevant properties.

(5) is true iff

(6) $s = t$ iff s and t share all relevant properties.

(6) is true on the relevant indiscernibility interpretation of identity. So (5) is true. So (4), the principle underlying the traditional interpretation of identity, is true on the relevant indiscernibility interpretation of identity. We note that this line of thought shares much in common with Carnap's (1947) discussion of a *neutral metalanguage*.

If the principle underlying the traditional interpretation of identity is true, then why is it such a poor guide to the appropriate axiomatisation of identity in **R**? Why does it seem to motivate irrelevant axioms? Consider the *sense* in which this principle is true. It is a metalinguistic principle, mentioning but not using the expressions "s", "t" and "$s = t$". Our rendering of (4) as (5) depends on the metalanguage being in some sense a *relevance* metalanguage as opposed to a classical metalanguage. When the traditional interpretation was first articulated (§2), we were working in a classical metalanguage. This pushed us to consider the extensional semantics, developed in a classical metalanguage, for **R**. If we had a sufficiently well understood first order relevance system in which to carry out our meta-theory, we might not have been led in the same unhappy direction by the traditional interpretation of identity.

We were also working in a classical metalanguage in our considerations of the relevant indiscernibility interpretation of **R**. Why were we not in the same danger of being misled? The pertinent difference in the two cases is that in the case of the traditional interpretation, the guiding principle has no natural object language rendering, while in the case of the relevant indiscernibility interpretation, the guiding principle was given the following object language rendering: $s = t$ iff s and t share all relevant properties. Since the object language is a *relevance* object language, the principle motivating the relevant indiscernibility interpretation is already articulated as a relevance principle, and is thus less likely to lead us into irrelevancies than the principle motivating the traditional interpretation.

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NOTES

[1] It is tricky to express such a principle for a first order language that contains no individual constants, since "$x = y$" has no truth value. But there is an attenuated sense in which "$x = y$" *can* be taken to be true or false: it is true or false relative to an assignment of values to the individual variables. Similarly "x" and "y" can be taken to refer, relative to an assignment of values to the variables. As we proceed, we allow ourselves to treat variables as referring expressions and open formulas as bearers of truth or falsity, relative to some assignment of values to the variables.

[2] Suppose that the assignment of individuals to variables is the same for each world. Then, if "x" and "y" are both assigned the same individual, then "$x = y$" is true in each world, in which case $(x = y \to x = y)$ is true in each world, in which case $(p \to (x = y \to x = y))$ is true in each world. This argument can be checked against Fine's semantics for quantified relevance logics. Similarly if "x" and "y" are assigned distinct individuals, then "$x = y$" is false in each world, in which case $(x = y \to x = y)$ is true in each world, in which case $(p \to (x = y \to x = y))$ is true in each world. So in every model, $(p \to (x = y \to x = y))$ is true in each world.

[3] "$x = x$" would be true in every world, in which case so would $(p \to x = x)$.

[4] Geach does not consider Cambridge properties expressed by formulas of this conjunctive form. Robert Brandom suggested the idea to me. Dunn has directed me to another property expressed by such a formula, Fodor's (1987) property of being a fridgeon: x is a *fridgeon* at t iff x is a particle at t and my fridge is on at t.

[5] Below we propose a language that reserves some of its non-logical predicate constants for relevant properties. If F is one of *these* then, of course, it expresses a relevant property.

[6] We soften this stance at the end of §15, below.

[7] This follows from the results of Meyer and Routley (1974) and from the fact that $\mathbf{R}^{\forall\exists x}$ is a conservative extension of $\mathbf{R}^{x}$, where $\mathbf{R}^{x}$ is the logic axiomatised by the axioms of $\mathbf{R}$, in a quantifier and negation free first order language.

[8] Kremer (1989) shows that certain non-atomic formulas are relevant in $\mathbf{R}^{\forall\exists x=\mathrm{S}}$ but not in $\mathbf{R}^{\forall\exists x=}$: for example, $(G_1 x \,\&\, p) \to G_2 x$. So the which non-atomic formulas are relevant depends on the axiomatisation of identity.

[9] Dunn (1990) weakens his 1987 commitment to substitution.

[10] He initially characterises opaque contexts as ones in which the contribution of an occurrence of a singular term to the truth of a sentence depends on something other than

its referent. He argues that intersubstitutivity, *salve veritate*, of co-referential singular terms is a necessary and sufficient condition for the transparency of a context.

[11] Principle (1) does not dictate the proper treatment of the occurrence of "Cicero" in

(∗) Tracy believes that Cicero denounced Catiline

At one extreme, we could take this occurrence of "Cicero" to be like the occurrence of "cat" in "cattle", making no semantic contribution to (∗). (See Quine (1953).) A formal rendering of (∗) would then not contain the name "Cicero" as a constituent. At the other extreme, we could take the surface grammar of (∗) to be perfectly in order. Then we could, despite our intuitions to the contrary, infer "Tracy believes that *Tully* denounced Catiline" from (∗). This is implicit in Marcus's (1986) "object-centered" account of beliefs.

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