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RELEVANT PREDICATION: GRAMMATICAL CHARACTERISATIONS*

ABSTRACT. This paper reformulates and decides a certain conjecture in Dunn's 'Relevant Predication 1: The Formal Theory' (*Journal of Philosophical Logic* **16**, 347–381, 1987). This conjecture of Dunn's relates his object-language characterisation of a property's being *relevant in a variable x* to certain grammatical characterisations of relevance, analogous to some given by Helman, in 'Relevant Implication and Relevant Functions' (to appear in *Entailment: The Logic of Relevance and Necessity*, vol. 2, by Alan Ross Anderson, Nuel Belnap, and J. Michael Dunn *et al.*) In the course of the investigation this paper also investigates Kit Fine's semantics for quantified relevance logics, which appears in his appropriately titled 'Semantics for Quantified Relevance Logics' (*Journal of Philosophical Logic* **17**, 27–59, 1988).

1. BACKGROUND AND INTRODUCTION

Dunn (1987) investigates the notion of relevant predication in the context of relevance logic; after suggesting an object-language definition specifying when a formula Ax is of a "*kind that determines relevant properties* (with respect to the variable x)", he leaves us with a technical conjecture relating his object-language characterisation of relevance to various possible meta-linguistic characterisations – in particular, to some characterisations given in Helman (to appear; see also Helman 1977). In this paper we restate this conjecture in various forms and show which forms of the conjecture are true and which are false, thereby linking Dunn's and Helman's research projects.

Dunn's project

Dunn motivates his object-language definition of relevance by considering the following pair of statements:

(1) Socrates is such that he is wise

(2) Reagan is such that Socrates is wise

Despite the surface similarities of these statements – and the classical logician's temptation to treat (2) as a kind of degenerate case of the

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logical structure exhibited by (1) – Dunn takes it that these statements are quite different. These differences are brought out by the "strict analogy" between (1) and (2) and the following two statements

(1′) If anyone is Socrates then he is wise.

(2′) If anyone is Reagan then Socrates is wise.

The argument for (1′) is taken to be the following:

(1^) Socrates is wise. Therefore, if (x = Socrates) then x is wise.

But the corresponding argument for (2′) is a clear case of irrelevance. The argument for (2′) would be:

(2^) Socrates is wise. Therefore, if (x = Reagan) then Socrates is wise.

(1^) is taken to be an instance of the *indiscernibility of identicals*:

(1#) $Fc \vdash (x = c) \rightarrow Fx.$

This is at least plausibly a relevant principle. Meanwhile, (2^) seems to be an instance of the dreaded

(2#) $A \vdash B \rightarrow A.$

Dunn notes that we cannot count as a relevant theorem the indiscernibility of identity in its unrestricted form:

$$A[c/x] \rightarrow .\ (x = c) \rightarrow A.$$

This is because, if A does not contain x free (for instance if $A = p$) or even if A is relevantly equivalent to a formula which does not contain x free (for instance if $A = p \mathbin{\&} (p \vee Fx)$) we obtain

$$p \rightarrow .\ (x = c) \rightarrow p,$$

which is too close to

$$p \rightarrow .\ q \rightarrow p$$

for comfort. Whatever instances of indiscernibility we *do* count as theorems we at least want

$$A[c/x] \rightarrow .\ (x = c) \rightarrow A$$

to hold when A is a *relevant* property of c – whatever properties we take to be *relevant*. This is at least some of the motivation – though by no means all of it; for a fuller story we refer the reader to Dunn (1987) – for the following two object-language definitions:

(1) $(\varrho x A)c =_{\mathrm{df}} \forall x((x = c) \rightarrow A)$.
Here $(\varrho x A)c$ should be read as "c relevantly has the property of being (an x) such that A". The use "$(\varrho x A)$" is here meant to remind us of lambda abstraction and the use of "$(\lambda x A)$".

(2) "A is of a kind to determine relevant properties (with respect to x)" $=_{\mathrm{df}} \forall x(Ax \rightarrow \forall y(y = x \rightarrow Ay))$, where y is not free in A.

The second definition corresponds exactly to an object language definition of "Str(A, x)" that he suggests later in the paper, where he defines

$$\mathrm{Str}(A, x) =_{\mathrm{df}} \forall x(Ax \rightarrow \forall y(y = x \rightarrow Ay))$$

(where y is not free in A).

This later definition occurs in the wake of a discussion of several meta-linguistic grammatical characterisations of what variables a formula is said to interestingly *depend* on, or what variables a formula is *strict* in. His general hypothesis is that these meta-linguistic characterisations somehow pick out the notion of the relevance of a property, as does his object-language definition. His specific technical conjecture is this:

CONJECTURE. Suppose we take as axioms all formulas of the form Str(A, x), where A is an atomic sentence, and x is a variable occurring in A. Then for all formulas A (compound as well), Str(A, x) is a theorem iff A is (provably relevantly equivalent to a formula B which is) strict in x in the . . . refined [sense] due to Helman. [See Helman, to appear, and 1977. See also Dunn, 1987, Note 12.]

Helman's project

Though the concerns of Helman (to appear) are quite different from the concerns of Dunn (1987), Helman becomes interested, as Dunn does, in when a formula interestingly depends on a variable. One of

Helman's concerns is to present an interpretation of $R_{\rightarrow \&}$ which relies on an analogy between the provability of implicational formulas and the definability of functions of certain types. More specifically, Helman formalises the notion of proof from hypothesis (in a number of logics) by a system of typed lambda-abstractions.

First he reminds us that, given a term, t and a variable x, lambda-abstraction allows us to form the function

$$(\lambda xt)$$

which takes on the value $t[a/x]$ for any argument a. Then he turns our attention to the possibility of lambda-abstraction within a *typed* language, where a *type* is any sentence formed from propositional variables, using only the conditional. Formally, he begins with a collection of infinitely many variables of each type. He then defines the collection of λ-terms as follows:

(1) each variable of type A is a λ-term of type A;
(2) if t is a λ-term of type $A \rightarrow B$ and u is a λ-term of type A, then (tu) is a λ-term of type B; here we think of ourselves as applying the *function* t to the *argument* u;
(3) if t is a λ-term of type B, and x is a variable of type A, then λxt is a λ-term of type $A \rightarrow B$; here, the function λxt takes arguments of type A, and spits out terms of type B.

Next, Helman claims that we can regard the terms of our typed language as natural deduction proofs for a pure implicational logic: a term proves the sentence that is its type, with the types of its free variables as undispatched hypotheses. And so, closed terms ought to represent proofs of the theorems of some interesting implicational logic. Fortunately, they do: Helman shows that the set of types of closed λ-terms is the same as the set of theorems of $H_{\rightarrow}$, the pure-implicational part of intuitionistic logic. For this reason, Helman denotes the set of closed λ-terms "$\lambda H_{\rightarrow}$".

Consider the following λ-term, where x is a variable of type B, and where y is a variable of type A:

$$\lambda y \lambda xy.$$

This closed λ-term, which is of type $A \rightarrow (B \rightarrow A)$, represents a proof of $A \rightarrow (B \rightarrow A)$; note that the vacuousness of the abstraction λx

corresponds to the intuition that the formula B is not used in proving A from the hypotheses A and B. Such examples, together with an interest in relevance logic as opposed to intuitionistic logic, motivates the following two definitions:

(1) an abstract, λxt, is *vacuous* when x is not free in t;

(2) $\lambda R_{\rightarrow}$ is the class of λ-terms containing no vacuous abstracts.

The second definition only really makes sense in light of the following theorem:

> A sentence (in the language of $R_{\rightarrow}$) is a theorem of $R_{\rightarrow}$ iff it is the type of some λ-term in $\lambda R_{\rightarrow}$.

The considerations which I have been outlining motivate the thought that, given a function t, and a variable x, t is *relevant* in x (or x is relevant to t) just in case x occurs freely in t (in which case the abstract, λxt, is not vacuous). This thought must, however, be refined when we want to extend our considerations to $R_{\rightarrow\&}$. In order to extend our considerations. Helman enlarges the class of λ-terms to the full class of *terms* by adding the following operators to our language:

(1) pairing: if t and u are terms with the types A and B, respectively, then the pair (t, u) is a term of type $A \& B$;
(2) left and right projections: if t is a term of type $A \& B$, then its left and right projections, pt and qt, are terms of type A and type B, respectively.

It no longer suffices to characterise the functions relevant in x as those functions in which x occurs freely, since, if we did, the function (x, y) would be relevant in x and y, despite the fact that the term $\lambda x \lambda y(x, y)$ is of type $A \rightarrow (B \rightarrow (A \& B))$ for some A and B – and this is *not* the type of a theorem of R.

What Helman eventually does is to characterise *two* interesting senses in which a function depends on a variable; the first sense is related to the logic $U_{\rightarrow\&}$, and the second to $R_{\rightarrow\&}$. (U, as noted in Helman 1977, stems from tinkering with the natural deduction systems for relevance logic in Anderson and Belnap 1975 (§27.2,

p. 348), and takes its name and axiomatic form from Chidgey 1970.) In order to characterise the first sense, Helman defines the set of variables *strict* in a term t, st(t), as follows:

(1)	$\mathrm{st}(x)$	$=$	$\{x\}$;
(2)	$\mathrm{st}((t, u))$	$=$	$\mathrm{st}(t) \cap \mathrm{st}(u)$;
(3)	$\mathrm{st}((tu))$	$=$	$\mathrm{st}(t) \cup \mathrm{st}(u)$;
(4)	$\mathrm{st}(\lambda xt)$	$=$	$\mathrm{st}(t) - \{x\}$.

An abstract λxt is *strict* when x is strict in t.
If we define $\lambda U_{\to\&}$ as the set of closed terms containing no abstracts that are not strict, we find that a sentence in the language of $U_{\to\&}$ is a theorem of $U_{\to\&}$ iff it is the type of a term in $\lambda U_{\to\&}$.

Turning his attention to $R_{\to\&}$, Helman defines x to be *used evenly* in t if x is free in t and no free occurrence of x in t appears in one half of a subterm (u, v) of t which has no free occurrence of x in the other half. Then λxt is *relevant* if x is used evenly in t. Finally if we define $\lambda R_{\to\&}$ to be the set of closed terms containing no abstracts that are not relevant, we find that a sentence in the language of $R_{\to\&}$ is a theorem of $R_{\to\&}$ iff it is the type of a term in $\lambda R_{\to\&}$.

What is of special interest to us is the association of strictness with U, and of even use, or relevance, with R. Helman's characterisations of strictness and relevance are for terms in his language, but they have analogs for formulas in the language of quantified relevance logic. We will define these analogs in §2, and discover an association of our analog of strictness with U, and of our analog of relevance with R.

To begin our project

The logic for which Dunn is interested in carrying out this investigation is quantified relevance logic, with "=" introduced by means of a number of axioms. In order to draw the connections between Dunn's and Helman's projects, we will carry out the investigation for a number of related logics, involving two ways of adding quantification and "=" to R (the relevance logic of Anderson and Belnap 1975) and two ways of adding quantification and "=" to U.

The axioms we will use are

(1) the standard axioms for relevance logic (which can be found in Anderson and Belnap 1975 §27.1.1);
(2) the following quantificational axioms:

(Q1) $(\forall x)(A) \to A[v/x]$

(Q2) $(\forall x)(A \to B) \to (A \to (\forall x)B)$, x not free in A;

(Q3) $(\forall x)(A \vee B) \to (A \vee (\forall x)B)$, x not free in A;

Note: These are only three of the quantificational axioms laid out in Anderson, Belnap and Dunn (to appear, §38.2), where Relevance Logic with quantification is discussed and described. We omit the other two axioms for "∀" given there, since these axioms, namely

$$(\forall x)(A \to B) \to ((\forall x)A \to (\forall x)B), \quad \text{and}$$

$$((\forall x)A \,\&\, (\forall x)B) \to (\forall x)(A \,\&\, B),$$

can be derived in using (Q1), (Q2), (Q3) and the axioms for R, and the rule of generalisation, which we will take as a rule for our systems. These last two axioms are included in Anderson, Belnap and Dunn (to appear, §38.2), since that particular exposition does not include the rule of generalisation; in the exposition given there, the rule is restricted to axioms, and is a rule for deriving axioms. These approaches are equivalent. We omit the axioms for "∃" given in Anderson, Belnap and Dunn (to appear, §38.2) since it will make our exposition simpler to think of "∃" as a defined expression in our language.
(3) The following axioms for "=":

(Refl) $x = x$;

(Sym) $x = y \to y = x$;

(Trans) $x = y \to (y = z \to x = z)$;

the possibly "unstable" substitution axiom:

(Sub) $(Ay \,\&\, x = y) \to Ax$;[1]

and an axiom added for the purposes of this investigation, the strict axiom

(Str) Str(A, x) where A is an atomic sentence and x is a variable occurring in A.

((Str) can be re-written as

$$\forall x(Ax \rightarrow \forall y(y = x \rightarrow Ay))$$

(where y is not free in A).)

(4) the U axiom:

(U) $(A \& \mathbf{t}) \rightarrow (A \rightarrow B) \rightarrow . A \rightarrow B$;

(see Chidgey 1970, and Anderson and Belnap 1975, §27.2, p. 348).

The four systems with which we will be concerned have three rules:

(1) modus ponens;

(2) conjunction; and

(3) the rule of generalisation: From $A[v/x]$ to infer $(\forall x)(A)$, where v is not free for x in $A[v/x]$.

These four systems are the following:

(1) $R^{\forall\exists x=}$ ("Relevance Logic with Quantifiers and '='"): all axioms except (U) and (Sub);

(2) $R^{\forall\exists x=S}$ ("Relevance Logic with Quantifiers and '=' and Substitution"): $R^{\forall\exists x=}$ + (Sub);

(3) $U^{\forall\exists x=}$ ("U with Quantifiers and '='"): $R^{\forall\exists x=}$ + (U);

(4) $U^{\forall\exists x=S}$ ("U with Quantifiers and '=' and Substitution"): $U^{\forall\exists x=}$ + (Sub).

For the purposes of this paper, we will simply refer to these as R, RS, U and US respectively, unless otherwise noted. (The system with which Dunn's paper is concerned is, in this terminology, $R^{\forall\exists x=S}$.) Before attending to our conjecture we point out the following.

THEOREM 1. *$R^{\forall\exists x=S}$ and $R^{\forall\exists x=}$ are conservative extensions of $R^{\forall\exists x}$, which is itself a conservative extension of R.*

Proof. We leave it to the reader to show that $R^{\forall\exists x}$ is a conservative extension of R. The proof that $R^{\forall\exists x=S}$ and $R^{\forall\exists x=}$ are conservative extensions of $R^{\forall\exists x}$ relies on the observation that if a formula A is a theorem of $R^{\forall\exists x=S}$ or $R^{\forall\exists x=}$, then the formula A' is a theorem of $R^{\forall\exists x}$, where A' is the result of replacing in A every instance of the sub-formula $x = y$ with the following: the conjunction of all formulas of the form $\forall z_1, \ldots, \forall z_k\ (Fz_1 \ldots x \ldots z_k \leftrightarrow Fz_1 \ldots y \ldots z_k)$, where F is a predicate (possibly 0-place, in which case $Fz_1 \ldots x \ldots z_k = F$) occurring in A.

THEOREM 2. *$R < RS < U = US$.*

Proof. (1) $U = US$:
A straightforward induction on the length of the formula A will show that every instance of the substitution axiom is a theorem of U.

(2) $RS < U$:
Note that

$$((p \,\&\, \mathbf{t}) \to (p \to q)) \to (p \to q) \quad (*)$$

is a theorem (indeed an axiom) of U, but it is not a theorem of RS. (It is straightforward to show using Urquhart's semi-lattice semantics for $R_{\&\to}$ – see Urquhart 1972 and 1973, as well as Anderon, Belnap and Dunn, to appear, §47 – that (*) is not a theorem of R. So it is not a theorem of $R^{\forall\exists x}$, since $R^{\forall\exists x}$ is a conservative extension of R. And so it is not a theorem of $R^{\forall\exists x=S}$ since $R^{\forall\exists x=S}$ is a conservative extension of $R^{\forall\exists x}$.)

(3) $R < RS$:
Note that

$$(x = y \,\&\, ((Fy \,\&\, p) \to p))) \to ((Fx \,\&\, p) \to p)$$

is a theorem of RS but not a theorem of R (see Appendix 1).

2. RE-STATEMENT OF THE CONJECTURE

We are now almost ready to re-state, in our terms, the conjecture along with some of its relatives. Before we do, we define "inspectably weakly relevant in x", and "inspectably strongly relevant in x" for formulas in the language of $R^{\forall\exists x=}$; these are the analogs of Helman's *strictness* and Helman's *relevance*, respectively.

DEFINITION 3. Inspectably weakly relevant in x:

(1) All atomic formulas with x free are inspectably weakly relevant in x.

(2) if A and B are inspectably weakly relevant in x, then so are $A \& B$, $A \vee B$ and $\neg A$;

(3) if either A or B is inspectably weakly relevant in x, so is $A \rightarrow B$;

(4) if A is inspectably weakly relevant in x and y is a variable distinct from x, then $\forall y(A)$ and $\exists y(A)$ are inspectably weakly relevant in x.

DEFINITION 4. Inspectably strongly relevant in x:

(1) All atomic formulas with x free are inspectably strongly relevant in x;

(2) if A and B are inspectably strongly relevant in x, then so are $A \& B$, $A \vee B$, $\neg A$, and $A \rightarrow B$.

(3) if A is inspectably strongly relevant in x, and x is not free in B then $A \rightarrow B$ and $B \rightarrow A$ are inspectably strongly relevant in x;

(4) if A is inspectably strongly relevant in x, and y is a variable distinct from x, then $\forall y(A)$ and $\exists y(A)$ are inspectably strongly relevant in x.

Now, given any logic L (chosen among R, RS and U), we define:

DEFINITION 5. A formula A is "strongly (weakly) L-relevant in x" iff it is L-equivalent to a formula which is inspectably strongly (weakly) relevant in x.

Note that inspectably strong relevance in x entails inspectably weak relevance in x, and strong L-relevance in x entails weak L-relevance in x. Now, given any logic L, we can simply state the conjecture for strong (weak) L-relevance:

CONJECTURE FOR STRONG (WEAK) L-RELEVANCE:
For every formula A, $\vdash_L \mathrm{Str}(A, x)$ iff A is strongly (weakly) L-relevant in x.

Note that we have here six conjectures: one for each sense in which a formula can be said to be "relevant" in a variable, and for each logic. What we have done is first defined *inspectably* relevance, and then L-relevance, so that we could economically restate Dunn's conjecture that $\mathrm{Str}(A, x)$ is a theorem iff A is *provably relevantly equivalent to a formula B which is* strict (i.e., relevant) in x.

In what follows we will first prove the strong L-conjecture for R, RS, and U. We will then suggest that this proof is too easy and rather uninteresting. This will motivate a slightly amended definition of strong and weak L-relevance, and we will see that this re-definition makes things more interesting. In our concluding remarks, we will suggest that this research *not only* brings together the research projects of Helman and Dunn, but *also* gives us reasons to doubt the "stability" of the substitution axiom (at least in the context of relevance logic), and therefore of any (relevance) system that requires it as an axiom.

3. SOME INITIAL RESULTS

We are now ready to state some initial results.

THEOREM 6. *For $L = R$, RS or U, if A is inspectably strongly relevant in x, then $\vdash_L Str(A, x)$.*

Proof. Induction on the complexity of A.

COROLLARY 7. *If A is strongly L-relevant in x, then $\vdash_L Str(A, x)$.*

THEOREM 8. *If $\vdash_L Str(A, x)$, then A is strongly L-relevant in x.*

Proof. Suppose $\vdash_L \mathrm{Str}(A, x)$. Then, $\vdash_L Ax \rightarrow \forall y(y = x \rightarrow Ay)$. Also note that $\vdash_L \forall y(y = x \rightarrow Ay) \rightarrow Ax$, since $\vdash_L (x = x \rightarrow Ax) \rightarrow Ax$.

So, A is L-equivalent to $\forall y(y = x \rightarrow Ay))$, which is inspectably strongly relevant in X. QED

Corollary 7 and Theorem 8 give us the conjecture for strong L-relevance, for $L = R$, RS, or U. This is where we suggest that things seems almost too easy, and rather uninteresting. The problem is that this last proof almost "cheats". The formula A is indeed shown to be L-equivalent to a formula, say A', which is inspectably strongly relevant in x, but the disturbing thing is that, whether or not A has "=" in its vocabulary, A' does. And "=" is a rather strong piece of vocabulary to bring to bear in showing A to be strongly L-relevant. What would happen if we required A' to have no more vocabulary than A? This question suggests the following redefinition of strong and weak L-relevance.

DEFINITION 9. A formula A is "strongly (weakly) L-relevant in x" iff it is L-equivalent to a formula A' which is inspectably strongly (weakly) relevant in x, and *which has no more vocabulary than A.*

For our purposes, the vocabulary of a formula is taken to be the set of predicates in the formula, including possibly "=". Note that is it still the case that L-relevance entails L-strictness.

Given these new definitions, the conjectures for strong and weak L-relevance take on new meaning, and are not so easily dealt with. (Before we continue, note that Theorem 6 still holds with the new definition of strong L-relevance.)

4. MAIN RESULTS

In this section we will simply summarize the main results to be proven in this paper. As we have already noted, we have two conjectures for each logic; so we have six conjectures in all. Here then are the results, and where they are proven:

(1) Conjecture for strong R-relevance: true; i.e., any formula, A, is strongly R-relevant in x iff $\vdash_R \mathrm{Str}(A, x)$. See §5.

(2) Conjecture for weak R-relevance: false. Although $\vdash_R \mathrm{Str}(A, x)$ entails that A is weakly R-relevant in x (since strong R-relevance entails weak R-relevance) the converse is not true. For example, the formula

$$(Fx \mathbin{\&} p) \rightarrow Fx$$

is weakly R-relevant in x, but it is not true that $\vdash_R \mathrm{Str}((Fx \mathbin{\&} p) \rightarrow Fx, x)$. See §7.

(3) Conjecture for strong U-relevance: true. See §5.

(4) Conjecture for weak U-relevance: true. See §7.

Notice that the last two results entail that, in the logic U, strong and weak relevance are equivalent.

(5) Conjecture for strong RS-relevance: false. It is true that for every formula, A, which is strongly RS-relevant in x we have $\vdash_{RS} \mathrm{Str}(A, x)$. But the formula

$$(Fx \mathbin{\&} p) \rightarrow Fx$$

is *not* strongly RS-relevant in x, although we *do* have $\vdash_{RS} \mathrm{Str}((Fx \mathbin{\&} p) \rightarrow Fx, x)$. See §6.

(6) Conjecture for weak RS-relevance: true. See §8.

5. PROOF OF THE CONJECTURE FOR STRONG L-RELEVANCE, FOR L = R OR U

We now proceed to prove the conjecture for strong L-relevance for $L = R$ or U.

LEMMA 10. *If the language we are working with has a finite, but non-empty set of non-logical predicates of positive degree, then there is a formula, $x \approx y$, with just two free variables, x and y, and with the following properties*:

(1) *$x \approx y$ is inspectably strongly relevant in x and in y*;

(2) *"=" is not in the vocabulary of $x \approx y$*;

(3) *For $L = R$ or U, and for all formulas A, if $\vdash_L A$, then $\vdash_L A'$, where A' the result of replacing every instance of "$u = v$" with "$u \approx v$" where u and v are any two variables.*

Proof. Let k be the greatest degree of all of the non-logical predicates of the language. Add variables $z_1, \ldots, z_k$ to the language.
Let $S = \{Fz_1, \ldots, z_{\deg(F)}$: F is a non-logical predicate of the language, of positive degree.$\}$
Let $T = \{A[x/z_i] \leftrightarrow A[y/z_i]$: $A \in S$ and $i = 1, \ldots, k$, and z_i occurs in A.$\}$
Finally, let $x \approx y =_{\mathrm{df}} \forall z_1, \ldots, \forall z_k\ (\Lambda T)$, where ΛT is taken to be the conjunction of all formulas in T. Clearly, $x \approx y$ has property (2). To see that it has property (1), note that every conjunct in ΛT is relevant in both x and y. The proof that it has property (3) is by induction on the length of proof of A.

LEMMA 11. *If $\vdash_L A$, where $L = R$ or U, and where the vocabulary of A includes at least one non-logical predicate of positive degree, then there is a formula $x \approx_A y$ such that*

(1) *$x \approx_A y$ is inspectably strongly relevant in x and y;*
(2) *the vocabulary of $x \approx_A y$ is a subset of the vocabulary of A, and, furthermore, does not contain "$=$";*
(3) *$\vdash_L A'$, where A' is the same as A except, for all variables u and v, every instance of "$u = v$" has been replaced by "$u \approx_A v$".*

Proof. Let Lang^A be the language whose non-logical predicates are just the non logical predicates used in A. Then, by obvious conservative extension theorems $\vdash_L A$, where "$\vdash_L$" is now taken to be provability using only Lang^A. Clearly, Lang^A has a non-empty and finite set of non-logical predicates of positive degree. So, we can define $x \approx_A y$ as the $x \approx y$ in Lemma 10 (restricting attention to Lang^A). Clearly $x \approx_A y$ will have the desired properties.

LEMMA 12. *For $L = R$, RS, or U, if $\vdash_L x = x \rightarrow A$, then the vocabulary of A contains at least one predicate of strictly positive degree, including possibly "$=$".*

Proof. To prove this lemma, we are going to define a function ϕ from formulas in the language to the set $\{-1, 0, 1\}$.

For atomic formulas:

(1) If F is a predicate of strictly positive degree, including possibly, "$=$", let $\phi(Fx_1, \ldots, x_n) = 1$;
(2) If p is a sentence letter – i.e., a predicate letter of degree 0 – then let $\phi(p) = 0$.

For complex formulas:

(1) $\phi(\neg A) = -\phi(A)$;

(2) $\phi(A \,\&\, B) = \min(\phi(A), \phi(B))$;

(3) $\phi(A \vee B) = \max(\phi(A), \phi(B))$;

(4) $\phi(A \rightarrow B) = \max(-\phi(A), \phi(B))$ if $\phi(A) \leqslant \phi(B)$;

(5) $\phi(A \rightarrow B) = \min(-\phi(A), \phi(B))$ if $\phi(A) > \phi(B)$;

(6) $\phi(\forall xA) = \phi(A)$.

(The idea of proving the theorem this way was a suggestion of Nuel Belnap's. The definition of the function ϕ is taken almost verbatim from the technical appendix of Dunn 1987.) The following facts are straightforward to show:

(1) If A is an axiom of R, RS, or U, then $\phi(A) = 0$ or 1.

(2) If $\phi(A) = 0$ or 1, and if $\phi(A \rightarrow B) = 0$ or 1, then $\phi(B) = 0$ or 1.

(3) If $\phi(A) = 0$ or 1, and if $\phi(B) = 0$ or 1, then $\phi(A \,\&\, B) = 0$ or 1.

(4) If $\phi(A) = 0$ or 1, then $\phi(\forall xA) = 0$ or 1.

Therefore, if A is a theorem of R, RS, of U, then $\phi(A) = 0$ or 1. Now we can also show by induction on the complexity of A, that if A contains no predicates of strictly positive degree, then $\phi(A) = 0$. Therefore, if A contains no predicates of strictly positive degree, $\phi(x = x \rightarrow A) = -1$. Therefore, if A contains no predicates of strictly positive degree, $x = x \rightarrow A$ is neither a theorem of R, RS, nor U. QED

Finally, we obtain

THEOREM 13. *For $L = R$ or U, if $\vdash_L Str(A, x)$ then A is strongly L-relevant in x.*

Proof. If "=" is in the vocabulary of A then the proof follows immediately from the proof of Theorem 8. Assume that "=" is not in the vocabulary of A. Now, $\vdash_L \text{Str}(A, x)$. Therefore $\vdash_L \forall x(Ax \rightarrow \forall y (x = y \rightarrow Ay))$. Therefore $\vdash_L x = x \rightarrow (Ax \rightarrow Ax)$. Therefore by Lemma 12, the vocabulary of A contains at least one non-logical predicate of positive degree; and therefore so does the vocabulary of $\forall x(Ax \rightarrow \forall y(x = y \rightarrow Ay))$. Therefore, by Lemma 11, there is a formula $x \approx y$

(1) that is relevant in x and in y;
(2) the vocabulary of which is contained in the vocabulary of A; and
(3) which is such that

$$\vdash_L \forall x(Ax \rightarrow \forall y(x \approx y \rightarrow Ay)),$$

(since "$=$" is not in the vocabulary of A).

Now, let $A' =_{\text{df}} \forall y(x \approx y \rightarrow Ay)$. Note that A' is inspectably strongly relevant in x. Also note that the vocabulary of A' is contained in the vocabulary of A. Also recall
$\vdash_L \forall x(Ax \rightarrow \forall y(x \approx y \rightarrow Ay))$.
Therefore $\vdash_L \forall x(Ax \rightarrow A'x)$.
Note as well that $\vdash_L \forall y(x \approx y \rightarrow Ay) \rightarrow (x \approx x \rightarrow Ax)$. Also, $\vdash_L (x \approx x \rightarrow Ax) \rightarrow Ax$, since $\vdash_L x \approx x$. So $\vdash_L \forall x(A'x \rightarrow Ax)$.
Therefore, $\vdash_L A \leftrightarrow A'$.
Therefore, A is L-equivalent to A'. QED

Theorem 13 and Corollary 7 together give us the conjecture for strong L-relevance, for $L = U$ or R.

6. WHAT ABOUT RS?

We have proven true the conjecture for strong R-relevance and the conjecture for strong U-relevance. But this proof does not work for the conjecture for strong RS-relevance. Why not? The crucial lemma in the proof of the other two conjectures is Lemma 10, where we prove that, when working with a finite language, we can always replace every occurrence of $x = y$ in every theorem with $x \approx y$. This proof is carried out by induction on the length of the proof of the theorem, A. This induction does *not* go through in RS since there are instances of the substitution axiom – which is not an axiom of R or of U and so which does not need to be worried about in the proofs for R and U – in which we *cannot* replace "$x = y$" with "$x \approx y$"; and so, we cannot even prove the first step of the induction. To see an example of such an instance of the substitution axiom, suppose we are working in a language with just one non-logical predicate, say F, of degree 1, and with just one sentence letter p. Then "$x \approx y$" is

$$\forall z(Fx \leftrightarrow Fy).$$

Note that

$$A \;=\; (x = y \,\&\, ((Fy \,\&\, p) \to p)) \to ((Fx \,\&\, p) \to p)$$

is an axiom of RS, but

$$A' \;=\; (\forall z(Fx \leftrightarrow Fy) \,\&\, ((Fy \,\&\, p) \to p)) \to ((Fx \,\&\, p)$$

is not a theorem of RS (see Appendix 1).

Of course, we have not yet shown that the conjecture for strong RS-relevance is false. But it is, as we will shortly see. First, however, we show some preliminary results involving RS.

LEMMA 14. *For $L = RS$ or U, if $\vdash_L x = x \to (Ax \to Ax)$, then $\vdash_L x = y \to (Ax \to Ay)$. (In which case $\vdash_L Str(A, x)$.)*

Proof. Suppose $\vdash_L x = x \to (Ax \to Ax)$.
We will show $\vdash_L x = y \to (Ax \to Ay)$ by deriving

$$x \;=\; y \to (Ax \to Ay)$$

in a Fitch-style natural deduction (using the rules for R and the axioms for RS, which are all theorems of U). Here goes:

1. $x = x \to (Ax \to Ax)_{\varnothing}$
2. $\quad \underline{x = y_1}$
3. $\quad y = x_1$ (from 2., the Sym axiom, and MP)
4. $\quad x = x_1$ (from 2., 3., Trans, and MP)
5. $\quad (Ax \to Ax)_1$ (from 1., 4., and MP)
6. $\quad y = x \,\&\, (Ax \to Ax)_1$ (from 3., 5., and &-introduction)
7. $\quad (Ax \to Ay)_1$ (from 6., Sub, and MP)
8. $x = y \to (Ax \to Ay)_{\varnothing}$ (from 2., 7., and the rule for conditional proof.)

QED

LEMMA 15. *For $L = RS$ or U, if A is inspectably weakly relevant in x, then* $\vdash_L x = x \rightarrow (Ax \rightarrow Ax)$.

Proof. By induction on the complexity of A.

COROLLARY 16. *For $L = RS$ or U, if A is weakly L-relevant in x, then* $\vdash_L Str(A, x)$.

We now require only one more lemma to show that the conjecture for strong RS-relevance is false.

LEMMA 17. *If "=" is not in the vocabulary of A, then strong (weak) R- and RS-relevance in x are equivalent for A.*

Proof. For the purposes of this proof, we will refer to Relevance Logic with Quantifiers and "=" as $R^{\forall\exists x=}$, and Relevance Logic with Quantifiers, "=" and Substitution as $R^{\forall\exists x=S}$, and Relevance Logic with Quantifiers but without "=" as $R^{\forall\exists x}$. First we recall that both $R^{\forall\exists x=}$ and $R^{\forall\exists x=S}$ are conservative extensions of $R^{\forall\exists x}$ (see Theorem 1).

Therefore, if B is a formula whose vocabulary does not include "=", then:

$$(\vdash_{R^{\forall\exists x=S}} B) \Leftrightarrow (\vdash_{R^{\forall\exists x}} B) \Leftrightarrow (\vdash_{R^{\forall\exists x=}} B). \quad (*)$$

Now, suppose that A is a formula which is strongly (weakly) $R^{\forall\exists x=S}$-relevant in x, and the vocabulary of which does not include "=". Then, there is some formula A', which is inspectably strongly (weakly) relevant in x, and which uses no more vocabulary than A, and which is such that $\vdash_{R^{\forall\exists x=S}} A \leftrightarrow A'$. Since A' uses no more vocabulary than A, and "=" is not in the vocabulary of A, "=" is not in the vocabulary of $A \leftrightarrow A'$. So, by (*),

$$\vdash_{R^{\forall\exists x=}} A \leftrightarrow A'.$$

Therefore, A is strongly (weakly) $R^{\forall\exists x=}$-relevant in x. So, we see that for formulas without "=" in their vocabulary, strong (weak) $R^{\forall\exists x=S}$-relevance entails strong (weak) $R^{\forall\exists x=}$-relevance. The converse is proven similary. QED

We are now ready to show:

THEOREM 18. *The conjecture for strong RS-relevance is false.*

Proof. Let Ax be the formula "$(Fx \,\&\, p) \rightarrow Fx$". Suppose Ax is strongly RS-relevant in x. Then Ax is strongly R-relevant in x, since "$=$" is not in its vocabulary (see Lemma 17). Therefore $\vdash_R \text{Str}(A, x)$.

However, we can show that $\text{Str}(A, x)$ is *not* a theorem of R (see Appendix 1). Therefore Ax is *not* strongly RS-relevant in x. However, Ax *is* inspectably weakly relevant (and therefore weakly RS-relevant). Therefore $\vdash_{RS} \text{Str}(A, x)$. (See Corollary 16.) QED

7. THE CONJECTURES FOR WEAK R-RELEVANCE AND WEAK U-RELEVANCE

We have decided the conjectures for strong L-relevance; we now turn to the conjectures for weak L-relevance. The conjectures for weak R-relevance and U-relevance can be dealt with in a trice.

THEOREM 19. *The conjecture for weak R-relevance is false.*

Proof. Let Ax be the formula "$(Fx \,\&\, p) \rightarrow Fx$". Ax is inspectably weakly relevant in x and therefore weakly R-relevant in x. However, as we saw in the proof of Theorem 18, $\text{Str}(Ax, x)$ is not a theorem of R. QED

THEOREM 20. *The conjecture for weak U-relevance is true.*

Proof. Suppose A is weakly U-relevant in x. Then $\vdash_U \text{Str}(A, x)$, by Corollary 16. On the other hand, suppose $\vdash_U \text{Str}(A, x)$. Then A is strongly U-relevant in x by Theorem 13. Now recall that for any $L = R$, RS, or U, strong L-relevance entails weak L-relevance. Therefore A is weakly U-relevant. QED

8. THE CONJECTURE FOR WEAK RS-RELEVANCE

We now go on to prove the conjecture for weak RS-relevance. Corollary 16 tells us that for every formula A, if A is weakly RS-relevant in x, then $\vdash_{RS} \text{Str}(A, x)$. To prove the converse, we first make the following definition and state an important lemma.

DEFINITION 21. A formula B *contains an interesting identity* iff B has a sub-formula of the form "$u = v$", where u and v are distinct variables. For example, the formula $(\exists x)(x = y)$ contains an interesting identity, while the formula $(\exists x)(\forall y)(x = x \rightarrow (y = y \,\&\, Fxy))$ doesn't.

LEMMA 22. *If a formula B does not contain an interesting identity, then if* $\vdash_{RS} B$ *then* $\vdash_R B$.

Proof. Although this Lemma is intuitively "obvious", its proof takes us sufficiently far afield and is sufficiently involved to be the topic of another paper. In this paper, we will give a sketch of the proof; see Appendix 2.

THEOREM 23. *If* $\vdash_{RS} Str(A, x)$ *then A is weakly RS-relevant.*

Proof. Suppose that $\vdash_{RS}\mathrm{Str}(A, x)$. Now if "=" is in the vocabulary of A, then the result follows from the proof of Theorem 8. So suppose that "=" is not in the vocabulary of A. So we have $\vdash_{RS}(\forall x)(\forall y)(Ax \rightarrow (x = y \rightarrow Ay))$, where y is not free in A.
Therefore $\vdash_{RS} Ax \rightarrow (x = x \rightarrow Ax)$.
Since "=" is not in the vocabulary of A, the formula

$$Ax \rightarrow (x = x \rightarrow Ax)$$

contains no interesting identities.
Therefore $\vdash_R Ax \rightarrow (x = x \rightarrow Ax)$, by Lemma 22.
Now, by Lemma 11 (see §5) there is a formula $x \approx y$ such that

(1) $x \approx y$ is inspectably strongly relevant in x and y;
(2) the vocabulary of $x \approx y$ is a subset of the vocabulary of $Ax \rightarrow (x = x \rightarrow Ax)$, and, furthermore, does not contain "=";
(3) $\vdash_R B$, where B is the same as $Ax \rightarrow (x = x \rightarrow Ax)$ except, for all variables u and v, every instance of "$u = v$" has been replaced by "$u \approx v$".

Since "=" is not in the vocabulary of A, it follows from (2) that the vocabulary of $x \approx y$ is a subset of the vocabulary of A. So if we let $A'x =_{\mathrm{df}} x \approx x \rightarrow Ax$, then A' uses no more vocabulary than A. Furthermore, from (3) we see that

$$\vdash_R Ax \rightarrow (x \approx x \rightarrow Ax).$$

Therefore $\vdash_R Ax \rightarrow A'x$.
Therefore $\vdash_{RS} Ax \rightarrow A'x$. (*)

Also notice that $\vdash_R x \approx x$. (This is clear from the proofs of Lemmas 10 and 11.) From this it follows that

$$\vdash_R (x \approx x \to Ax) \to Ax$$

Therefore $\vdash_R A'x \to Ax$.
Therefore $\vdash_{RS} A'x \to Ax$. (**)
From (*) and (**) we see that A' is *RS*-equivalent to A. Also, A' is inspectably weakly relevant in x, since $x \approx x$ is inspectably strongly (and therefore weakly) relevant in x. Therefore A is weakly *RS*-relevant in x. QED

And this is enough to prove the conjecture for weak *RS*-relevance.

9. CONCLUDING REMARKS

Now that we have shown all of our promised conjectures, we draw attention to a couple of points. First notice that the broadest grammatical characterisation of relevance is strong relevance for *R*, and weak relevance for *U* (and for *RS*). These were our analogs to Helman's *relevance* and Helman's *strictness*, respectively. Recall that Helman's relevance plays an important role in characterising theorems in *R*; our grammatical analog to it plays an important role in characterising relevant formulas in *R*. Recall that Helman's strictness plays an important role in characterising theorems in *U*; our grammatical analog to it plays an important role in characterising relevant formulas in *U*. Whether there is a deeper association between Dunn's ideas and Helman's is an issue for further research.

We also draw attention to the logic for which Dunn originally set out his conjecture, *RS*. We suggest that there is reason to doubt the "stability" of that logic, and, therefore, of the substitution axiom itself. We begin our discussion by asking, what is the meaning of "$x = y$", and what, as a result, are the most suitable axioms which with to introduce "$=$" into a relevance logic. The stock answer to the first question is, of course, that x and y refer to the same entity. We are uncomfortable with this answer, especially in light of the existing semantics for quantified relevance logic; see Fine (1988) – we discuss this semantics in some depth in Appendix 2. (The problem is that it is quite possible, given a model, and given any two of the model's individuals, i_1 and i_2, for one point of the model to validate $i_1 = i_2$, and another to validate $i_1 \neq i_2$; and so it is unclear whether i_1 and i_2 are in fact the same object. Furthermore, it is difficult to see how one might amend the semantics so that whether or not a given point validates $c_1 = c_2$ depends on the individuals assigned to c_1 and c_2.) Another answer to the question of the meaning of "=" is that "$x = y$" simply "says" that "x" and "y" are interchangeable in all – or maybe just some – contexts. Another way of putting this is that "$x = y$" is just a short-hand for the conjunction of all formulas of the form "$Ax \leftrightarrow Ay$". But this isn't very satisfying, since we would then have

$$x = y \rightarrow (p \rightarrow p) \quad (*)$$

which offends our relevance intuitions. And what is it about (*) that offends these intuitions? It is that p is not, in any intuitive sense, relevant in either x or y.

This suggests an understanding of "$x = y$" which is more in keeping with our relevance intuitions: "$x = y$" is a shorthand for the conjunction of all formulas of the form "$Ax \leftrightarrow Ay$", *where A is "relevant" in x*. Here we still have an intuitive notion of relevance in mind – a notion for which we have two rival grammatical characterisations: inspectably weak relevance and inspectably strong relevance.

If this is our intuitive understanding of "$x = y$", with what axioms are we best to introduce "=" into quantified relevance logic? The following three should be relatively uncontroversial.

(Refl) $x = x$

(Sym) $x = y \rightarrow y = x$;

(Trans) $x = y \rightarrow (y = z \rightarrow x = z)$.

In addition to this we should introduce at least one axiom which embodies our understanding of "$x = y$". The following suggests itself:

$$x = y \rightarrow (Ax \rightarrow Ay), \quad \text{where } y \text{ is not free in } A \text{ and where } A \text{ is "relevant" in } x.$$

Substituting our two grammatical characterisations of relevance in for "relevant" yields two rival logics:

(1) $R^{\forall\exists x=}$ with

(Strong) $x = y \rightarrow (Ax \rightarrow Ay)$, where y is not free in A and where A is inspectably strongly relevant in x;

and

(2) $R^{\forall\exists x=}$ with

(Weak) $x = y \rightarrow (Ax \rightarrow Ay)$, where y is not free in A and where A is inspectably weakly relevant in x.

We will call these logics R^{strong} and R^{weak}, respectively.

The possibility that *strong* relevance is the best characterisation of relevance is motivated by a preference for *R* over *RS*; and the possibility that *weak* relevance is the best characterisation of relevance is motivated by a preference for *RS* over *R*. If our intuition is right, and if an axiom like

$x = y \rightarrow (Ax \rightarrow Ay)$, where y is not free in A and where A is "relevant" in x,

is supposed to embody our understanding of "$x = y$" then we should expect *R* – which tells us that "relevance" is strong relevance – to be equivalent to R^{strong}, and *RS* – which tells us that "relevance" is weak relevance – to be equivalent to R^{weak}.

We note with some happiness that *R* is in fact equivalent to R^{strong}. However, we suspect that R^{weak} is *strictly weaker* than *RS*, and *this* is the source of our suspicion that *RS* is in some sense unstable. After all, it seems that *RS* tells us that relevance is weak relevance and hence that "$x = y$" is shorthand for the conjunction of all formulas of the form "$Ax \leftrightarrow Ay$", where A is *weakly* relevant. And it seems that R^{weak} provides us with the *same* understanding of "=". Doesn't it seem then that there's something amiss if R^{weak} is strictly weaker than *RS*? Unfortunately, we do not yet have a proof of our conjecture to that effect (or a disproof of it) and so we leave the following unproved conjecture:

R^{weak} is strictly weaker than *RS*.

APPENDIX 1. CONSTANT DOMAIN SEMI-LATTICE SEMANTICS

The most natural move to make in an attempt to come up with a semantics for $R^{\forall\exists x}$ is to take Routley and Meyer's relational semantics for R – as provided in Routley and Meyer (1973), and Routley, Plumwood, Meyer and Brady (1982), and as discussed in Anderson, Belnap and Dunn (to appear, §48) – and to add a domain D of objects, and to say (intuitively) of a point, a, that $a \models \forall x Ax$ just in case $a \models Au$, for every object "u" in the domain. (This is spelled out formally in Anderson, Belnap and Dunn (to appear, §48.9).) Unfortunately, as is shown in Fine 1986, $R^{\forall\exists x}$ is not complete for such a semantics. However, if we apply the same idea to Urquhart's semi-lattice semantics for $R_{\to\&}$ – which can be found in Urquhart (1972 and 1973), and in Anderson, Belnap and Dunn (to appear, §47) – we *do* obtain a semantics for which $R^{\forall\exists x}_{\to\&}$ is sound and complete. We spell it out formally with the following definition.

DEFINITION 24. A quintuple $M = \langle K, 0, \cup, D, V\rangle$ is a "constant domain semi-lattice model for $R^{\forall\exists x}_{\to\&}$", iff it satisfies the following:

(1) $\langle K, 0, \cup\rangle$ is a semi-lattice.
(2) D is a non-empty set.
(3) We can allow actual elements of this domain enter in to formulas (thought of as finite sequences) in place of variables. A D-sentence is then such a formula with no free variables. Then we stipulate that V is a function from $K \times \{Fd_1 \ldots d_n\colon F$ is an n-place predicate term and $d_1, \ldots, d_n \in D\}$ to {true, false}. Note that we allow 0-place predicate letters; if F is 0-place we will say that $Fd_1 \ldots d_n = F$.

DEFINITION 25. For every model $M = \langle K, 0, \cup, D, V\rangle$, and every $a \in K$, and every D-sentence A, we define $a \models_M A$ as follows:

(1) For atomic D-sentences, $Fd_1 \ldots d_n$, $a \vDash_M Fd_1 \ldots d_n$ iff $V(Fd_1 \ldots d_n) = \text{true}$.
(2) $a \vDash_M A \,\&\, B$ iff $a \vDash_M A$ and $A \vDash_M B$.
(3) $a \vDash_M A \rightarrow B$ iff $(\forall b \in K)(b \vDash_M A \Rightarrow b \cup a \vDash_M B)$.
(4) $a \vDash_M (\forall x)(A)$ iff $(\forall d \in D)(a \vDash A[d/x])$.
(5) $a \vDash_M (\exists x)(A)$ iff $(\exists d \in D)(a \vDash A[d/x])$.

If A is a formula with free variables in it, we say that $a \vDash_M A$, just in case $a \vDash_M (\forall x_1) \cdots (\forall x_n)A$, where $x_1 \ldots x_n$ are the free variables in A.

We say that $M \vDash A$ iff $0 \vDash_M A$.

We say that a formula A is valid (with respect to a constant domain semi-lattice semantics) iff $(\forall M)(M \vDash A)$.

We now state, without proof, the following theorem.

THEOREM 26. *A formula A is valid iff it is a theorem of* $R^{\forall\exists x}_{\rightarrow\&}$.

If we add the following two clauses to Definition 25, we obtain a semantics for which $R^{\forall\exists x}$ is not complete, but sound:

(6) $a \vDash_M A \vee B$ iff $a \vDash_M A$ or $a \vDash_M B$;
(7) $a \vDash_M \neg A$ iff it is not the case that $a \vDash_M A$.

THEOREM 27. *If A is a theorem of* $R^{\forall\exists x}$, *then A is valid with respect to a constant domain semi-lattice semantics.*

Proof. by induction on the length of proof of A.

Indeed, we can state a more general theorem.

THEOREM 28. *Suppose that L is an axiomatic extension of* $R^{\forall\exists x}$. *Suppose M is a constant domain semi-lattice model, such that, for every axiom A of L,* $M \vDash A$. *Then, for every theorem B of L,* $M \vDash B$.

Proof. by induction on the length of proof, in L, of B.

Using this semantics, we can show that

(i) $(x = y \,\&\, ((Fy \,\&\, p) \rightarrow p))) \rightarrow ((Fx \,\&\, p) \rightarrow p)$, and

(ii) $(x = y \rightarrow (((Fx \,\&\, p) \rightarrow Fx) \rightarrow ((Fy \,\&\, p) \rightarrow Fy))$, and are not theorems of $R^{\forall\exists x=}$, and that

(iii) $(\forall z(Fx \leftrightarrow Fy) \,\&\, ((Fy \,\&\, p) \to p)) \to ((Fx \,\&\, p) \to p)$

is not a theorem of $R^{\forall\exists x=S}$.

APPENDIX 2. A SKETCH OF THE PROOF OF LEMMA 22, AND MORE ON SEMANTICS

Recall the statement of Lemma 22.

LEMMA 22. *If a formula B does not contain an interesting identity, then if* $\vdash_{RS} B$ *then* $\vdash_R B$.

This lemma, which we prove semantically, would be quite easy to prove if we had a constant domain semantics for $R^{\forall\exists x}$, as we shall shortly see. Unfortunately, we don't. However, we do have a constant domain semi-lattice semantics for $R^{\forall\exists x}_{\to\&}$, which we will use to prove the following theorem, which is a version of Lemma 22, restricted to formulas in the language of $R^{\forall\exists x=}_{\to\&}$.

THEOREM 29. *If a formula B, in the language of* $R^{\forall\exists x=}_{\&}$, *does not contain an interesting identity, then if* $\vdash_{R^{\forall\exists x=S}_{\to\&}} B$, *then*

$$\vdash_{R^{\forall\exists x=}_{\to\&}} B.$$

Proof. Suppose $\neg(\vdash_{R^{\forall\exists x=}_{\to\&}} B)$. Then for some constant domain semi-lattice model $M = \forall K, 0, \cup, D, V\exists$, such that M validates every axiom of $R^{\forall\exists x=}_{\to\&}$, we have $\neg(M \vDash B)$. We define a new model M' thus:

$$M' = \langle K, 0, \cup, D, V' \rangle$$

where

$$V'(a, Fd_1 \ldots d_n) = \text{false if } F \text{ is "=" and } d_1 \neq d_2,$$
$$= V(a, Fd_1 \ldots d_n) \text{ otherwise.}$$

Notice that, as far as atomic formulas are concerned, M' agrees with M at every point of K, except that at no point does M' validate $d_1 = d_2$, where d_1 and d_2 are distinct. As it turns out, M' validates every axiom, not only of $R^{\forall\exists x=}_{\rightarrow\&}$, but also of $R^{\forall\exists x=S}_{\rightarrow\&}$. Furthermore, it can be shown by induction on the complexity of C that for every formula C, and for every $a \in K$, if C does not contain an interesting identity, then $a \models C$ iff $a \models' C$. And so, if C is a formula which does not contain any interesting identities, $M \models C$ iff $M' \models C$.

And so $\neg(M' \models B)$. And so, since M' is an $R^{\forall\exists x=S}_{\rightarrow\&}$-model, $\neg(\vdash_{R^{\forall\exists x=S}_{\rightarrow\&}} B)$. QED

Unfortunately, as I have already mentioned, such a proof is not available to us if we use the currently known full semantics for $R^{\forall\exists x}$, which is provided by Fine (1988),[2] and which is not a constant domain semantics. Shortly, we shall see what the problem is.

In SQRL, Fine defines a *possible model* **A** to be an 11-tuple, $(T, S, \mathbf{D}, l, \cdot, \uparrow, \downarrow, \rightarrow, \phi)$, where:

(i) T (theories) is a set;
(ii) S (saturated theories) is a subset of T;
(iii) $\mathbf{D}$ (relative domain) is a function from T into sets (he uses D (domains) for Rg($\mathbf{D}$), I (individuals) for $\cup D$, and $\approx$ (domain equivalence) for $\{\langle t, u\rangle \in T \times T: \mathbf{D}_u = \mathbf{D}_t\}$);
(iv) l (logics) is a function from D into T, with $\mathbf{D}_{l(\alpha)} = \alpha$ for all $\alpha \in D$;
(v) $\cdot$ (fusion) is a partial function from $T \times T$ into T, which is defined, for arguments t and u, only when $t \approx u$, and which then takes a value $v \approx t$;
(vi) $-$ (complementation) is a unary operation on S for which $-a \approx a$ whenever $a \in S$;
(vii) $\geqslant$ is a binary relation on T, holding between t and u from T only when $t \approx u$;
(viii) $\uparrow$ (the up operator) is a partial function from $T \times D$ into T. It is defined, for given arguments t and α, only when $\alpha \supseteq \mathbf{D}_t$, and its value is a u for which $\mathbf{D}_u = \alpha$;
(ix) $\downarrow$ (the down operator) is a partial function from $T \times D$ into T. It is defined, for given arguments t and α, only when $\mathbf{D}_t \supseteq \alpha$, and its value is a u for which $\mathbf{D}_u = \alpha$;
(x) $\rightarrow$ (the across operator) is a partial function from $T \times \{\{i, j\}: i \text{ and } j \text{ are distinct elements of } I\}$ into T – it is

defined on T and $\{i, j\}$ just in case the distinct $i, j \in \mathbf{D}_t$, and its value is then a $u \approx t$;

(xi) ϕ (valuation) is a relation that holds between a theory $t \in T$ and an $n + 1$-tuple $\langle R, i_1, \ldots, i_n \rangle$ consisting of an n-place predicate R and individuals $i_1, \ldots, i_n$ from $\mathbf{D}_t$.

Fine often writes – and we will often write:

(1) l for $l(\alpha)$;

(2) $t\uparrow^{\beta}$ or t^{β} or simply $t\uparrow$ for $\uparrow(t, \beta)$;

(3) $t\downarrow_{\beta}$ or t_{β} or simply $t\downarrow$ for $\downarrow(t, \beta)$;

(4) ${}^{i,j}\vec{t}$ or simply $\vec{t}$ for $\rightarrow(t, \{i, j\})$.

(5) tu for $t \bullet u$.

Fine uses – and we will use – the following conventions:

(1) $t, s, u, \ldots$ for members of T;
(2) $a, b, c, \ldots$ for members of S;
(3) $\alpha, \beta, \gamma, \ldots$ for members of D;
(4) $i, j, k, \ldots$ for members of $\imath$.

Fine defines a possible model **A** to be an *actual* model if it satisfies some thirty-five conditions he lists in the next few pages. Now, Fine at first shows soundness and completeness with respect to actual models for a certain minimal logic, which he calls *BQ*. But he also shows that if we require actual models to satisfy an additional nine conditions listed at the end of his paper, we obtain a sound and complete semantics for $R^{\forall\exists x}$. In light of this, we will define a *Fine-model* to be an actual model which satisfies these last nine conditions. Henceforth this paper will be concerned with Fine-models, rather than with what Fine calls actual models.

Fine defines truth with respect to a Fine-model in the following way; see SQRL, §2. First, given a Fine-model **A** and a language L, we add the individuals, $\imath$, to the language (as constants) to obtain the enlarged language L^+. Given a formula A of L^+, let $\imath(A) = \{i \in \imath\colon i$ occurs (as a name) in $A\}$. We say that A is *defined at* the point t of **A** if $\mathbf{D}_t \supseteq \imath(A)$; and we let $S(t) = \{A\colon A$ is a sentence of L^+ which is defined at $t\}$. Finally, relative to the Fine-model **A**, the relation $\models$ of truth – we will call this "Fine-truth" – is to hold between a theory t of **A** and a sentence A of $S(t)$. It is defined by the following clauses:

(i) $t \models Ri_1 \ldots i_n$ iff $\phi(t, \langle R, i_1, \ldots, i_n \rangle)$, for R a n-place predicate;
(ii) $t \models B \,\&\, C$ iff $t \models B$ and $t \models C$;
(iii) $t \models \neg B$ iff $(\forall a \geqslant t)(\text{not } -a \models B)$;
(iv) $t \models B \rightarrow C$ iff $(\forall u \approx t)(u \models B \Rightarrow tu \models C)$;
(v) $t \models \forall x Bx$ iff $(\exists t\uparrow)(\exists i \in \mathbf{D}_{t\uparrow} \backslash \mathbf{D}_t)(t\uparrow \models Bi)$.

We will say that a sentence A is *Fine-true* in a Fine-model $\mathbf{A}$ – in symbols, $\mathbf{A} \models A$ – if $l \models A$ for every l at which A is defined. We will say that a formula $Av_1 \ldots v_n$ with free variables $v_1, \ldots, v_n$ is *Fine–true* in every Fine-model $\mathbf{A}$ if, for any individuals $i_1, \ldots, i_n \in I$, and for any l at which $Ai_1 \ldots i_n$ is defined, $l \models Ai_1 \ldots i_n$, and that a formula of the given language L is *Fine-valid* – $\models A$ – if it is Fine-true in every Fine-model.

Not too surprisingly, Fine proves the following theorem.

THEOREM 30. For any formula A, $\vdash_{R^{\forall\exists x}} A$ iff $\models A$.

This is an immediate consequence of his Theorem 20 in SQRL.

Now we will see why a proof such as the proof of Theorem 29 is not available using Fine's semantics. The main step in the proof was this: given a model, M, we defined a model M' which was just like M except that it did not validate $d_1 = d_2$ at any point, when d_1 and d_2 were distinct members of D. The natural way to do the same proof with Fine's semantics would be, given a model $\mathbf{A}$, to define a model $\mathbf{A}'$ which is exactly like $\mathbf{A}$ except that $\phi'(t, \langle =, i_1, i_2 \rangle)$ is false whenever i_1 is distinct from i_2. Unfortunately, the resulting possible model $\mathbf{A}'$ *would not be a Fine-model*, since it would not satisfy Fine's condition $V(ix)$, which is the following; (see SQRL §1):

$$\phi(t, R, i_1, \ldots, i_n) \Rightarrow \phi(\vec{t}, R, i'_1, \ldots, i'_n), \text{ where } \rightarrow \text{ is } j, k\!\rightarrow \text{ for distinct } j, k \in \mathbf{D}_t \text{ and, for } p = 1, 2, \ldots, n, \; i'_p = i_p \text{ if } i_p \neq j \text{ or } k, \text{ and } i'_p = j \text{ or } k \text{ if } i_p = j \text{ or } k.$$

What this condition says is that if a point t validates $Ri_1 \ldots i_n$, then the point ${}^{j,k}\vec{t}$ validates anything of the form $Ri'_1 \ldots i'_n$ where $Ri'_1 \ldots i'_n$ is just $Ri_1 \ldots i_n$ with as any number of j's in it replaced by k's and any number of k's in it replaced by j's. Now, this condition throws a spanner in the works for the following reasons: if $\mathbf{A}'$ is to be a model for $R^{\forall\exists x=S}$, then we must have $l(\alpha) \vDash i = i$, where $i \in \alpha$ and where α has at least one other member, say j. So we must have $\phi'(l(\alpha), \langle =, i, i \rangle)$. Therefore, if **A**′ is to satisfy Fine's condition $V(ix)$, we must have $\phi'({}^{i,j}\vec{l}(\alpha), \langle =, i, j \rangle)$. But we wanted to define ϕ' so that it was *not* the case that $\phi'({}^{i,j}\vec{l}(\alpha), \langle =, i, j \rangle)$. And this is the problem with Fine's condition $V(ix)$.

How do we deal with this problem? It would be nice to just delete this condition, but, unfortunately, the new semantics would not be complete.

What we actually do is weaken the condition, and change the definition of truth. Formally, we proceed as follows.

DEFINITION 31. A possible model $\mathbf{A} = (T, S, \mathbf{D}, l, \cdot, \uparrow, \downarrow, \rightarrow, \phi)$ is a **-model* iff it satisfies

(1) all of the conditions satisfied by Fine-models, save, perhaps, Fine's condition $V(ix)$; and
(2) the following condition, which we will call condition $V(ix)^*$:

> $\phi(t, R, i_1, \ldots, i_n) \Rightarrow \phi(\vec{t}, R, i'_1, \ldots, i'_n)$, where $\rightarrow$ is $j, k\rightarrow$ for distinct $j, k \in \mathbf{D}_t$ and where $i'_1, \ldots, i'_n$ is just like $i_1, \ldots, i_n$ except that either all occurrences of j in $i_1, \ldots, i_n$ have been replaced by k, or vice versa.

Notice that every Fine-model is a *-model, by that not every *-model is a Fine model, since the condition $V(ix)^*$ is strictly weaker than Fine's condition $V(ix)$.

To define truth for a *-model, **A**, we proceed very much as for a Fine-model. The important difference is in the inductive definition of

$$t \vDash \forall x Bx$$

First, given a *-model **A** and a language L, we add the individuals, I, to the language (as constants) to obtain the enlarged language L^+. Given a formula A of L^+, let $I(A) = \{i \in I\colon i \text{ occurs (as a name) in } A\}$. We say that A is *defined at* the point t of **A** if $\mathbf{D}_t \supseteq I(A)$; and we let $S(t) = \{A\colon A \text{ is a sentence of } L^+ \text{ which is defined at } t\}$. Finally, relative to a *-model **A**, the relation $\vDash^*$ of *-truth is to hold between a theory t of **A** and a sentence A of $S(t)$. It is defined by the following clauses:

(i) $t \models^* Ri_1 \ldots i_n$ iff $\phi(t, \langle R, i_1, \ldots, i_n \rangle)$, for R a n-place predicate;
(ii) $t \models^* B \,\&\, C$ iff $t \models B$ and $t \models C$;
(iii) $t \models^* \neg B$ iff $(\forall a \geqslant t)(\text{not } -a \models B)$;
(iv) $t \models^* B \rightarrow C$ iff $(\forall u \approx t)(u \models B \Rightarrow tu \models C)$;
(v)* $t \models^* \forall x Bx$ iff the following two conditions are satisfied:
(int) $(\exists t\uparrow)(\exists i \in \mathbf{D}_{t\uparrow} \backslash \mathbf{D}_t)(t\uparrow \models^* Bi)$, and
(ext) $(\forall i \in \mathbf{D}_t)(t \models^* Bi)$.[3]

We will say that a sentence A is **-true* in a *-model **A**, or that **A** *-validates A – in symbols, $\mathbf{A} \models^* A$ – if $l \models^* A$ for every l at which A is defined. We will say that a formula $Av_1 \ldots v_n$ with free variables $v_1, \ldots, v_n$ is **-true* in a *-model **A** if, for any individuals $i_1, \ldots, i_n \in I$, and for any l at which $Ai_1 \ldots i_n$ is defined, $l \models^* Ai_1 \ldots i_n$, and that a formula of the given language L is **-valid* – $\models^* A$ – if it is *-true in every *-model.

Notice the diffrence between the definitions of Fine-truth and of *-truth. Fine's intuitive gloss on the way he defines $t \models \forall x Bx$ is that $\forall x Bx$ is true just in case Bi is true for some *arbitrary* object i. Fine takes it that the requirement that i not be in $\mathbf{D}_t$ guarantees that it is indeed arbitrary. We can think of Fine's condition – i.e., that $(\exists t\uparrow)(\exists i \in \mathbf{D}_{t\uparrow} \backslash \mathbf{D}_t)(t\uparrow \models Bi)$ – for the truth of $\forall x Bx$ at t, as an *intensional* condition on the pair (t, B). We have added the condition $(\forall i \in \mathbf{D}_t)(t \models^* Bi)$, i.e., that Bi also be true for every *actual* object, i. This condition can be thought of as an *extensional* condition on the pair (t, B). This explains the use of "(int)" and "(ext)" to name these conditions.[4]

We are now ready to state the following important theorem.

THEOREM 32. *For any formula* A, $\vdash_{R^{\forall\exists x}} A$ *iff* $\models^* A$.

Proof. ($\Leftarrow$): Suppose that it is not the case that $\vdash_{R^{\forall\exists x}} A$. What we need is a *-model, **A**, such that $\neg(\mathbf{A} \models^* A)$. Let **A** be the "canonical model for $R^{\forall\exists x}$, $\mathbf{A}_{R^{\forall\exists x}}$" as defined by Fine in SQRL. Fine shows that **A** is a Fine-model. So $\neg(\mathbf{A} \models A)$. As we have already noted, every Fine-model is a *-model. So **A** is a *-model. Now we point out that for *any* Fine-model, $\models^*$ and $\models$ are the same; see Lemma 33. Therefore $\neg(\mathbf{A} \models^* B)$.

($\Rightarrow$): This is the detailed part, which we omit. The proof is somewhat similar to Fine's completeness proof in SQRL.

LEMMA 33. *If* $\mathbf{A} = (T, S, \mathbf{D}, l, \bullet, \uparrow, \downarrow, \rightarrow, \phi)$ *is a Fine-model, then for every* $t \in T$ *and for every formula* A *of the language* L^+ *(where* L *is the language we are working in),*

$$t \vDash^* A \text{ iff } t \vDash A.$$

Proof. We note that Fine shows that, for every $t \in T$, and every formula B, if $t \vDash (\forall x)(Bx)$, then $(\forall i \in \mathbf{D}_t)(t \vDash Bi)$; see Corollary 6 of SQRL, §2. Using this, the result can be proven by induction on the complexity of A.

We need one more theorem before we can prove Lemma 22.

THEOREM 34. *Suppose* L *is an axiomatic extension of* $R^{\forall\exists x}$, *and that* $\mathbf{A}$ *is a *-model, and that every axiom of* L *is *-true in* $\mathbf{A}$. *Then every theorem of* L *is *-true in* $\mathbf{A}$.

Proof. Again, we omit the proof.

LEMMA 22. *If a formula* B *does not contain an interesting identity, then if* $\vdash_{R^{\forall\exists x=S}} B$ *then* $\vdash_{R^{\forall\exists x=}} B$.

Proof. Suppose $\neg(\vdash_{R^{\forall\exists x=}} B)$. Then for some *-model, $\mathbf{A} = (T, S, \mathbf{D}, |, \bullet, \uparrow, \downarrow, \rightarrow, \phi)$, such that $\mathbf{A}$ *-validates every axiom of $R^{\forall\exists x=}$, we have $\neg(\mathbf{A} \vDash^* B)$. We define a new model $\mathbf{A}'$ thus:

$$\mathbf{A}' = (T, S, \mathbf{D}, |, \bullet, \uparrow, \downarrow, \rightarrow, \phi')$$

where, for any $t \in T$, and any n-place predicate letter R, and any individuals $i_1, \ldots, i_n$,

$\phi'(t, \langle R, i_1, \ldots, i_n\rangle)$ iff (1) $\phi'(t, \langle R, i_1, \ldots, i_n\rangle)$, and
(2) $Ri_1 \ldots i_n$ is not of the form "$i = j$" where i and j are distinct individuals.

Note the following:

(1) as far as atomic formulas are concerned, $\mathbf{A}'$ agrees with $\mathbf{A}$ at every $t \in T$, except that at no t does $\mathbf{A}'$ *-validate $i_1 = i_2$, where i_1 and i_2 are distinct individuals;

(2) $\mathbf{A}'$ is a *-model (though it might not be a Fine-model);

(3) $\mathbf{A}'$ *-validates all of the axioms of $R^{\forall\exists x=S}$.

(4) it can be shown by induction on the complexity of C that for every formula C, and for every $t \in T$, if C does not contain an interesting identity, then $t \models^{*\prime} C$ iff $t \models^{*} C$. And so, if C is a formula which does not contain any interesting identities, $\mathbf{A} \models^{*} C$ iff $\mathbf{A}' \models^{*} C$.

And so $\neg(\mathbf{A}' \models B)$. And so, since $\mathbf{A}'$ is a *-model, $\neg(\vdash_{R^{\forall\exists x=s}} B)$.

QED

NOTES

* Many thanks go to Nuel Belnap who suggested this research topic, and whose comments and inspiration were most appreciated.

1 The sense in which this axiom might be "unstable" is explicated in the concluding remarks.

2 Henceforth we will call this paper "SQRL" – for "Semantics for Quantified Relevance Logic."

3 The reasons for calling these conditions "(int)" and "(ext)" will soon become apparent.

4 The thought of these as intensional and extensional conditions was given to me, in conversation, by Michael Dunn.

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