Strong completeness of S4 wrt the real line

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Abstract

In the topological semantics for modal logic, S4 is well-known to be complete wrt the rational line and wrt the real line: these are special cases of S4's completeness wrt any dense-in-itself metric space. The construction used to prove completeness can be slightly amended to show that S4 is not only complete, but strongly complete, wrt the rational line. But no similarly easy amendment is available for the real line. In another paper, we prove that S4 is, in fact, strongly complete wrt any dense-in-itself metric space, the real line being a particular case. In the current paper, we give a proof of strong completeness tailored to the particular case of the real line: we believe that it usefully clarifies matters to work through a particular and important example. We proceed in two steps: first, we show that S4 is strongly complete wrt the space of finite and infinite binary sequences, equipped with the Scott topology; and then we show that there is an interior map from the real line to this space.

Keywords: modal logic, topological semantics, strong completeness, real line.

In the topological semantics for modal logic ([5, 6, 8]), S4 is well-known to be complete wrt the class of all topological spaces, as well as wrt a number of particular topological spaces, notably the rational line, $\mathbb{Q}$, and the real line, $\mathbb{R}$. The resuts for $\mathbb{Q}$ and $\mathbb{R}$ are special cases of the fact that S4 is complete wrt any dense-in-itself metric space: see [8], Theorem XI, 9.1, which is derived from [5, 6]. It is customary to strengthen completeness to *strong* completeness, i.e., the claim that any consistent set of formulas is satisfiable at some point in the space in question. As long as the language is countable,

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the construction used to prove completeness can be slightly amended to show that S4 is not only complete, but strongly complete, wrt $\mathbb{Q}$. But no similarly easy amendment is available for $\mathbb{R}$ or for $\mathcal{C}$: until [4], the questions of strong completeness wrt $\mathbb{R}$ and wrt $\mathcal{C}$ were open. In [4] we prove that S4 is strongly complete wrt to *any* dense-in-itself metric space – and therefore wrt $\mathbb{R}$. In the current paper, we give a proof of strong completeness tailored to the particular case of the real line: we believe that it usefully clarifies matters to work through a particular and important example.

Completeness wrt any given dense-in-itself metric space X is typically proved by showing that any finite rooted reflexive transitive Kripke frame is the image of an interior map from X. When $X = \mathbb{Q}$, strengthening completeness to *strong* completeness is accomplished by slightly amending the construction to show that any *countable* rooted reflexive transitive Kripke frame is the image of an interior map from $\mathbb{Q}$. But this strategy is not generalizable: because of the Baire Category Theorem, the countable rooted reflexive transitive Kripke frame $\langle \mathbb{N} \leq \rangle$, for example, is *not* the image of any interior map from $\mathbb{R}$ (I owe this observation to Guram Bezhanishvili, David Gabelaia, and Valentin Shehtman): see [4], Section 3, for details.

To show that S4 is strongly complete wrt $\mathbb{R}$, we proceed in two steps. First we show that S4 is strongly complete wrt the space $2^{\leq\omega}$ of finite and infinite binary sequences, equipped with the Scott topology: see Section 2, in particular Lemma 2.4.[1] Then we show that there is an interior map from $\mathbb{R}$ to $2^{\leq\omega}$: see Section 3, in particular Lemma 3.1. Thus S4 is strongly complete wrt $\mathbb{R}$.

1 Basics

We begin by fixing notation and terminology. We assume a propositional language with a countable set PV of propositional variables; standard Boolean connectives &, $\vee$ and $\neg$; and one modal operator, $\Box$. A finite set of formulas is *consistent* iff either it is empty or the negation of the conjunction of the formulas in it is not a theorem of S4; and an infinite set of formulas is *consistent* iff every finite subset is consistent.

A *Kripke frame* is an ordered pair $\langle X, R\rangle$, where X is a nonempty set and $R \subseteq X \times X$. We will somewhat imprecisely identify X with $\langle X, R\rangle$, letting

[1] I owe to Nick Bezhanishvili the observation that the topology used in Section 2 is the Scott topology.

context or fiat determine R. A Kripke frame X is reflexive [transitive] iff R is: for the rest of this paper, we will assume that all Kripke frames are reflexive and transitive. A Kripke frame is *rooted* iff $(\exists r \in W)(\forall w \in W)(rRw)$. A subset O of X is *open* iff $(\forall x, y \in X)(x \in O \;\&\; xRy \Rightarrow y \in O)$. A subset C of X is *closed* iff $X - C$ is open. The *interior* of a set $S \subseteq X$ is the largest open subset of S: $Int(S) =_{\mathrm{df}} \{x \in S : \forall y \in X, xRy \Rightarrow y \in S\}$. The *closure* of a set $S \subseteq X$ is the smallest closed superset of S: $Cl(S) =_{\mathrm{df}} X - Int(X - S)$. A *topological space* is an ordered pair $\langle X, \tau\rangle$, where X is a nonempty set and $\tau \subseteq \mathcal{P}(X)$ is a topology on X. We will somewhat imprecisely identify X with $\langle X, \tau\rangle$, letting context or fiat determine τ. Thus, for example, we identify $\mathbb{R}$ with $\langle \mathbb{R}, \tau_{\mathbb{R}}\rangle$, where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$. We take the basics of point-set topology to be given, in particular the notion of the interior and closure, $Int(S)$ and $Cl(S)$, of a subset S of a topological space.

A *Kripke model* [*topological model*] is an ordered pair $M = \langle X, V\rangle$, where X is a Kripke frame [topological space] and $V : PV \to \mathcal{P}(X)$. We will use the term *model* to cover Kripke models and topological models. For any model $M = \langle X, V\rangle$, V is extended to all formulas as follows: $V(\neg A) = X - V(A)$; $V(A \;\&\; B) = V(A) \cap V(B)$; $V(A \vee B) = V(A) \cup V(B)$; and $V(\Box A) = Int(V(A))$. If Γ is a nonempty set of formulas, then $V(\Gamma) =_{\mathrm{df}} \bigcap_{A\in\Gamma} V(A)$; if Γ is empty, then $V(\Gamma) =_{\mathrm{df}} X$.

Suppose that Γ is a set of formulas. If X is a Kripke frame or topological space and $x \in X$, then we say that Γ is *satisfiable at x in X* iff there is some model $M = \langle X, V\rangle$ such that $x \in V(\Gamma)$; and we say that Γ is *satisfiable in X* iff Γ is satisfiable at some x in X. We say that S4 is *complete* wrt X iff every finite consistent set of formulas is satisfiable in X, and *strongly complete* wrt to X iff every consistent set of formulas is satisfiable in X.

The following completeness theorem follows from [8], Theorem XI, 9.1, (vii), which itself derived from [5, 6]:

Theorem 1.1. S4 *is complete wrt* $\mathbb{R}$.

Theorem 1.1 is well-known: there are new and more accessible proofs in [1, 2, 7]. The current paper's main result is a special case of the main theorem, Theorem 1.2, in [4]:

Theorem 1.2. S4 *is* strongly *complete wrt* $\mathbb{R}$.

Before we prove Theorem 1.2, we recall the standard notion of an *interior map*, also sometimes called a *p-morphism*. A function from a topological space or Kripke frame to a topological space or Kripke frame is *continuous* iff the preimage of every open set is open; is *open* iff the image of every open set is open; and is an *interior map* iff it is a continuous open surjection. Suppose that $M = \langle X, V\rangle$ and $M' = \langle X', V'\rangle$ are models, and that f is an interior map from X to X'. Then f is an *interior map from M to M'* iff, for every $p \in PV$ and $x \in X$, $x \in V(p)$ iff $f(x) \in V'(p)$. The following lemma and corollary are standard:

Lemma 1.3. *If f is an interior map from $M = \langle X, V\rangle$ to $M' = \langle X', V'\rangle$, then for every formula B and $x \in X$, $x \in V(B)$ iff $f(x) \in V'(B)$.*

Corollary 1.4. *Suppose that each of X and X' is a Kripke frame or topological space, and that there is an interior map from X to X'. Then if Γ is satisfiable in X' then Γ is satisfiable in X.*

Given Corollary 1.4, we can divide the work of proving Theorem 1.2 into two parts. The first part is mainly logical: we show that S4 is strongly complete for the space $2^{\leq\omega}$ of finite and infinite binary sequences, equipped with the Scott topology (Lemma 2.4). The second part is purely topological: we show that there's an interior map from $\mathbb{R}$ to $2^{\leq\omega}$. In fact, we will proceed by showing that there's an interior map from the open unit interval, $\mathcal{I} = (0, 1)$ to $2^{\leq\omega}$: this suffices since there are many interior maps from $\mathbb{R}$ to $\mathcal{I}$.

2 The space $2^{\leq\omega}$

For each $n \geq 0$, let 2^n be the set of binary sequences (sequences of 0's and 1's) of length n. Let $2^{<\omega} =_{\mathrm{df}} \bigcup_{n=0}^{\infty} 2^n$, i.e., $2^{<\omega}$ is the set of finite binary sequences. We write *length*(b) for the length of $b \in 2^{<\omega}$. Let 2^ω be the set of infinite binary sequences or order type ω. And let $2^{\leq\omega} =_{\mathrm{df}} 2^{<\omega} \cup 2^\omega$. We use Λ for the the empty binary sequence, i.e., the binary sequence of length 0. We use b, b', etc., to range over $2^{<\omega}$; $\mathbf{b}, \mathbf{b}'$, etc., to range over 2^ω; and $\boldsymbol{b}, \boldsymbol{b}'$, etc., to range over $2^{\leq\omega}$. If $b \in 2^{<\omega}$ and $\boldsymbol{b} \in 2^{\leq\omega}$, then we write $b^\frown\boldsymbol{b}$ for b concatenated with $\boldsymbol{b}$. We will write $b0$ and $b1$ for $b^\frown\langle 0\rangle$ and $b^\frown\langle 1\rangle$. Given any $\mathbf{b} \in 2^\omega$ and any $n \in \mathbb{N}$, the finite binary sequence $\mathbf{b}|_n$ is the initial segment of length n of $\mathbf{b}$. Thus $\mathbf{b}|_0 = \Lambda$. Given $b \in 2^{<\omega}$ and $\boldsymbol{b} \in 2^{\leq\omega}$, we say $b \leq \boldsymbol{b}$ iff b is an initial segment of $\boldsymbol{b}$ and $b < \boldsymbol{b}$ iff both $b \leq \boldsymbol{b}$ and $b \neq \boldsymbol{b}$. We will also use '$\leq$' for $\leq$ restricted to $2^{<\omega}$.

We identify $2^{<\omega}$ with the *infinite binary tree*, i.e., the countably infinite rooted transitive reflexive Kripke frame $\langle 2^{<\omega}, \leq\rangle$. We can represent any *branch* of the tree $2^{<\omega}$ with as an infinite binary sequence $\mathbf{b} \in 2^{\omega}$: $\mathbf{b}$ represents the branch whose nodes are $\mathbf{b}|_0$, $\mathbf{b}|_1$, $\mathbf{b}|_2$, ...

We impose a topology on $2^{\leq\omega}$, by taking as a basis all the sets of the following form, where $b \in 2^{<\omega}$: $[b] =_{\mathrm{df}} \{\boldsymbol{b'} \in 2^{\leq\omega} : b \leq \boldsymbol{b'}\}$. It is easy to check that this is the Scott topology on $2^{\leq\omega}$: See [11], p 95, for a definition of the Scott topology on any partially ordered set. The main task of the current section is to prove that S4 is strongly complete wrt $2^{\leq\omega}$ – see Lemma 2.4.

The following result, due orignally to Dov Gabbay and independently discovered by Johan van Benthem, is well-known; for a proof see [3], Theorem 1:

Lemma 2.1. *Any finite rooted reflexive transitive Kripke frame is the image of* $2^{<\omega}$ *under some interior map.*

Together with the fact that any finite consistent set Γ of formulas is satisfiable in some finite rooted reflexive transitive Kripke frame, Lemma 2.1 entails that S4 is complete wrt $2^{<\omega}$. The proof of 2.1 can easily be strengthened to prove

Lemma 2.2. *Any countable rooted reflexive transitive Kripke frame is the image of* $2^{<\omega}$ *under some interior map.*

Together with the fact that any consistent set Γ of formulas is satisfiable in some countable rooted reflexive transitive Kripke frame, Lemma 2.1 entails

Lemma 2.3. S4 *is strongly complete wrt* $2^{<\omega}$.

The remainder of this section uses Lemma 2.3 to prove

Lemma 2.4. S4 *is strongly complete wrt* $2^{\leq\omega}$.

Proof. Let Γ be a consistent set of formulas. Given Lemma 2.3, Γ is satisfiable in $2^{<\omega}$. So there is a Kripke model $M = \langle 2^{<\omega}, V\rangle$ such that $V(\Gamma) \neq \emptyset$. We will define a $V^* : PV \to 2^{\leq\omega}$ and to show that, in the topological model $M^* = \langle 2^{\leq\omega}, V^*\rangle$, we have $V^*(\Gamma) \neq \emptyset$.

First, we assign sets $\Delta_{\boldsymbol{b}}$ and $\Sigma_{\boldsymbol{b}}$ of formulas to each $\boldsymbol{b} \in 2^{\leq\omega}$. If $\boldsymbol{b} \in 2^{<\omega}$ then $\Delta_{\boldsymbol{b}} = \Sigma_{\boldsymbol{b}} =_{\mathrm{df}} \{A : \boldsymbol{b} \in V(A)\}$. Note, if $\boldsymbol{b} \in 2^{<\omega}$, then $\Sigma_{\boldsymbol{b}}$ is consistent; $\Sigma_{\boldsymbol{b}}$ is also complete in the following sense: for every formula A, either $A \in \Sigma_{\boldsymbol{b}}$ or $\neg A \in \Sigma_{\boldsymbol{b}}$. If $\boldsymbol{b} \in 2^{\omega}$, then let $\Delta_{\boldsymbol{b}} =_{\mathrm{df}} \bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} \Sigma_{\boldsymbol{b}|_m} = \{A : (\exists n \in \mathbb{N})(\forall m \geq n)(\boldsymbol{b}|_m \in V(A))\}$. Note that $\Delta_{\boldsymbol{b}}$ is consistent, so that we can let $\Sigma_{\boldsymbol{b}}$ be any complete consistent superset of $\Delta_{\boldsymbol{b}}$. We highlight some obvious points about the $\Sigma_{\boldsymbol{b}}$:

Claim 1. Suppose that $b \in 2^{<\omega}$, $\boldsymbol{b}' \in 2^{\leq\omega}$, $b \leq \boldsymbol{b}'$ and $\Box A \in \Sigma_b$. Then $\Box A \in \Sigma_{\boldsymbol{b}'}$.

Claim 2. Suppose that $\mathbf{b} \in 2^{\omega}$ and $\Box A \in \Sigma_{\mathbf{b}}$. Then $(\exists n \in \mathbb{N})(\forall \boldsymbol{b}' \in 2^{\leq\omega})(\mathbf{b}|_n \leq \boldsymbol{b}' \Rightarrow \Box A \in \Sigma_{\boldsymbol{b}'})$.

Define $V^*(p) = \{\boldsymbol{b} \in 2^{\leq\omega} : p \in \Sigma_{\boldsymbol{b}}\}$. Now we will show that, for every formula A,

$(*)$ for every $\boldsymbol{b} \in 2^{\leq\omega}$, $\boldsymbol{b} \in V^*(A)$ iff $A \in \Sigma_{\boldsymbol{b}}$.

The proof is by induction on the construction of A. If $A \in PV$ then $(*)$ follows from the fact that $V'(p) = \{x \in X : p \in \Sigma_{\boldsymbol{b}(x)}\}$; and if A is of the form $\neg B$, $(B \,\&\, C)$ or $(B \vee C)$, then $(*)$ follows from the fact that each $\Sigma_{\boldsymbol{b}(x)}$ is consistent and complete. So suppose that A is of the form $\Box B$ and that

$(*_B)$ for every $\boldsymbol{b} \in 2^{\leq\omega}$, $\boldsymbol{b} \in V^*(B)$ iff $B \in \Sigma_{\boldsymbol{b}}$.

We want to show,

$(*_{\Box B})$ for every $\boldsymbol{b} \in 2^{\leq\omega}$, $\boldsymbol{b} \in V^*(\Box B)$ iff $\Box B \in \Sigma_{\boldsymbol{b}}$.

Proof of $(\Rightarrow)$. Choose $\boldsymbol{b} \in 2^{\leq\omega}$ and assume that $\boldsymbol{b} \in V^*(\Box B)$. So there is some $b' \in 2^{<\omega}$ such that $\boldsymbol{b} \in [b'] \subseteq V^*(B)$. So $[b'] \cap 2^{<\omega} \subseteq V(B)$, by $(*_B)$ and the definition of the Σ_b. So $[b'] \cap 2^{<\omega} \subseteq V(\Box B)$, by the definition of $V(\Box B)$. Also $b' \leq \boldsymbol{b}$. So $\Box A \in \Sigma_{\boldsymbol{b}}$, by Lemma 1.

Proof of $(\Leftarrow)$. Choose $\boldsymbol{b} \in 2^{\leq\omega}$ and assume that $\Box B \in \Sigma_{\boldsymbol{b}}$. Then there is an $m \in \mathbb{N}$ such that $(\forall n \geq m)(\Box B \in \Sigma_{\boldsymbol{b}|_n})$. So $\Box B \in \Sigma_{\boldsymbol{b}|_m}$. So $\boldsymbol{b}|_m \in V(\Box B)$. So, for every $b' \in 2^{<\omega}$, if $\boldsymbol{b}|_m \leq b'$ then $b' \in V(B)$. So, for every $b' \in 2^{<\omega}$, if $\boldsymbol{b}|_m \leq b'$ then $B \in \Sigma_{b'}$. But then, by the definition of $\Sigma_{\mathbf{b}''}$ for $\mathbf{b}'' \in 2^{\omega}$, we have for every $\mathbf{b}'' \in 2^{<\omega}$, if $\boldsymbol{b}|_m \leq \mathbf{b}''$ then $B \in \Sigma_{\mathbf{b}''}$. So, by $(*_B)$, for every $\boldsymbol{b}^* \in [\boldsymbol{b}|_m]$, $\boldsymbol{b}^* \in V^*(B)$. So $\boldsymbol{b} \in [\boldsymbol{b}|_m] \subseteq V^*(\Box B)$, as desired.

Given $(*)$, to see that Γ is satisfiable in $2^{\leq\omega}$, simply choose $b \in 2^{<\omega}$ with $b \in V(\Gamma)$. Note: $\Gamma \subseteq \Sigma_b$, so that $b \in V^*(\Gamma)$, by $(*)$. $\Box$

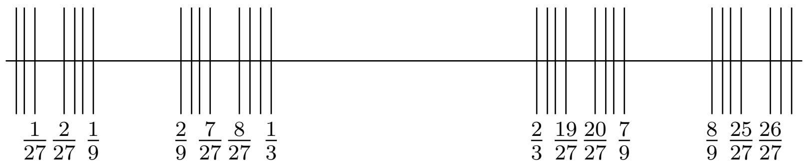

Figure 1: The Cantor set without the endpoints 0 and 1.

3 An interior map from $\mathcal{I} = (0, 1)$ to $2^{\leq\omega}$

Our remaining work is purely topological: we want to prove

Lemma 3.1. *There is an interior map from* $\mathbb{R}$ *to* $2^{\leq\omega}$.

Let $\mathcal{I} = (0, 1)$ be the open unit interval. As noted in the introductory remarks, it suffices to prove

Lemma 3.2. *There is an interior map from* $\mathcal{I}$ *to* $2^{\leq\omega}$.

We will prove Lemma 3.2 by partitioning $\mathcal{I}$ into nonempty pairwise disjoint sets $X_{\boldsymbol{b}}$, one for each $\boldsymbol{b} \in 2^{\leq\omega}$. We will then define $\mathbf{F} : \mathcal{I} \to 2^{\leq\omega}$ as follows: $\mathbf{F}(x) =$ the unique $\boldsymbol{b} \in 2^{\leq\omega}$ such that $x \in X_{\boldsymbol{b}}$. The trick is to do this in such a way that $\mathbf{F}$ is an interior map.

First, some preliminaries. For subsets of $\mathcal{I}$, we will interpret interior, Int, and closure, Cl, as relativized to $\mathcal{I}$. Let $\mathcal{C}$ be the Cantor set without the endpoints 0 and 1. So $\mathcal{C}$ is the set of all real numbers that have a ternary expansion of the form $0.a_1a_2a_3 \ldots a_k \ldots$ where each a_k is either 0 or 2, and where not all the a_k's are 0 (so that $0 \notin \mathcal{C}$) and not all the a_k's are 2 (so that $1 \notin \mathcal{C}$): we will find it useful to represent real numbers as ternary expansions. Figure 1 pictorially represents $\mathcal{C}$, which is closed (in the space $\mathcal{I}$).

$\mathcal{C}$ can be got from progressively deleting open intervals from $\mathcal{I} = (0, 1)$ as follows: delete the open interval $(0.1, 0.2)$, which is the middle third of $\mathcal{I}$, leaving $(0, 0.1] \cup [0.2, 1)$. Then delete the middle thirds of each of these: delete the open interval $(0.01, 0.02)$ from $(0, 0.1]$ and delete the open interval $(0.21, 0.22)$ from $[0.2, 1)$: this leaves $(0, 0.01] \cup [0.02, 0.1] \cup [0.2, 0.21] \cup [0.22, 1)$. More precisely, a *middle third* is any open interval of the form $(0.a_1a_2 \ldots a_n1, 0.a_1a_2 \ldots a_n2)$, where $n \geq 0$ and where $a_k = 0$ or 2 for all

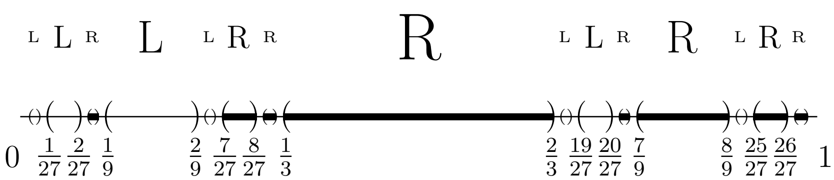

Figure 2: Labelling deleted middle thirds with L and R. The labels appear above the labelled middle thirds: for clarity, we have written the labels of larger middle thirds in larger fonts. The set $\mathcal{R}$ is represented by thicker lines.

$k \leq n$. It is well-known that if we take what's left undeleted after we carry out this process of deleting middle thirds *ad infinitum*, then we get $\mathcal{C} = \mathcal{I} - \bigcup\{J : J \text{ is a middle third}\}$. Label the deleted middle thirds with L and R, for *left* and *right*, as in Figure 2. And let $\mathcal{L}$ be the union of the middle thirds labeled L, and $\mathcal{R}$ be the union of the middle thirds labeled R.

Now suppose that $J \subseteq \mathcal{I}$ is an open interval. Let $f_J : \mathcal{I} \to J$ be the unique increasing linear function from $\mathcal{I}$ onto J. We define $\mathcal{L}(J)$, $\mathcal{R}(J)$, and $\mathcal{C}(J)$ as the images under f_J of $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{C}$ respectively. Thus $\mathcal{L}(J)$, $\mathcal{R}(J)$, and $\mathcal{C}(J)$ are copies of $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{C}$, respectively. Finally, suppose that $O \subseteq \mathcal{I}$ is open. We say that an open interval $J \subseteq O$ is a *maximal open interval in* O iff, for any open interval $J' \subseteq O$, if $J \cap J' \neq \emptyset$ then $J' \subseteq J$. Note that O is the disjoint union of the maximal open intervals in O. We define

$$\mathcal{L}(O) = \bigcup_{J \text{ is a maximal open interval in } O} \mathcal{L}(J),$$

and similarly for $\mathcal{R}(O)$ and $\mathcal{C}(O)$. So $\mathcal{L}(O)$ is the union of copies of $\mathcal{L}$, and similarly for $\mathcal{R}(O)$ and $\mathcal{C}(O)$. Note the following:

Lemma 3.3. *1. $\mathcal{L}(O)$, $\mathcal{R}(O)$, and $\mathcal{C}(O)$ are pairwise disjoint;*

2. $\mathcal{L}(O)$ and $\mathcal{R}(O)$ are open;

3. $O = \mathcal{L}(O) \dot\cup \mathcal{R}(O) \dot\cup \mathcal{C}(O)$; and

4. *$Cl(\mathcal{L}(O)) - \mathcal{L}(O)$*
   $= Cl(\mathcal{R}(O)) - \mathcal{R}(O)$
   $= Cl(\mathcal{C}(O))$
   $= Cl(O) - (\mathcal{L}(O) \cup \mathcal{R}(O))$.

5. *If J is a maximal open interval in O, then $J \cap \mathcal{C}(O)$ is nonempty.*

If $S \subseteq \mathcal{I}$ and there is some open interval $J \subseteq S$, then we define the *width* of S as follows: $width(S) = sup\{length(J) : J$ is an open interval and $J \subseteq S\}$. Note the following:

Lemma 3.4. *If O is an open subset of $\mathcal{I}$, then $width(\mathcal{R}(O)) = width(O)/3$ and $width(\mathcal{L}(O)) = width(O)/9$.*

Our next task will be to define nonempty open $O_b \subseteq \mathcal{I}$ and other nonempty sets $X_b \subseteq \mathcal{I}$ for each $b \in 2^{<\omega}$, and also to define nonempty sets $X_{\mathbf{b}} \subseteq \mathcal{I}$ for each $\mathbf{b} \in 2^{\omega}$. Once this has been done, we will have a partition of $\mathcal{I}$ into sets $X_{\boldsymbol{b}}$ for each $\boldsymbol{b} \in 2^{\leq\omega}$. We will define $\mathbf{F} : 2^{\leq\omega} \to \mathcal{I}$ as follows: $\mathbf{F}(\boldsymbol{b})$ = the unique $x \in \mathbb{R}$ such that $x \in X_{\boldsymbol{b}}$. And we will show that $\mathbf{F}$ is an interior map.

Define the O_b, for $b \in 2^{<\omega}$, recursively as follows:

$$\begin{aligned} O_\Lambda &=_{\text{df}} \mathcal{I} \\ O_{b0} &=_{\text{df}} \mathcal{L}(O_b) \\ O_{b1} &=_{\text{df}} \mathcal{R}(O_b) \end{aligned}$$

For $b \in 2^{<\omega}$, we define $X_b =_{\text{df}} \mathcal{C}(O_b)$: If $b = \Lambda$, then X_b is simply $\mathcal{C}$, the Cantor set without endpoints; and if b is some other finite binary sequence, then X_b is a union of infinitely many copies of $\mathcal{C}$. Note that each $\mathcal{L}(O_{\boldsymbol{b}})$ and $\mathcal{R}(O_{\boldsymbol{b}})$ is open in $\mathcal{I}$; that each $\mathcal{L}(O_{\boldsymbol{b}})$, $\mathcal{R}(O_{\boldsymbol{b}})$, and $\mathcal{C}(O_b)$ is nonempty; and that, $(\forall b \in 2^{<\omega})(O_b = O_{b0} \mathbin{\dot\cup} O_{b1} \mathbin{\dot\cup} X_b)$. Note the following facts about the O_b and the X_b:

Lemma 3.5. *1. X_b and O_b are nonempty, for each $b \in 2^{<\omega}$.*

2. *If $b \leq b'$ then $X_{b'} \subseteq O_{b'} \subseteq O_b$.*

3. *If $b < b'$ then $X_b \cap X_{b'} = X_b \cap O_{b'} = \emptyset$.*

4. *If $b' \not\leq b \not\leq b'$ then $O_b \cap O_{b'} = \emptyset$.*

5. *If $b \not\leq b'$ then $O_b \cap X_{b'} = \emptyset$.*

6. *If* $b \neq b'$ *then* $X_b \cap X_{b'} = \emptyset$.

7. $width(O_b) \leq 1/3^{length(b)}$.

Lemma 3.6. $(\forall b, b' \in 2^{<\omega})(b \leq b' \Rightarrow Cl(X_b) \subseteq Cl(X_{b'}))$.

Proof. The fact that $(\forall b \in 2^{\omega})(Cl(X_b) \subseteq Cl(X_{b0}))$ follows immediately from the following, for any $b \in 2^{\omega}$:

1. $O_{b0} = O_{b00} \mathbin{\dot\cup} O_{b01} \mathbin{\dot\cup} X_{b0}$ (Lemma 3.3, Clause 3),

2. $Cl(X_{b0}) = Cl(O_{b0}) - (O_{b00} \cup O_{b01})$ (Lemma 3.3, Clause 4), and

3. $Cl(X_b) = Cl(O_{b0}) - O_{b0}$ (Lemma 3.3, Clause 4).

Similarly $(\forall b \in 2^{\omega})(Cl(X_b) \subseteq Cl(X_{b1}))$. This suffices for the lemma. □

For $\mathbf{b} \in 2^{\omega}$, define $X_{\mathbf{b}} =_{\mathrm{df}} \bigcap_{n\in\mathbb{N}} O_{\mathbf{b}|_n}$.

Lemma 3.7. $\mathcal{I} = \dot{\bigcup}_{\boldsymbol{b}\in 2^{\leq\omega}} X_{\boldsymbol{b}}$.

Proof. The $X_{\boldsymbol{b}}$ are pairwise disjoint, by Lemma 3.5. To see that $\mathcal{I} = \bigcup_{\boldsymbol{b}\in 2^{\leq\omega}} X_{\boldsymbol{b}}$, suppose that $x \in \mathcal{I}$, but suppose that $x \notin X_b$ for any $b \in 2^{<\omega}$. It will suffice to find a $\mathbf{b} \in 2^{\omega}$ such that $x \in X_{\mathbf{b}}$: we will inductively define $b_n \in 2^{<\omega}$, each of length n, so that $b_0 \leq b_1 \leq \ldots \leq b_n \leq b_{n+1} \leq \ldots$, and so that $x \in O_{b_n}$ for each n. Let $b_0 = \Lambda$, the empty sequence. Assume that $x \in O_{b_n}$. Then $x \in O_{b_n0} \mathbin{\dot\cup} O_{b_n1} \mathbin{\dot\cup} X_{b_n}$. But $x \notin X_{b_n}$. So x is a member of exactly one of O_{b_n0} and O_{b_n1}. Let b_{n+1} be whichever of b_n0 and b_n1 is such that $x \in O_{b_{n+1}}$. Note that each b_n has length n, that $b_0 \leq b_1 \leq \ldots \leq b_n \leq b_{n+1} \leq \ldots$ and that $x \in O_{b_n}$ for each n. Let $\mathbf{b}$ be the unique member of 2^{ω} such that $\mathbf{b}|_n = b_n$. Then note that $x \in \bigcap_n O_{\mathbf{b}|_n} = X_{\mathbf{b}}$, as desired. □

Given Lemma 3.7, every $x \in \mathcal{I}$ is in exactly one of the $X_{\boldsymbol{b}}$. Let $\mathbf{F}(x) =_{\mathrm{df}}$ the unique $\boldsymbol{b} \in 2^{\leq\omega}$ such that $x \in X_{\boldsymbol{b}}$. Our final task is to show that $\mathbf{F}$ is an interior map.

$\mathbf{F}$ is continuous since the preimage of $[b]$, where $b \in 2^{<\omega}$, is O_b. We want to prove that $\mathbf{F}$ is both open and a surjection: see Corollary 3.11. First, some preliminary lemmas. For $S \subseteq \mathcal{I}$, we use $Img(S)$ for the image of S under $\mathbf{F}$.

Lemma 3.8. *Suppose that* $J \subseteq \mathcal{I}$ *is an open interval,* $b \in Img(J) \cap 2^{<\omega}$, $b' \in 2^{<\omega}$ *and* $b \leq b'$. *Then* $b' \in Img(J)$.

Proof. Choose $x \in J$ with $\mathbf{F}(x) = b$. Then $x \in X_b$. So $x \in Cl(X_{b'})$, by Lemma 3.6. So there is some $y \in X_{b'} \cap J$. So $b' \in Img(J)$, since $\mathbf{F}(y) = b'$. $\square$

Lemma 3.9. *Suppose that $J \subseteq \mathcal{I}$ is an open interval, $b \in Img(J) \cap 2^{<\omega}$, $\mathbf{b}' \in 2^\omega$ and $b \leq \mathbf{b}'$. Then $\mathbf{b}' \in Img(J)$.*

Proof. Let $n = length(b)$, so that $b = \mathbf{b}'|_n$. We will now inductively choose open intervals $J_0, J_1, ... \subseteq J \cap O_b$ and points $x_0 \in J_0, x_1 \in J_1, \dots$ so that $F(x_k) = \mathbf{b}'|_{n+k}$, for each $k \geq 0$.

First, choose $x_0 \in J$ such that $\mathbf{F}(x_0) = b = \mathbf{b}'|_n$. Since $x_0 \in J \cap O_{b|_n}$, we can choose an open interval J_0 so that $x_0 \in J_0$ and $Cl(J_0) \subseteq J \cap O_{b|_n}$. Suppose that we have chosen an open interval J_k and a point $x_k \in J_k$ with $F(x_k) = \mathbf{b}'|_{n+k}$. Then $\mathbf{b}'|_{n+k} \in Img(J_k)$. So $\mathbf{b}'|_{n+k+1} \in Img(J_k)$, by Lemma 3.8. So there is an $x_{k+1} \in Img(J_k)$ with $\mathbf{F}(x_{k+1}) = \mathbf{b}'|_{n+k+1}$. Note that $x_{k+1} \in X_{\mathbf{b}'|_{n+k+1}} \subseteq O_{\mathbf{b}'|_{n+k+1}}$. So $x_{k+1} \in J_k \cap O_{\mathbf{b}'|_{n+k+1}}$. So we can choose an open interval J_{k+1} with $x_{k+1} \in J_{k+1}$ and $Cl(J_{k+1}) \subseteq J_k \cap O_{\mathbf{b}'|_{n+k+1}}$.

Note: $Cl(J_{k+1}) \subseteq J_k$ for each $k \geq 0$. So $\langle Cl(J_k) \rangle_k$ is a decreasing sequence of closed intervals. So $\bigcap_k Cl(J_k)$ is nonempty. Also, $\bigcap_k Cl(J_k) \subseteq J$ and $\bigcap_k Cl(J_k) \subseteq \bigcap_k O_{\mathbf{b}'|_{n+k}}$. So there is a point $x \in \bigcap_k Cl(J_k) \subseteq J \cap X_{\mathbf{b}'}$. So $\mathbf{F}(x) = \mathbf{b}'$ and $x \in J$. So $\mathbf{b}' \in Img(J)$. $\square$

Lemma 3.10. *Suppose that $J \subseteq \mathcal{I}$ is an open interval and $\mathbf{b} \in Img(J) \cap 2^\omega$. Then there is a $b' \in Img(J) \cap 2^{<\omega}$ with $b' \leq \mathbf{b}$.*

Proof. Suppose that $J \subseteq \mathcal{I}$ is an open interval and $\mathbf{b} \in Img(J) \cap 2^\omega$. Choose $x \in J$ with $\mathbf{F}(x) = \mathbf{b}$, and choose a positive real number d so that $(x-d, x+d) \subseteq J$. Choose $n \in \mathbb{N}$ with $1/3^n < d$ and let $b' = \mathbf{b}|_n \in 2^{<\omega}$. Note that $x \in O_{b'} \cap (x-d, x+d)$; also, $width(O_{b'}) < d$, by Lemma 3.5, clause (7). Let J' be any maximal open interval in $O_{b'}$ with $x \in J'$, and note two things about J': (1) J' has length $\leq width(O_{b'}) < d$, since J' is an open interval and $J' \subseteq O_{b'}$; and (2) $J' \cap \mathcal{C}(O_{b'})$ is nonempty, by Lemma 3.3, Clause (5). By (2), there is an $x' \in J' \cap X_{b'}$, and by (1) $J' \subseteq (x-d, x+d)$. So $x' \in J$ and $\mathbf{F}(x') = b'$. So $b' \in Img(J)$. $\square$

Corollary 3.11. $\mathbf{F}$ *is an open surjection.*

Consider any interval $J \subseteq \mathcal{I}$. By Lemma 3.8 and 3.9, if $b \in Img(J) \cap 2^{<\omega}$ then $[b] \subseteq Img(J)$. So $\bigcup_{b \in Img(J) \cap 2^{<\omega}} [b] \subseteq Img(J)$.

Also note that if $\boldsymbol{b} \in Img(J)$, then there is some $b' \leq \boldsymbol{b}$ such that $b' \in Img(J) \cap 2^{<\omega}$: this follows from Lemma 3.10 if $\boldsymbol{b} \in 2^\omega$; and it is trivial if

$\boldsymbol{b} \in 2^{<\omega}$, since we can just let $b' = \boldsymbol{b}$. Thus, if $\boldsymbol{b} \in Img(J)$ then there exists $b' \in 2^{<\omega}$ with $\boldsymbol{b} \in [b'] \subseteq Img(J)$. Thus $Img(J) \subseteq \bigcup_{b \in Img(J) \cap 2^{<\omega}} [b]$. Thus $Img(J) = \bigcup_{b \in Img(J) \cap 2^{<\omega}} [b]$.

So the image of any open interval $J \subseteq \mathcal{I}$ is open. So the function $\mathbf{F}$ is open. Also, $Img(\mathcal{I}) = \bigcup_{b \in Img(\mathcal{I}) \cap 2^{<\omega}} [b] \supseteq [\Lambda] = 2^{\leq\omega}$. So $\mathbf{F}$ is a surjection. $\square$

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