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# Topological-frame products of modal logics

**Abstract.** The simplest bimodal combination of unimodal logics $\mathrm{L}_1$ and $\mathrm{L}_2$ is their *fusion*, $\mathrm{L}_1 \otimes \mathrm{L}_2$, axiomatized by the theorems of $\mathrm{L}_1$ for $\Box_1$ and of $\mathrm{L}_2$ for $\Box_2$, and the rules of modus ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution. Shehtman introduced the *frame product* $\mathrm{L}_1 \times \mathrm{L}_2$, as the logic of the products of certain Kripke frames: these logics are two-dimensional as well as bimodal. Van Benthem, Bezhanishvili, ten Cate and Sarenac transposed Shehtman's idea to the topological semantics and introduced the *topological product* $\mathrm{L}_1 \times_t \mathrm{L}_2$, as the logic of the products of certain topological spaces. For almost all well-studies logics, we have $\mathrm{L}_1 \otimes \mathrm{L}_2 \subsetneq \mathrm{L}_1 \times \mathrm{L}_2$, for example, $\mathrm{S4} \otimes \mathrm{S4} \subsetneq \mathrm{S4} \times \mathrm{S4}$. Van Benthem *et al* show, by contrast, that $\mathrm{S4} \times_t \mathrm{S4} = \mathrm{S4} \otimes \mathrm{S4}$. It is straightforward to define the product of a topological space and a frame: the result is a topologized frame, i.e., a set together with a topology and a binary relation. In this paper, we introduce topological-frame products $\mathrm{L}_1 \times_{tf} \mathrm{L}_2$ of modal logics, providing a complete axiomatization of $\mathrm{S4} \times_{tf} \mathrm{L}$, whenever L is a Kripke complete Horn axiomatizable extension of the modal logic D: these extensions include T, S4 and S5, but not K or K4. We leave open the problem of axiomatizing $\mathrm{S4} \times_{tf} \mathrm{K}$, $\mathrm{S4} \times_{tf} \mathrm{K4}$, and other related logics. When L = S4, our result confirms a conjecture of van Benthem *et al* concerning the logic of products of Alexandrov spaces with arbitrary topological spaces.

*Keywords*: Bimodal logic, multimodal logic, combining modal logics, topological semantics, Kripke semantics.

## 1. Introduction

Consider two modal logics $\mathrm{L}_1$ and $\mathrm{L}_2$ in a unimodal language, i.e., a language with one modal operator $\Box$. There are many natural ways to combine $\mathrm{L}_1$ and $\mathrm{L}_2$ to make a bimodal logic, i.e., a logic in a language with two modal operators, $\Box_1$ and $\Box_2$. The simplest is to take their *fusion*, $\mathrm{L}_1 \otimes \mathrm{L}_2$, axiomatized by the theorems of $\mathrm{L}_1$ for $\Box_1$ and of $\mathrm{L}_2$ for $\Box_2$, and the rules of modus ponens, necessitation for $\Box_1$ and for $\Box_2$, and substitution.

Shehtman [12] and van Benthem *et al* [14] introduce combinations which are 2-dimensional as well as bimodal. More specifically, given two Kripke frames $\langle W_1, R_1\rangle$ and $\langle W_2, R_2\rangle$, Shehtman defines a particular birelational frame $\langle W_1 \times W_2, R_1', R_2'\rangle$ as their product.[1] He then introduces the *frame*

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[1]The precise definition of the product of two frames, as well as other definitions and

 

*product* $L_1 \times L_2$ of unimodal logics $L_1$ and $L_2$, as the bimodal logic of the products of Kripke frames for $L_1$ and $L_2$. For almost all well-studied logics, $L_1 \otimes L_2 \subsetneq L_1 \times L_2$.[2] Shehtman also introduces the *commutator* of two logics: $[L_1, L_2] =_{df} L_1 \otimes L_2 + com_{\supset} + com_{\subset} + chr$, where

$$\begin{array}{rcll} com_{\supset} & = & \Box_1\Box_2 p \supset \Box_2\Box_1 p & \text{(left commutativity),} \\ com_{\subset} & = & \Box_2\Box_1 p \supset \Box_1\Box_2 p & \text{(right commutativity), and} \\ chr & = & \Diamond_1\Box_2 p \supset \Box_2\Diamond_1 p & \text{(Church-Rosser).} \end{array}$$

For logics that are Kripke complete and Horn axiomatizable, $L_1 \times L_2 = [L_1, L_2]$ ([3], Theorem 5.9). These logics include those on the following list of well-studied logics: K; $D = K + \Diamond\top$; $K4 = K + (\Box p \supset \Box\Box p)$; $D4 = D + (\Box p \supset \Box\Box p)$; $D5 = D + (\Diamond p \supset \Box\Diamond p)$; $D45 = D4 + (\Diamond p \supset \Box\Diamond p)$; $T = K + (\Box p \supset p)$; $S4 = T + (\Box p \supset \Box\Box p)$; and $S5 = S4 + (\Diamond p \supset \Box\Diamond p)$. For convenience we write $LIST = \{K, K4, D, D4, D5, D45, S4, S5\}$.

Van Benthem *et al* transpose Shehtman's idea to the topological semantics for modal logics extending S4, and introduce the *topological product* $L_1 \times_t L_2$, as the logic of the products of certain topological spaces. We always have $L_1 \otimes L_2 \subseteq L_1 \times_t L_2 \subseteq L_1 \times L_2$, for $L_1, L_2 \supseteq S4$. The main theorem of [14] is that $S4 \times_t S4 = S4 \otimes S4$. It is easy to show, by contrast, that $S5 \otimes S5 \subsetneq S5 \times_t S5 = S5 \times S5$ and $S4 \otimes S5 \subsetneq S4 \times_t S5 \subsetneq S4 \times S5$. (See Section 3, below.)

It is straightforward to define the product of a topological space and a Kripke frame: the result is a topologized frame, i.e., a set together with a topology and a binary relation.[3] In this paper, we introduce the topological-frame product $L_1 \times_{tf} L_2$ of modal logics, analogous to $L_1 \times L_2$ and $L_1 \times_t L_2$, as the logic of the products of certain topological spaces with certain frames. Our main result is that, if L is a Kripke complete Horn axiomatizable extension of D, then $S4 \times_{tf} L = S4 \otimes L + com_{\supset} + chr$. Thus, for such L, the product $S4 \times_{tf} L$ is the *e-commutator* of S4 and L, as defined in [10], where e-commutators are given a Kripke semantics. We note that our result fails when L = K or K4: we leave the axiomatization of $S4 \times_{tf} K$, $S4 \times_{tf} K4$ and related logics as an open problem.

Two remarks before we get to the details. First, van Benthem *et al* conjecture that the logic of products of Alexandrov spaces with arbitrary

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terminological/notational conventions, are given in Section 2, below.

[2]There are trivial cases where $L_1 \otimes L_2 = L_1 \times L_2$, e.g. when either $L_1$ or $L_2$ is inconsistent or is one of $Ver = K + \Box\bot$ or $CL = K + (p \equiv \Box p)$.

[3]Our 'topologized frames' are the topological frames or topological Kripke frames of, for example, [1]. We coin a slightly different expression here, since the expression 'topological Kripke frame' is first introduced in [2] for something more specific.

topological spaces is $\mathrm{S4} \otimes \mathrm{S4} + com_{\subset} + chr$. As we shall see, Alexandrov spaces are a notational variant of transitive-reflexive Kripke frames. Thus, along the way, we confirm the conjecture of van Benthem *et al*, since it is equivalent to the claim that $\mathrm{S4} \times_{tf} \mathrm{S4} = \mathrm{S4} \otimes \mathrm{S4} + com_{\supset} + chr$.

Second, we mention the recent work of Kudinov [7, 8, 9]. In the unimodal case, *neighbourhood semantics* is a well-known generalization of topological semantics, allowing a topological-like semantics for logics much weaker than S4, indeed much weaker than K. Kudinov defines *neighbourhood products* $\mathrm{L}_1 \times_n \mathrm{L}_2$ of modal logics, axiomatizing a large family of these. See Remark 3.6 and Problem 3.7, below, for more on neighbourhood products.

## 2. Technical details

For each $n \geq 1$, let $\mathcal{L}_n$ be a propositional language with a set $PV$ of propositional variables; standard Boolean connectives &, $\vee$ and $\neg$; a propositional constant $\top$; and $n$ modal operators, $\Box_1, \ldots, \Box_n$. For $n = 1$, we often write $\Box$ instead of $\Box_1$. We use standard definitions of the Boolean connective $\supset$, the propositional constant $\bot$, and the modal operators $\Diamond_i$. A set L of formulas of $\mathcal{L}_n$ is a *normal $n$-modal logic* ($n$-logic) iff every propositional tautology is in L, $(\Box_i(p \supset q) \supset (\Box_i p \supset \Box_i q)) \in \mathrm{L}$ for each $i = 1, \ldots, n$, and L is closed under modus ponens, necessitation for each $\Box_i$, and substitution. Denote the smallest $n$-logic by $\mathrm{K}_n$. Say that the $n$-logic $\mathrm{L}_1$ *extends* or *is an extension of* the $n$-logic $\mathrm{L}_2$ iff $\mathrm{L}_1 \supseteq \mathrm{L}_2$.

In general, for any $n$-logic L and any set $\Delta$ of formulas of $\mathcal{L}_n$, define the extension $\mathrm{L} + \Delta$ of L as the smallest $n$-logic $\mathrm{L}'$ such that $\mathrm{L} \cup \Delta \subseteq \mathrm{L}'$. If $\Delta = \{A_1, \ldots, A_k\}$, we write $\mathrm{L} + \Gamma = \mathrm{L} + A_1 + \ldots + A_n$. For $n = 1$, we have already mentioned a list of extensions of K.

Given 1-logics $\mathrm{L}_1, \ldots, \mathrm{L}_n$ all formulated in $\mathcal{L}_1$, their *fusion* $\mathrm{L}_1 \otimes \ldots \otimes \mathrm{L}_n$ is the the smallest set of formulas of $\mathcal{L}_n$ that contains $\cup_{i=1}^{n} \mathrm{L}'_i$ and is closed under modus ponens, necessitation for each $\Box_i$, and substitution – where $\mathrm{L}'_i$ is the set of formulas of $\mathcal{L}_n$ got by replacing each occurrence of $\Box_1$ in each formula in $\mathrm{L}_i$ by $\Box_i$. Note that $\mathrm{L}_1 \otimes \ldots \otimes \mathrm{L}_n$ is a normal $n$-modal logic. If L is a 1-logic, we write $\otimes^n \mathrm{L}$ for $\mathrm{L} \otimes \ldots \otimes \mathrm{L}$ repeated $n$ times. Note that $\otimes^n \mathrm{K} = \mathrm{K}_n$.

**Topoframes.** Here we generalize both the Kripke and the topological semantics: this allows us to introduce various products of modal logics. We assume familiarity with the basics of topology. An *$n$-topoframe* is an $(n+1)$-tuple $\mathcal{X} = \langle X, Y_1, \ldots, Y_n \rangle$, where each $Y_i$ is either a topology on $X$ or a binary relation on $X$. If $Y_i$ is a topology, then we say that a set $O \subseteq X$ is *$i$-open*, *$i$-closed*, etc., if $O$ is open, closed, etc., in the topological space

$\langle X, Y_i \rangle$. We say that $\mathcal{X}$ is an *n-frame* iff every $Y_i$ is a binary relation on $X$ and that $\mathcal{X}$ is an *n-space* iff every $Y_i$ is a topology on $X$. If $\mathcal{X} = \langle X, Y_1, Y_2 \rangle$ where $Y_1$ is a topology on $X$ and $Y_2$ is a binary relation on $X$, then we call $\mathcal{X}$ a *topologized frame*. Suppose that $S \subseteq X$. For any binary relation $Y$ on $X$, define $Int_Y(S) =_{\mathrm{df}} \{x \in X : \forall y \in X(xYy \Rightarrow y \in S)\}$. And for any topology $Y$ on $X$, write $Int_Y(S)$ for the topological interior, according to $Y$, of $S$.

**Rooted frames.** Suppose that $\mathcal{X} = \langle X, Y_1, \ldots, Y_n \rangle$ is an $n$-frame. A *path* from $x \in X$ to $x' \in X$ is a sequence $x_1, \ldots, x_m$ such that $x_1 = x$ and $x_m = x'$, and, for every $k \in \{1, \ldots, m-1\}$, $x_k Y_i x_{k+1}$ for some $i \in \{1, \ldots, n\}$. We say that $r \in X$ is a *root* of $\mathcal{X}$ iff, for every $x \in X$, there is a path from $r$ to $x$. We say that $\mathcal{X}$ is *rooted* iff it has a root.

**Subframes.** An $n$-frame $\mathcal{X}' = \langle X', Y_1', \ldots, Y_n' \rangle$ is a *subframe* of an $n$-frame $\mathcal{X} = \langle X, Y_1, \ldots, Y_n \rangle$ iff $X' \subseteq X$ and each $Y_i' = Y_i \cap (X' \times X')$. Write $\mathcal{X}' \sqsubseteq \mathcal{X}$. For any $n$-frame $\mathcal{X}$, let $\mathsf{SF}(\mathcal{X}) =_{\mathrm{df}} \{\mathcal{X}' : \mathcal{X}' \sqsubseteq \mathcal{X}\}$. For any class $\mathsf{F}$ of $n$-frames let $\mathsf{SF}(\mathsf{F}) =_{\mathrm{df}} \{\mathcal{X} : \exists \mathcal{X}' \in \mathsf{F}, \mathcal{X} \sqsubseteq \mathcal{X}'\}$.

**Models.** An *n-model* is an $(n+2)$-tuple

$$\mathcal{M} = \langle X, Y_1, \ldots, Y_n, V \rangle,$$

where $\mathcal{X} = \langle X, Y_1, \ldots, Y_n \rangle$ is an $n$-topoframe and $V : PV \to \mathcal{P}(X)$. We extend $V$ to all formulas of $\mathcal{L}_n$ by defining $V_{\mathcal{M}}$ as follows:

$$\begin{array}{rcl} V_{\mathcal{M}}(A) & = & V(A), \text{ if } A \in PV \\ V_{\mathcal{M}}(\top) & = & X \\ V_{\mathcal{M}}(\neg A) & = & X - V_{\mathcal{M}}(A) \\ V_{\mathcal{M}}(A \ \&\ B) & = & V_{\mathcal{M}}(A) \cap V_{\mathcal{M}}(B) \\ V_{\mathcal{M}}(A \vee B) & = & V_{\mathcal{M}}(A) \cup V_{\mathcal{M}}(B) \\ V_{\mathcal{M}}(\Box_i A) & = & Int_{Y_i}(V_{\mathcal{M}}(A)). \end{array}$$

We often suppress the subscripted $\mathcal{M}$ and simply write $V(A)$. We say that $\mathcal{M} \vDash A$ iff $V(A) = X$. Given an $n$-topoframe $\mathcal{X} = \langle X, Y_1, \ldots, Y_n \rangle$, we say that $\mathcal{X} \vDash A$ iff $\mathcal{M} \vDash A$ for each $n$-model $\mathcal{M} = \langle X, Y_1, \ldots, Y_n, V \rangle$. And if $\Delta$ is a set of formulas of $\mathcal{L}_n$, we say that $\mathcal{X} \vDash \Delta$ iff $\mathcal{X} \vDash A$ for each $A \in \Delta$. We read $\vDash$ as *validates*. If $\Delta$ is a set of formulas of $\mathcal{L}_n$, then $\mathsf{Fr}(\Delta)$ and $\mathsf{Top}(\Delta)$ are the following classes of $n$-frames and $n$-spaces: $\mathsf{Fr}(\Delta) = \{\mathcal{X}$ is an $n$-frame: $\mathcal{X} \vDash \Delta\}$ and $\mathsf{Top}(\Delta) = \{\mathcal{X}$ is an $n$-space: $\mathcal{X} \vDash \Delta\}$. Given any class $\mathsf{X}$ of $n$-topoframes, $\mathsf{Log}(\mathsf{X}) =_{\mathrm{df}} \{A : (\forall \mathcal{X} \in \mathsf{X})(\mathcal{X} \vDash A)\}$. Note that $\mathsf{Log}(\mathsf{X})$ is an $n$-logic. We say that an $n$-logic $\mathrm{L}$ is *sound* [*complete*] for a class $\mathsf{X}$ of $n$-topoframes iff $\mathrm{L} \subseteq \mathsf{Log}(\mathsf{X})$ [$\mathsf{Log}(\mathsf{X}) \subseteq \mathrm{L}$]. We say that $\mathrm{L}$ is sound [complete] for an $n$-topoframe $\mathcal{X}$ iff $\mathrm{L}$ is sound [complete] for $\{\mathcal{X}\}$. We say that an $n$-logic L is *Kripke complete* [*topologically complete*] iff there is some class $\mathsf{X}$ of $n$-frames [$n$-spaces] such that $\mathrm{L} = \mathsf{Log}(\mathsf{X})$. Note, then, that L is Kripke complete [topologically complete] iff $\mathrm{L} = \mathsf{Log}(\mathsf{Fr}(\mathrm{L}))$ [$\mathrm{L} = \mathsf{Log}(\mathsf{Top}(\mathrm{L}))$]. It is well-known that $\mathrm{K}_n = \otimes^n\mathrm{K}$ is the smallest Kripke complete $n$-logic. It is also well-known that S4 is the smallest topologically complete 1-logic: this result easily generalizes to the claim that $\otimes^n\mathrm{S4}$ is the smallest topologically complete $n$-logic. Thus, the topological semantics – i.e., the current semantics restricted to $n$-spaces – is best seen as a semantics for logics $\mathrm{L} \supseteq \otimes^n\mathrm{S4}$.

**Product topoframes.** Given two 1-topoframes $\mathcal{X}_1 = \langle X_1, Y_1\rangle$ and $\mathcal{X}_2 = \langle X_2, Y_2\rangle$, we define the *product* $\mathcal{X}_1 \times \mathcal{X}_2$ as the topoframe $\langle X_1 \times X_2, Y_1', Y_2'\rangle$, where $Y_i'$ is a binary relation [topology] on $X_1 \times X_2$ iff $Y_i$ is a binary relation [topology] on $X_i$ and

| | | |
|---|---|---|
| If $Y_1$ is a binary relation on $X_1$ | then | $\langle a, b\rangle Y_1' \langle u, v\rangle$ iff $aY_1u$ and $b = v$. |
| If $Y_2$ is a binary relation on $X_2$ | then | $\langle a, b\rangle Y_2' \langle u, v\rangle$ iff $bY_2v$ and $a = u$. |
| If $Y_1$ is a topology on $X_1$ | then | $\{\{O \times \{v\}\} : O \in Y_1$ and $v \in X_2\}$ is a basis for $Y_1'$. |
| If $Y_2$ is a topology on $X_2$ | then | $\{\{\{a\} \times O\} : a \in X_1$ and $O \in Y_2\}$ is a basis for $Y_2'$. |

We call $\mathcal{X}_1 \times \mathcal{X}_2$ a *product topoframe*. If both $\mathcal{X}_1$ and $\mathcal{X}_2$ are frames, then $\mathcal{X}_1 \times \mathcal{X}_2$ is a *product frame* as defined in [12]. If both $\mathcal{X}_1$ and $\mathcal{X}_2$ are topological spaces, then $\mathcal{X}_1 \times \mathcal{X}_2$ is a *product space* as defined in [14]. If $\mathcal{X}_1$ is a topological space and $\mathcal{X}_2$ is a frame, then $\mathcal{X}_1 \times \mathcal{X}_2$ is a topologized frame, as defined above: we call such topologized frames, *product topologized frames*. If $\mathsf{X}$ and $\mathsf{X}'$ are classes of 1-topoframes, then $\mathsf{X} \times \mathsf{X}' =_{\mathrm{df}} \{\mathcal{X} \times \mathcal{X}' : \mathcal{X} \in \mathsf{X}$ and $\mathcal{X}' \in \mathsf{X}'\}$. Now we define three products of 1-modal logics $\mathrm{L}_1$ and $\mathrm{L}_2$: the *frame product* $\mathrm{L}_1 \times \mathrm{L}_2$ of [12], the *topological product* $\mathrm{L}_1 \times_t \mathrm{L}_2$ of [14], and the *topological frame product* $\mathrm{L}_1 \times_{tf} \mathrm{L}_2$:

$$\begin{aligned}
\mathrm{L}_1 \times \mathrm{L}_2 &=_{\mathrm{df}} \mathsf{Log}(\mathsf{Fr}(\mathrm{L}_1) \times \mathsf{Fr}(\mathrm{L}_2)).\\
\mathrm{L}_1 \times_t \mathrm{L}_2 &=_{\mathrm{df}} \mathsf{Log}(\mathsf{Top}(\mathrm{L}_1) \times \mathsf{Top}(\mathrm{L}_2)).\\
\mathrm{L}_1 \times_{tf} \mathrm{L}_2 &=_{\mathrm{df}} \mathsf{Log}(\mathsf{Top}(\mathrm{L}_1) \times \mathsf{Fr}(\mathrm{L}_2)).
\end{aligned}$$

These notions can be generalized to various ways of getting a topoframe product of any list $\mathrm{L}_1, \ldots, \mathrm{L}_n$ of multimodal logics, inserting $\mathsf{Top}(\mathrm{L}_i)$ in some dimensions and $\mathsf{Fr}(\mathrm{L}_i)$ in others. Frame products have been extensively studied: see [4, 3]. Much less is known about topological and topological-frame products, though, as noted above, Kudinov [7, 8, 9] has made significant progress on *neighbourhood products*, a natural generalization of topological products.

**Alexandrov topologies.** A topology on $X$ is *Alexandrov* iff the arbitrary intersection of open sets is open, alternatively, iff every point has a least open neighbourhood. Given any reflexive transitive relation $Y$ on $X$, say that a set $O \subseteq X$ is *open* (in the frame $\langle X, Y \rangle$) iff $O$ is closed under the relation $Y$, i.e., $\forall a \in O, \forall b \in X, (aYb \Rightarrow b \in O)$. Note that the family $\tau(Y)$ of open subsets of $X$ is an Alexandrov topology and moreover that $Int_{\tau(Y)}(S) = Int_Y(S)$, for any $S \subseteq X$. Given any topology $Y$ on $X$, define the reflexive transitive relation $R(Y)$, the *specialization preorder induced by* $Y$: $uR(Y)v$ iff $u \in Cl(\{v\})$, where $Cl$ is topological closure according to $Y$. Note: if $Y$ is a topology on $X$ then, $Y$ is Alexandrov iff $\tau(R(Y)) = Y$ iff, for any $S \subseteq X$, $Int_{R(Y)}(S) = Int_Y(S)$.

Thus, in effect, we can identify topoframes $\mathcal{X} = \langle X, Y_1, \ldots, Y_n \rangle$ and $\mathcal{X}' = \langle X, Y_1', \ldots, Y_n' \rangle$, where for each $i$, either $Y_i = Y_i'$; or $Y_i$ is a reflexive transitive relation and $Y_i' = \tau(Y_i)$; or $Y_i$ is an Alexandrov topology and $Y_i' = R(Y_i)$. And, for logics stronger than $\otimes^n$S4, we can think of the topological semantics as a generalization of the frame semantics.

The identification of an Alexandrov space with its corresponding reflexive transitive frame carries through under products. Suppose that $\mathcal{X}_1 = \langle X_1, Y_1 \rangle$ is a topological space, that $\mathcal{X}_2 = \langle X_2, Y_2 \rangle$ is an Alexandrov space, and that $\mathcal{X}_2^*$ is the reflexive transitive frame $\langle X_2, R(Y_2) \rangle$. Then, if $\mathcal{X}_1 \times \mathcal{X}_2 = \langle X_1 \times X_2, {Y_1}', {Y_2}' \rangle$. Then $\mathcal{X}_1 \times \mathcal{X}_2^* = \langle X_1 \times X_2, {Y_1}', R({Y_2}') \rangle$. Let $\mathsf{Alex}$ be the class of Alexandrov 1-spaces. Note: If $\mathsf{X} \subseteq \mathsf{Alex}$ and $\mathsf{X}^*$ is the corresponding class of reflexive transitive 1-frames, then, for any class $\mathsf{X}'$ of 1-topoframes, $\mathsf{Log}(\mathsf{X}' \times \mathsf{X}) = \mathsf{Log}(\mathsf{X}' \times \mathsf{X}^*)$ and $\mathsf{Log}(\mathsf{X} \times \mathsf{X}') = \mathsf{Log}(\mathsf{X}^* \times \mathsf{X}')$.

## 3. Results

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|---|---|
| In general | $\mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times \mathrm{L}_2$, |
| | $\mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times_{tf} \mathrm{L}_2$, and |
| | $\mathrm{L}_1 \otimes \mathrm{L}_2 \subseteq \mathrm{L}_1 \times_t \mathrm{L}_2$. |
| If $\mathrm{L}_1 \supseteq$ S4 then | $\mathrm{L}_1 \times_{tf} \mathrm{L}_2 \subseteq \mathrm{L}_1 \times \mathrm{L}_2$. |
| If $\mathrm{L}_1, \mathrm{L}_2 \supseteq$ S4 then | $\mathrm{L}_1 \times_t \mathrm{L}_2 \subseteq \mathrm{L}_1 \times_{tf} \mathrm{L}_2$. |
| | |
| Also, since $\mathsf{Top}$(S5) $\subseteq \mathsf{Alex}$ | S4 $\times_t$ S5 = S4 $\times_{tf}$ S5; and |
| | S5 $\times_t$ S5 = S5 $\times_{tf}$ S5 = S5 $\times$ S5. |

Recall Shehtman's [12] definition, given in Section 1, of the *commutator*, $[\mathrm{L}_1, \mathrm{L}_2]$, of $\mathrm{L}_1$ and $\mathrm{L}_2$. It is often easy to construct models to show that $\mathrm{L}_1 \otimes \mathrm{L}_2 \subsetneq [\mathrm{L}_1, \mathrm{L}_2]$, for example when $\mathrm{L}_1, \mathrm{L}_2 \in$ LIST. For the main result concerning frame products, we need the notion of a *Horn axiomatizable* logic. The following characterization is lifted almost verbatim from [3], Section 5.1,

p. 228. Consider the first-order classical language with equality and a binary predicate $R$. A formula in this language is *positive* iff it is built up from atoms using only & and $\vee$. A *universal Horn sentence* is a sentence of the form $\forall x \forall y \forall z_1 \ldots \forall z_n (B \supset Rxy)$, where $B$ is a positive formula. A formula in our modal language $\mathcal{L}_1$ is a *Horn formula* iff there is a universal Horn sentence $A_H$ such that $\mathcal{X} \vDash A$ iff $\mathcal{X} \vDash A_H$, for every 1-frame $\mathcal{X}$. A formula in our modal language $\mathcal{L}_1$ is *variable-free* if it contains no propositional variables, i.e., its only atomic subformula is $\top$. A 1-logic L is *Horn axiomatizable* iff it is axiomatizable by only Horn and variable-free formulas, i.e., if $\mathrm{L} = \mathrm{K} + \Delta$, where $\Delta$ is a set of Horn and variable-free formulas. All the logics in LIST are Horn axiomatizable. The main result for frame-product logics is

**Theorem 3.1.** ([3], Theorem 5.9) $\mathrm{L}_1 \times \mathrm{L}_2 = [\mathrm{L}_1, \mathrm{L}_2]$, *if* $\mathrm{L}_1, \mathrm{L}_2$ *are Kripke complete and Horn axiomatizable.*

Much less is known about topological products. It is a nontrivial theorem of [14] that

**Theorem 3.2.** $\mathrm{S4} \times_t \mathrm{S4} = \mathrm{S4} \otimes \mathrm{S4}$.

On the relationship between frame and topological products, the main result of [6] is

**Theorem 3.3.** *If* $\mathrm{L}_1$ *and* $\mathrm{L}_2$ *are Kripke complete extensions of* S4*, then* $\mathrm{L}_1 \times_t \mathrm{L}_2 = \mathrm{L}_1 \times \mathrm{L}_2$ *iff* $\mathrm{L}_1 \supsetneq \mathrm{S5}$ *or* $\mathrm{L}_2 \supsetneq \mathrm{S5}$ *or* $\mathrm{L}_1 = \mathrm{L}_2 = \mathrm{S5}$.

Our focus here is on topological-frame products: we provide completeness results for $\mathrm{S4} \times_{tf} \mathrm{L}$, when L is a Kripke complete Horn axiomatizable extension of D. As already noted, $\mathrm{S4} \times_{tf} \mathrm{S5} = \mathrm{S4} \times_t \mathrm{S5}$, so we also provide a completeness result for the topological product $\mathrm{S4} \times_t \mathrm{S5}$.

For any 1-logics $\mathrm{L}_1$ and $\mathrm{L}_2$, Kurucz and Zakharyaschev [10] define the *e-commutator* of $\mathrm{L}_1$ and $\mathrm{L}_2$ as follows:

$$[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathrm{L}_1 \otimes \mathrm{L}_2 + com_{\supset} + chr.$$

Shehtman [13] suggests the term *semiproducts* for such logics. Our main theorem is

**Theorem 3.4.** $\mathrm{S4} \times_{tf} \mathrm{L} = [\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$*, if* L *is a Kripke complete Horn axiomatizable extension of* D*.*

**Remark 3.5.** The claim that $[\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}} \subseteq \mathrm{S4} \times_{tf} \mathrm{L}$ is a soundness claim and follows from the more general fact that $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} \subseteq \mathrm{L}_1 \times_{tf} \mathrm{L}_2$, which is easy to check.

**Remark 3.6.** Theorem 3.4 fails for L = K and L = K4. For a counterexample to the claim that $\mathrm{S4} \times_{tf} \mathrm{K} \subseteq [\mathrm{S4}, \mathrm{K}]^{\mathsf{EX}}$, note that $(\Box_2\bot \to \Box_1\Box_2\bot) \in (\mathrm{S4} \times_{tf} \mathrm{K}) - [\mathrm{S4}, \mathrm{K}]^{\mathsf{EX}}$. Ditto with K replaced by K4. Kudinov [9], Proposition 3.5, uses the same example (with the subscripts switched) to show that $\mathrm{K} \times_n \mathrm{K} \not\subseteq \mathrm{K} \otimes \mathrm{K}$, where $\mathrm{K} \times_n \mathrm{K}$ is the *neighbourhood product* of K with itself.

To see that $(\Box_2\bot \to \Box_1\Box_2\bot) \notin [\mathrm{S4}, \mathrm{K4}]^{\mathsf{EX}}$, consider the 2-frame $\mathcal{X} = \langle X, Y_1, Y_2\rangle$, where $X = \{0, 1, 2\}$, $Y_1 = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\}$, and $Y_2 = \{\langle 1, 2\rangle\}$. Since $Y_1$ is reflexive and transitive and $Y_2$ is (trivially) transitive, $\mathcal{X} \vDash \mathrm{S4} \otimes \mathrm{K4}$. It is easy to check that $\mathcal{X} \vDash com_\supset$ and $\mathcal{X} \vDash chr$, but that $\mathcal{X} \nvDash (\Box_2\bot \to \Box_1\Box_2\bot)$.

The fact that $(\Box_2\bot \to \Box_1\Box_2\bot) \in (\mathrm{S4} \times_{tf} \mathrm{K})$, follows from a much more general fact adapted from [9], Lemma 3.6, with the same proof:

1. if $A$ is a variable- and $\Box_1$-free formula, then $A \supset \Box_1 A \in \mathrm{S4} \times_{tf} \mathrm{K}$; and
2. if $A$ is a variable- and $\Box_2$-free formula, then $A \supset \Box_2 A \in \mathrm{S4} \times_{tf} \mathrm{K}$.

**Problem 3.7.** Axiomatize $\mathrm{S4} \times_{tf} \mathrm{L}$ when L = K or K4, or more generally when L is Kripke complete and Horn axiomatizable but not an extension of D. Kudinov [9] solves a similar problem, axiomatizing the neighbourhood product $\mathrm{L}_1 \times_n \mathrm{L}_2$ for a subset of Horn axiomatizable logics, the so-called *HTC-logics*, which include K, K4 and S4 but not S5. In particular, let $\Delta = \{A \supset \Box_1 A : A$ is a variable- and $\Box_1$-free formula$\} \cup \{A \supset \Box_2 A : A$ is a variable- and $\Box_2$-free formula$\}$. Then, $\mathrm{L}_1 \times_n \mathrm{L}_2 = (\mathrm{L}_1 \otimes \mathrm{L}_2) + \Delta$ for any HTC-logics $\mathrm{L}_1$ and $\mathrm{L}_2$, and if $\mathrm{L}_1, \mathrm{L}_2 \supseteq \mathrm{D}$, then $\mathrm{L}_1 \times_n \mathrm{L}_2 = (\mathrm{L}_1 \otimes \mathrm{L}_2)$. We conjecture, in the current case, that $\mathrm{S4} \times_{tf} \mathrm{K} = [\mathrm{S4}, \mathrm{K}]^{\mathsf{EX}} + \Delta$, and similarly for K4.

**Remark 3.8.** In private correspondence, Valentin Shehtman has emphasized a connection between certain bimodal logics and quantified modal logic. In particular, under a simple translation, $[\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ is the one-variable fragment of QS4, standard quantified S4 without identity. Given a quantified modal language with a unary predicate $\overline{p}$ for every $p \in PV$, translate $p$ to $\overline{p}x$; translate Boolean connectives to themselves; translate $\Box_1$ to $\Box$; and translate $\Box_2$ to $\forall x$. Then $A \in [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$ iff $\forall x\overline{A} \in \mathrm{QS4}$, where $\overline{A}$ is the translation of $A$. Given the main completeness theorem in [11] for QS4 in the topological semantics (Chapter XI, Proposition 10.2), we also have $A \in \mathrm{S4} \times_t \mathrm{S5} = \mathrm{S4} \times_{tf} \mathrm{S5}$ iff $\forall x\overline{A} \in \mathrm{QS4}$: thus $\mathrm{S4} \times_t \mathrm{S5} = \mathrm{S4} \times_{tf} \mathrm{S5} = [\mathrm{S4}, \mathrm{S5}]^{\mathsf{EX}}$. We know of no similar application of the topological completeness of QS4 for proving the more general Theorem 3.4.[4]

[4]The proof in the current paper of Theorem 3.4 does borrow some ideas from a com-

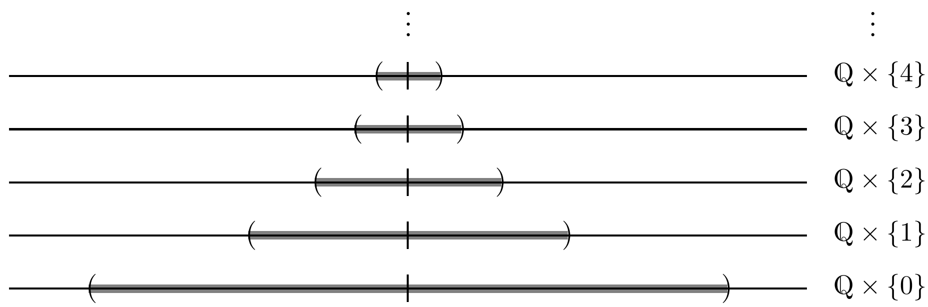


Figure 1. A countermodel for $com_{\subset}$ in $\mathcal{Q} \times \mathcal{N}$: $V(p)$ is in grey.

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Before our main task, a few easy claims. Here, $\mathrm{L} \in \mathrm{LIST} - \{\mathrm{S4}, \mathrm{S5}\}$:

$$\begin{array}{ccccccc} \mathrm{S4} \otimes \mathrm{S5} & \subsetneq & \mathrm{S4} \times_t \mathrm{S5} & = & \mathrm{S4} \times_{tf} \mathrm{S5} & \subsetneq & \mathrm{S4} \times \mathrm{S5}. \\ \mathrm{S4} \otimes \mathrm{S4} & = & \mathrm{S4} \times_t \mathrm{S4} & \subsetneq & \mathrm{S4} \times_{tf} \mathrm{S4} & \subsetneq & \mathrm{S4} \times \mathrm{S4}. \\ \mathrm{S4} \otimes \mathrm{L} & & & \subsetneq & \mathrm{S4} \times_{tf} \mathrm{L} & \subsetneq & \mathrm{S4} \times \mathrm{L}. \end{array}$$

If we replace '$\subsetneq$' by '$\subseteq$', then the above claims follow immediately from what has already been noted. As for the inequalities, first note that $com_{\supset} \in \mathrm{S4} \times_{tf} \mathrm{K}$, which can be easily checked; and $com_{\supset} \notin \mathrm{S4} \otimes \mathrm{S5}$, which is already known. Thus $\mathrm{S4} \otimes \mathrm{L} \subsetneq \mathrm{S4} \times_{tf} \mathrm{L}$, for $\mathrm{L} \in \mathrm{LIST}$. Second note that $com_{\subset} \in \mathrm{S4} \times \mathrm{K}$, which can be easily checked; and $com_{\subset} \notin \mathrm{S4} \times_{tf} \mathrm{S5}$, which we show below. Thus $\mathrm{S4} \times_{tf} \mathrm{L} \subsetneq \mathrm{S4} \times \mathrm{L}$, for $\mathrm{L} \in \mathrm{LIST}$. It remains to show that $com_{\subset} \notin \mathrm{S4} \times_{tf} \mathrm{S5}$.

Let $\mathcal{Q} = \langle \mathbb{Q}, \tau_{\mathbb{Q}} \rangle$, where $\tau_{\mathbb{Q}}$ is the standard topology on the set $\mathbb{Q}$ of rational numbers; and let $\mathcal{N} = \langle \mathbb{N}, R_u \rangle$, where $R_u$ is the universal binary relation on $\mathbb{N}$. Thus $\mathcal{Q} \vDash \mathrm{S4}$ and $\mathcal{N} \vDash \mathrm{S5}$. For each $n \in \mathbb{N}$, let

$$O_n = \{x \in \mathbb{Q} : \frac{-1}{n+1} < x < \frac{1}{n+1}\}.$$

Note that each $O_n$ is open in $\mathcal{Q}$. Write $\mathcal{Q} \times \mathcal{N}$ as $\langle \mathbb{Q} \times \mathbb{N}, \tau_{\mathbb{Q}}', R_u' \rangle$, and let $\mathcal{M}$ be a 2-model $\langle \mathbb{Q} \times \mathbb{N}, \tau_{\mathbb{Q}}', R_u', V \rangle$, where

$$V(p) = \bigcup_{n \in \mathbb{N}} O_n \times \{n\}.$$

---

pleteness proof for QS4, in particular, the proof of the main theorem in [5], that QS4 is complete for the rational line with a countable domain for the quantifiers.

See Figure 1 for a picture. It is easy to check that $V(\Box_2\Box_1 p) = \{0\} \times \mathbb{N}$ but that $V(\Box_1\Box_2 p) = \emptyset$.

## 4. p-morphisms

Let $\mathcal{X} = \langle X, Y_1, \ldots, Y_n\rangle$ and $\mathcal{X}' = \langle X', Y'_1, \ldots, Y'_n\rangle$ be $n$-topoframes, with $i \in \{1, \ldots, n\}$ and $\varphi : X \to X'$. Fix $i \in \{1, \ldots, n\}$. If each of $Y_i$ and $Y'_i$ is either an Alexandrov topology or a reflexive and transitive relation, then $\varphi$ is *$i$-continuous* iff the preimage of every set open in $\langle X', Y'_i\rangle$ is open in $\langle X, Y_i\rangle$; $\varphi$ is *$i$-open* iff the image of every set open in $\langle X, Y_i\rangle$ is open in $\langle X', Y'_i\rangle$; and $\varphi$ is an *$i$-p-morphism* from $\mathcal{X}$ to $\mathcal{X}'$ iff it is $i$-continuous and $i$-open. If each of $Y_i$ and $Y'_i$ is a binary relation on $X$, then $\varphi$ is *$i$-monotone* iff $(\forall a, b \in X)(aY_ib \Rightarrow \varphi(a)Y'_i\varphi(b))$;[5] is *$i$-lifting* iff $(\forall a \in X)(\forall c \in X')(\varphi(a)Y'_ic \Rightarrow (\exists b \in X)(aY_ib \ \&\ \varphi(b) = c))$;[6] and $\varphi$ is an *$i$-p-morphism* from $\mathcal{X}$ to $\mathcal{X}'$ iff it is $i$-monotone and $i$-lifting.

Finally, $\varphi$ is a *p-morphism* from $\mathcal{X}$ to $\mathcal{X}'$ iff $\varphi$ is an $i$-p-morphism for every $i \in \{1, \ldots, n\}$. We say that $\mathcal{X}'$ is a *p-morphic image of* $\mathcal{X}$ iff there is a *surjective* p-morphism from $\mathcal{X}$ to $\mathcal{X}'$.

**Lemma 4.1.** *Suppose that $\mathcal{X}$ and $\mathcal{X}'$ are $n$-topoframes and that $\mathcal{X}'$ is a p-morphic image of $\mathcal{X}$. Then $\mathcal{X} \models A$ iff $\mathcal{X}' \models A$, for every variable-free formula $A$.*

**Lemma 4.2.** *Suppose that $\mathcal{X}$ is an $n$-topoframe and that $\mathsf{X}$ is a class of $n$-topoframes each of which is a p-morphic image of $\mathcal{X}$. Then $\mathsf{Log}(\mathcal{X}) \subseteq \mathsf{Log}(\mathsf{X})$.*

**Lemma 4.3.** *Suppose that $\mathcal{X} = \langle X, Y_1, Y_2\rangle$ is a topologized frame and $\mathcal{X}' = \langle X', Y'_1, Y'_2\rangle$ is a rooted 2-frame, where $Y'_1$ is reflexive and transitive and $r$ is a root. Suppose that $\varphi : X \to X'$ is a p-morphism from $\mathcal{X}$ to $\mathcal{X}'$ with $r \in \varphi[X]$, the image of $X$ under $\varphi$. Then $\varphi$ is surjective.*

Proof. Suppose that $x \in X'$. We want to show that $x \in \varphi[X]$. Since $\mathcal{X}'$ is rooted, there is a path $x_1, \ldots, x_m$ such that $x_1 = r$ and $x_m = x$, and, for every $k \in \{1, \ldots, m-1\}$, either $x_kY'_1x_{k+1}$ or $x_kY'_2x_{k+1}$. It will suffice to show, by induction on $k \in \{1, \ldots, m\}$, that each $x_k \in \varphi[X]$. For the base case, we have $x_1 \in \varphi[X]$, since $x_1 = r$. For the inductive step, suppose that $x_k \in \varphi[X]$, with $k < m$. Then either $x_kY'_1x_{k+1}$ or $x_kY'_2x_{k+1}$. To show that

[5]Note: if $Y_i$ and $Y'_i$ are reflexive transitive relations, then $\varphi$ is $i$-monotone iff $\varphi$ is $i$-continuous.

[6]Note: if $Y_i$ and $Y'_i$ are reflexive transitive relations, then $\varphi$ is $i$-open iff $\varphi$ is $i$-lifting.

$x_{k+1} \in \varphi[X]$, we consider two cases.

**Case (1).** $x_k Y_1' x_{k+1}$. Since $\varphi$ is 1-open, $\varphi[X]$ is open in the frame $\langle X, Y_1' \rangle$. That is, $\varphi[X]$ is closed under the relation $Y_1'$:

$$\forall a \in \varphi[X], \forall b \in X', (a Y_1' b \Rightarrow b \in \varphi[X]).$$

So $x_{k+1} \in \varphi[X]$, since $x_k \in \varphi[X]$ and $x_k Y_1' x_{k+1}$.

**Case (2).** $x_k Y_2' x_{k+1}$. Choose $y \in X$ with $\varphi(y) = x_k$. Since $\varphi$ is 2-lifting,

$$(\forall a \in X)(\forall c \in X')(\varphi(a) Y_2' c \Rightarrow (\exists b \in X)(a Y_2 b \; \& \; \varphi(b) = c)).$$

So $(\exists b \in X)(y Y_2 b \; \& \; \varphi(b) = x_{k+1})$, since $\varphi(y) Y_2' x_{k+1}$. So $x_{k+1} \in \varphi[X]$, as desired. ∎

A final p-morphism lemma that will prove useful is a restatement of Lemma 6.2 (ii) in [5]:

**Lemma 4.4.** *Each countable rooted reflexive and transitive 1-frame is a p-morphic image of $\mathcal{Q}$.*

## 5. Expanding relativized product frames

Suppose that $\mathsf{F}$ is a class of subframes of product 2-frames. Given Kripke complete 1-logics $\mathrm{L}_1$ and $\mathrm{L}_2$, Kurucz and Zakharyaschev [10] define the $\mathsf{F}$-*relativized product*

$$(\mathrm{L}_1 \times \mathrm{L}_2)^{\mathsf{F}} =_{\mathrm{df}} \mathsf{Log}(\mathsf{F} \cap \mathsf{SF}(\mathsf{Fr}(\mathrm{L}_1) \times \mathsf{Fr}(\mathrm{L}_2))).$$

One class of particular interest in [10] is the class $\mathsf{EX}$ of *expanding relativized product frames* (ERPF's): a 2-frame $\mathcal{X} = \langle X, S_1, S_2 \rangle$ is an ERPF iff there are 1-frames $\mathcal{X}_1 = \langle X_1, R_1 \rangle$ and $\mathcal{X}_2 = \langle X_1, R_2 \rangle$ such that

- $\mathcal{X} \sqsubseteq \mathcal{X}_1 \times \mathcal{X}_2$, and
- for all $\langle x_1, x_2 \rangle \in X$ and $x \in X_1$, if $x_1 R_1 x$ then $\langle x, x_2 \rangle \in X$.

Figure 2 represents a product frame $\mathcal{X}_1 \times \mathcal{X}_2 \in \mathsf{Fr}(\mathrm{S4}) \times \mathsf{Fr}(\mathrm{S5})$ together with a subframe which is an ERPF.

We need the notion of a *subframe logic* for the next result: an $n$-logic L is a *subframe logic* iff $\mathsf{SF}(\mathsf{Fr}(\mathrm{L})) \subseteq \mathsf{Fr}(\mathrm{L})$. By Theorem 6 in [10],

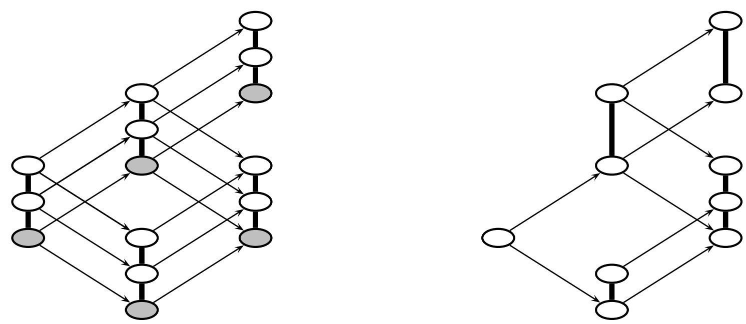

Figure 2. To the left is an example of a product frame of the form $\mathcal{X}_1 \times \mathcal{X}_2 = \langle X_1 \times X_2, R_1{}', R_2{}' \rangle \in \mathsf{Fr}(\mathrm{S4}) \times \mathsf{Fr}(\mathrm{S5})$. $R_1{}'$ is the reflexive transitive closure of the relation given by the thin diagonal arrows, and $R_2{}'$ is the reflexive transitive symmetric closure of the relation given by the thick vertical lines. A copy of $\mathcal{X}_1$ is shaded in grey. To the right is an expanding relativized product frame that is a subframe of $\mathcal{X}_1 \times \mathcal{X}_2$.

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**Lemma 5.1.** $(\mathrm{L}_1 \times \mathrm{L}_2)^{\mathsf{EX}} = [\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}$*, if* $\mathrm{L}_1 \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}, \mathrm{S5}\}$ *and* $\mathrm{L}_2$ *is a Kripke complete Horn axiomatizable subframe logic.*[7]

Lemma 5.1 fails when $\mathrm{L}_2$ is not a subframe logic, since $[\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}} \not\subseteq (\mathrm{S4} \times \mathrm{L})^{\mathsf{EX}}$. For example, if $\mathrm{L} \in \{\mathrm{D}, \mathrm{D4}\}$, then $\Diamond_2 \top \in [\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$ but $\Diamond_2 \top \notin (\mathrm{S4} \times \mathrm{L})^{\mathsf{EX}}$. But even when $\mathrm{L}_2$ is not a subframe logic, the claim that $(\mathrm{L}_1 \times \mathrm{L}_2)^{\mathsf{EX}} \subseteq [\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}$ still goes through, as long as $\mathrm{L}_2$ is Kripke complete and Horn axiomatizable. The proof of this in [10] gives us a result that we will find very useful. Let $\mathsf{CT}$ be the class of countable 2-frames. And let $\mathsf{SR}$ be the class of *strongly rooted* subframes of product 2-frames, in the following sense: a subframe $\mathcal{X}$ of a product frame $\mathcal{X}_1 \times \mathcal{X}_2$ is *strongly rooted* iff $\mathcal{X}_1$ has a root $r_1$ and $\mathcal{X}_2$ has a root $r_2$ such that $\langle r_1, r_2 \rangle$ is a root of $\mathcal{X}$. Then

**Lemma 5.2.** ([10]) *Suppose that* $\mathrm{L}_1 \in \{\mathrm{K}, \mathrm{K4}, \mathrm{T}, \mathrm{S4}, \mathrm{S5}\}$ *and* $\mathrm{L}_2$ *is Kripke complete and Horn axiomatizable. Then every countable rooted* $\mathcal{X} \in \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})$ *is the p-morphic image of some*

$$\mathcal{X}' \in \mathsf{EX} \cap \mathsf{CT} \cap \mathsf{SR} \cap \mathsf{SF}(\mathsf{Fr}(\mathrm{L}_1) \times \mathsf{Fr}(\mathrm{L}_2)).$$

Note the following immediate corollary to Lemmas 5.2 and 4.1, where $\mathsf{2SER}$ is the class of 2-frames $\langle X, R_1, R_2 \rangle$ such that $R_2$ is serial:

[7] In [10], the statement of the theorem omits the necessary stipulation that $\mathrm{L}_2$ be a subframe logic.

**Corollary 5.3.** *Suppose that* L *is a Kripke complete and Horn axiomatizable extension of* D*. Then every countable rooted* $\mathcal{X} \in \mathsf{Fr}([\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}})$ *is the p-morphic image of some*

$$\mathcal{X}' \in \mathsf{EX} \cap \mathsf{2SER} \cap \mathsf{CT} \cap \mathsf{SR} \cap \mathsf{SF}(\mathsf{Fr}(\mathrm{S4}) \times \mathsf{Fr}(\mathrm{L})).$$

Proof. Suppose that L is a Kripke complete and Horn axiomatizable extension of D and that $\mathcal{X}$ is countable and rooted and $\in \mathsf{Fr}([\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}})$. By Lemma 5.2, $\mathcal{X}$ is the p-morphic image of some $\mathcal{X}' \in \mathsf{EX} \cap \mathsf{CT} \cap \mathsf{SR} \cap \mathsf{SF}(\mathsf{Fr}(\mathrm{S4}) \times \mathsf{Fr}(\mathrm{Ł}))$. Note that $\mathcal{X} \vDash \Diamond_2 \top$, since $\mathrm{L} \supseteq \mathrm{D}$ and $\mathcal{X} \in \mathsf{Fr}([\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}})$. So $\mathcal{X}' \vDash \Diamond_2 \top$, by Lemma 4.1. So $\mathcal{X}' \in \mathsf{2SER}$. ∎

We add one more lemma about $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}$. Let $\mathsf{ROOTED}$ be the class of rooted 2-frames.

**Lemma 5.4.** *Suppose that* $\mathrm{L}_1$ *and* $\mathrm{L}_2$ *are Kripke complete and Horn axiomatizable* 1*-logics. Then*

$$[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathsf{Log}(\mathsf{CT} \cap \mathsf{ROOTED} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})).$$

Proof. By an argument nearly identical to the proof of Proposition 5.7 in [3], $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}$ is Kripke complete. That is, $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathsf{Log}(\mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}))$. The following three claims suffice for our result:

(1) $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathsf{Log}(\mathsf{CT} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}))$,

(2) $\mathsf{Log}(\mathsf{CT} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})) \subseteq \mathsf{Log}(\mathsf{CT} \cap \mathsf{ROOTED} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}))$, and

(3) $\mathsf{Log}(\mathsf{CT} \cap \mathsf{ROOTED} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})) \subseteq \mathsf{Log}(\mathsf{CT} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}))$.

*Re* Item (1). Let $\mathcal{L}_R$ be the first-order classical language with equality and a binary predicate $R$; and let $\mathcal{L}_{R_1,R_2}$ be the first-order classical language with equality and two binary predicates $R_1$ and $R_2$. Recall that each $\mathrm{L}_i = \mathrm{K} + \Delta_i$, where $\Delta_i$ is a set of Horn and variable-free formulas in the modal language $\mathcal{L}_1$, for $i = 1$ or 2. By the definition of *Horn formula* (page 7), for each Horn formula $A$ of the modal language $\mathcal{L}_1$, there is a sentence $A^*$ in $\mathcal{L}_R$ such that $\mathcal{X} \vDash A$ iff $\mathcal{X} \vDash A^*$, for every 1-frame $\mathcal{X}$. Similarly, by Lemma 5.6 in [3], for each variable-free formula $A$ of the modal language $\mathcal{L}_1$, there is a sentence $A^*$ in $\mathcal{L}_R$ such that $\mathcal{X} \vDash A$ iff $\mathcal{X} \vDash A^*$, for every 1-frame $\mathcal{X}$. For $i = 1$ or 2, let $\Gamma_i = \{A^* : A \in \Delta_i\}$. For $i = 1$ or 2, let $\Sigma_i$ be the set of sentences in $\mathcal{L}_{R_1,R_2}$ got by replacing $R$ in the sentences of $\Gamma_i$ with $R_i$.

Recall that $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathrm{L}_1 \otimes \mathrm{L}_2 + com_{\supset} + chr$. Thus, $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathrm{K}_2 + \Delta_1 + \Delta_2' + com_{\supset} + chr$, where $\Delta_2'$ the set of formulas of $\mathcal{L}_2$ got by replacing each occurrence of $\Box_1$ in each formula in $\Delta_2$ by $\Box_2$. Let $Com$ and $Chr$ be the following sentences of $\mathcal{L}_{R_1,R_2}$:

$$Com \quad \forall x \forall y \forall z (R_2xy \;\&\; R_1yz \supset \exists u(R_1xu \;\&\; R_2uz))$$
$$Chr \quad \forall x \forall y \forall z (R_2xy \;\&\; R_1xz \supset \exists u(R_1yu \;\&\; R_2zu))$$

As noted in [3], Section 5.1, for any 2-frame $\mathcal{X}$, we have $\mathcal{X} \vDash com_{\supset}$ iff $\mathcal{X} \vDash Com$ and $\mathcal{X} \vDash chr$ iff $\mathcal{X} \vDash Chr$. Thus, the class $\mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})$ of frames is defined by the following set of sentences in $\mathcal{L}_{R_1,R_2}$: $\Sigma_1 \cup \Sigma_2 \cup \{Com, Chr\}$. Thus $\mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})$ is first-order definable in the language $\mathcal{L}_{R_1,R_2}$. Theorem 1.6 in [3] says that if L is a $n$-modal logic and $\mathrm{L} = \mathsf{Log}(\mathsf{F})$ for some class $\mathsf{F}$ of frames definable in a first order language with $n$ binary relation symbols and equality, then L is determined by the class of its countable frames, i.e., $\mathrm{L} = \mathsf{Log}(\mathsf{CT} \cap \mathsf{Fr}(\mathrm{L}))$. Therefore, $[\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}} = \mathsf{Log}(\mathsf{CT} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}))$.

*Re* Items (2) and (3). Item (2) is trivial. As for Item (3), suppose that $A \notin \mathsf{Log}(\mathsf{CT} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}}))$, where $A$ is in the language $\mathcal{L}_2$. Then, for some frame $\mathcal{X} = \langle X, Y_1, Y_2 \rangle \in \mathsf{CT} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})$ and some model $\mathcal{M} = \langle X, Y_1, Y_2, V \rangle$, we have $V(A) \neq X$. Choose $x \in X - V(A)$ and let $\mathcal{X}' = \langle X', Y_1', Y_2' \rangle$ be the subframe of $\mathcal{X}$ with root $x$: $X' = \{y \in X : \text{there is a path from } x \text{ to } y\}$ and $Y_i'$ is $Y_i$ restricted to $X'$. And let $\mathcal{M}' = \langle X', Y_1', Y_2', V' \rangle$, where $V'(p) = V(p) \cap X'$ for each propopsitional variable $p$. By a standard inductive argument, we have $V'(B) = V(B) \cap X'$, for each formula $B$ of $\mathcal{L}_2$. So $x \in X' - V'(A)$. So $\mathcal{X}' \nvDash A$, where $\mathcal{X}' \in \mathsf{CT} \cap \mathsf{ROOTED} \cap \mathsf{Fr}([\mathrm{L}_1, \mathrm{L}_2]^{\mathsf{EX}})$. ■

## 6. Universal Horn sentences

Consider the first-order classical language with equality and a binary predicate $R$. We will appeal to the following claim about universal Horn sentences, as defined on page 7.

**Lemma 6.1.** *Suppose that $\Gamma$ is a set of universal Horn sentences in the first-order classical language and that $\mathcal{X} = \langle X, R \rangle$ is a frame. Then*

1. *There is a smallest relation $R^*$ on $X$ such that $R \subseteq R^*$ and $\mathcal{X}^* \vDash \Gamma$, where $\mathcal{X}^* = \langle X, R^* \rangle$; and*
2. *if $\mathcal{X}' = \langle X', R' \rangle$ is a frame with $\mathcal{X}' \vDash \Gamma$, then any monotone function from $\mathcal{X}$ to $\mathcal{X}'$ is also a monotone function from $\mathcal{X}^*$ to $\mathcal{X}'$.*

PROOF. The following proof is adapted from the proof of Lemma 5.8 in [3]. We start by defining a sequence $R_0 \subseteq R_1 \subseteq \ldots \subseteq R_n \ldots$ of relations on $X$ as follows:

$$\begin{aligned} R_0 &= R \\ R_{n+1} &= R_n \cup \{\langle a,b\rangle \in X \times X : \langle X, R_n\rangle \Vdash \exists z_1 \ldots \exists z_n B(a,b,z_1,\ldots,z_n) \\ &\quad \text{for some } B \text{ with } \forall x \forall y \forall z_1 \ldots \forall z_n (B \supset Rxy) \in \Gamma\}. \end{aligned}$$

Let

$$R^* =_{\mathrm{df}} \bigcup_{n \in \mathbb{N}} R_n.$$

Note, for Item (1) that $R^*$ is the smallest relation on $X$ such that $R \subseteq R^*$ and $\mathcal{X}^* \vDash \Gamma$, where $\mathcal{X}^* = \langle X, R^*\rangle$.

As for Item (2), suppose that $\varphi$ is a monotone function from $\mathcal{X}$ to $\mathcal{X}'$, where $\mathcal{X}' = \langle X', R'\rangle$ is a frame with $\mathcal{X}' \vDash \Gamma$. It will suffice to prove that $\varphi$ is a monotone function from each $\mathcal{X}_n = \langle X, R_n\rangle$ to $\mathcal{X}'$. By assumption, this is so for $n = 0$. Suppose that $\varphi$ is a monotone function from $\mathcal{X}_n$ to $\mathcal{X}'$, and consider $\mathcal{X}_{n+1}$. Suppose that $\langle a,b\rangle \in R_{n+1}$: we want to show that $\langle \varphi(a), \varphi(b)\rangle \in R'$. Since $\langle a,b\rangle \in R_{n+1}$, we have $\langle X, R_n\rangle \Vdash \exists z_1 \ldots \exists z_n B(a,b,z_1,\ldots,z_n)$, where $\forall x \forall y \forall z_1 \ldots \forall z_n (B \supset Rxy) \in \Gamma$. Choose $c_1, \ldots, c_n \in X$ such that $\mathcal{X}_n = \langle X, R_n\rangle \Vdash B(a,b,c_1,\ldots,c_n)$. Since $B$ is positive and since $\varphi$ is a monotone function from $\mathcal{X}_n$ to $\mathcal{X}'$, we have

$$\mathcal{X}' \Vdash B(\varphi(a), \varphi(b), \varphi(c_1), \ldots, \varphi(c_n)).$$

And since $\mathcal{X}' \Vdash \Gamma$ and

$$\forall x \forall y \forall z_1 \ldots \forall z_n (B \supset Rxy) \in \Gamma,$$

we have $\mathcal{X}' \Vdash R\varphi(a)\varphi(b)$. Thus $\langle \varphi(a), \varphi(b)\rangle \in R'$, as desired. ■

## 7. Proving Theorem 3.4

First note that if L is inconsistent, i.e., if L is the set of formulas of $\mathcal{L}_1$, then it is trivial that $\mathrm{S4} \times_{tf} \mathrm{L} = [\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$, since each of $\mathrm{S4} \times_{tf} \mathrm{L}$ and $[\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$ is then inconsistent, i.e., they are each the set of formulas of $\mathcal{L}_2$. So we will prove Theorem 3.4 for *consistent* Kripke complete Horn axiomatizable $\mathrm{L} \supseteq \mathrm{D}$. Note that, for any such L, there is a set $\Delta_{\mathrm{L}}$ of Horn formulas such that $\mathrm{L} = \mathrm{D} + \Delta_{\mathrm{L}}$: the reason is that the addition of variable-free formulas not already in D will produce the inconsistent logic in $\mathcal{L}_1$. Now, assume a first-order classical language with a binary predicate $R$ and equality, as on Page

7. For each Horn formula $A$, let $A_H$ be the corresponding universal Horn sentence in the first-order language. And for each consistent Kripke complete Horn axiomatizable $\mathrm{L} \supseteq \mathrm{D}$ with $\mathrm{L} = \mathrm{D} + \Delta_{\mathrm{L}}$, let $\Gamma_{\mathrm{L}} = \{A_H : A \in \Delta_{\mathrm{L}}\}$.

Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers. We use $\Lambda$ for the empty sequence in $\mathbb{N}^*$ and $\langle n \rangle$ for the sequence, of length 1, consisting of the natural number $n \in \mathbb{N}$. For $\overline{a} \in \mathbb{N}^*$, we write $\overline{a} = \langle a_0, \ldots, a_{\mathsf{ln}(\overline{a})-1} \rangle$, where $\mathsf{ln}(\overline{a})$ is the length of $\overline{a}$. For $\overline{a}, \overline{b} \in \mathbb{N}^*$, we write $\overline{a}\overline{b}$ for $\overline{a}$ concatenated with $\overline{b}$. And for $\overline{a} \in \mathbb{N}^*$ and $n \in \mathbb{N}$, we write $\overline{a}n$ for $\overline{a}$ concatenated with $\langle n \rangle$. Let $\lhd$ be the relation $\lhd =_{\mathrm{df}} \{\langle \overline{a}, \overline{a}n \rangle : \overline{a} \in \mathbb{N}^* \ \& \ n \in \mathbb{N}\}$. Let $\mathcal{N}^* = \langle \mathbb{N}^*, \lhd \rangle$. For each consistent Kripke complete Horn axiomatizable extension L of D, let $R_{\mathrm{L}}$ be the smallest relation on $\mathbb{N}^*$ such that $\lhd \subseteq R_{\mathrm{L}}$ and $\mathcal{N}_{\mathrm{L}} \vDash \Gamma_{\mathrm{L}}$, where $\mathcal{N}_{\mathrm{L}} = \langle \mathbb{N}^*, R_{\mathrm{L}} \rangle$: such an $R_{\mathrm{L}}$ exists, by Lemma 6.1, Item (1). Note that $\mathcal{N}_{\mathrm{L}} \in \mathsf{Fr}(\mathrm{L})$, since $\mathcal{N}_{\mathrm{L}} \vDash \Gamma_{\mathrm{L}}$. And by Lemma 6.1, Item (2), we get:

**Lemma 7.1.** *Suppose that $\mathcal{X} \in \mathsf{Fr}(\mathrm{L})$, where* L *is a consistent Kripke complete Horn axiomatizable extension of* D*. Then any monotone function from $\mathcal{N}^*$ to $\mathcal{X}$ is also a monotone function from $\mathcal{N}_{\mathrm{L}}$ to $\mathcal{X}$.*

The next lemma is *the* central lemma in the proof of Theorem 3.4. We devote subsections 7.1 and 7.2 to proving this lemma. Then, in subsection 7.3, we pull it all together and complete the proof of Theorem 3.4, for any consistent Kripke complete Horn axiomatizable extension of D.

**Lemma 7.2.** *Suppose that*

1. L *is a consistent Kripke complete Horn axiomatizable extension of* D*;*
2. $\mathcal{X}_1 \in \mathsf{Fr}(\mathrm{S4})$ *is a countable rooted 1-frame;*
3. $\mathcal{X}_2 \in \mathsf{Fr}(\mathrm{L})$ *is a countable rooted 1-frame;*
4. $\mathcal{X} \in \mathsf{EX} \cap \mathsf{2SER} \cap \mathsf{SF}(\mathcal{X}_1 \times \mathcal{X}_2)$*; and*
5. $\langle r_1, r_2 \rangle$ *is a root of $\mathcal{X}$, where $r_1$ is a root of $\mathcal{X}_1$ and $r_2$ is a root of $\mathcal{X}_2$.*

*Then there is a surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ onto $\mathcal{X}$.*

Assume Items (1)-(5) in the statement of Lemma 7.2. Write $\mathcal{X}_1$ as $\langle X_1, R_1 \rangle$; $\mathcal{X}_2$ as $\langle X_2, R_2 \rangle$; $\mathcal{X}_1 \times \mathcal{X}_2$ as $\langle X_1 \times X_2, {R_1}', {R_2}' \rangle$; $\mathcal{X}$ as $\langle X, S_1, S_2 \rangle$; and $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ as $\langle \mathbb{Q} \times \mathbb{N}^*, {\tau_{\mathbb{Q}}}', {R_{\mathrm{L}}}' \rangle$. For $x \in X_i$ we let $R_i(x) = \{y \in X_i : xR_iy\}$. And for $\langle u, v \rangle \in X$, we let $S_i(\langle u, v \rangle) = \{\langle u', v' \rangle \in X : \langle u, v \rangle S_i \langle u', v' \rangle\}$. A couple of observations.

1. $R_1$ is reflexive and transitive, since $\mathcal{X}_1 \in \mathsf{Fr}(\mathrm{S4})$. Thus, both ${R_1}'$ and $S_1$ are reflexive and transitive as well.

2. For $\langle u,v\rangle \in X$, $S_1(\langle u,v\rangle) = R_1(u) \times \{v\}$ and $S_2(\langle u,v\rangle) \subseteq \{u\} \times R_2(v)$.
   For a proof of this observation, suppose that $\langle u,v\rangle \in X$. For the second conjunct, suppose that $\langle x,y\rangle \in S_2(\langle u,v\rangle)$. Note that $S_2 \subseteq R_2{}'$, since $X \sqsubseteq X_1 \times X_2$. So $\langle x,y\rangle \in R_2{}'(\langle u,v\rangle)$. So $x = u$ and $vR_2y$. So $x \in \{u\}$ and $y \in R_2(v)$. So $\langle x,y\rangle \in \{u\} \times R_2(v)$, as desired. As for the first conjunct, the fact that $S_1(\langle u,v\rangle) \subseteq R_1(u) \times \{v\}$ is proved by a similar argument. To see that $R_1(u) \times \{v\} \subseteq S_1(\langle u,v\rangle)$, suppose that $\langle x,y\rangle \in R_1(u) \times \{v\}$. Then $uR_1x$ and $y = v$. So $\langle u,y\rangle \in X$ and $uR_1x$. So, since $X \in \mathsf{EX}$, we have $\langle x,y\rangle \in X$. Thus, since $X \sqsubseteq X_1 \times X_2$ and since $\langle u,v\rangle R_1{}'\langle x,y\rangle$, we also have $\langle u,v\rangle S_1\langle x,y\rangle$. So $\langle x,y\rangle \in S_1(\langle u,v\rangle)$, as desired.

Some more notation: for any ordered pair $\langle u,v\rangle$, we write $\mathsf{lft}(\langle u,v\rangle) = u$ for the left coordinate and $\mathsf{rt}(\langle u,v\rangle) = v$ for the right coordinate. We now set out to construct a surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ onto $\mathcal{X}$.

### 7.1. Constructing a surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ to $\mathcal{X}$: informal outline

We can think of $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ as a kind of big tree whose nodes are copies of $\mathbb{Q}$, one copy for each $\overline{a} \in \mathbb{N}^*$. Our construction will, in effect, unravel $\mathcal{X}$ into this big tree. In this subsection, we given an informal outline the construction, introducing useful concepts along the way. In subsection 7.2 we give a formally precise construction and prove that the resulting function is indeed a surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ to $\mathcal{X}$.

By Lemma 4.4, there is already a p-morphism, say $\varphi$, from $\mathcal{Q}$ onto $\mathcal{X}_1$: we will rely heavily on $\varphi$ as we proceed. In particular, we will define functions $\varphi_{\overline{a}} : \mathbb{Q} \to X$, each of which will be continuous and open as functions from $\mathcal{Q}$ to $\langle X, S_1\rangle$. The left coordinate of $\varphi_{\overline{a}}(q)$ will always be $\varphi(q)$: the trick will be to find a good right coordinate. Ultimately, the $\varphi_{\overline{a}}$'s will be constructed so that the following function will be our surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ to $\mathcal{X}$: $\psi(q,\overline{a}) = \varphi_{\overline{a}}(q)$.

The definition of $\varphi_{\Lambda}$ is simple: $\varphi_{\Lambda}(q) = \langle \varphi(q), r_2\rangle$. Thus, the $\Lambda^{th}$ copy of $\mathbb{Q}$ is mapped to a copy of $X_1$ in $X$, namely $X_1 \times \{r_2\}$. Note that $\varphi_{\Lambda}$ is a continuous open function from $\mathcal{Q}$ to $\langle X, S_1\rangle$.

The next step is to consider $\varphi_{\overline{a}}$ when $\overline{a}$ is an immediate successor of $\Lambda$, i.e., when $\overline{a} = \langle n\rangle$ for some $n \in \mathbb{N}$. First we enumerate $\mathbb{Q} \times X_2$:

$$\langle q_0, x_0\rangle, \langle q_1, x_1\rangle, \langle q_2, x_2\rangle, \ldots, \langle q_n, x_n\rangle, \ldots$$

We will worry, for now, about defining $\varphi_{\langle n\rangle}$ when $\langle \varphi(q_n), x_n\rangle \in X$. So we want to define $\varphi_{\langle n\rangle}(q)$, for every $q \in \mathbb{Q}$. Since the left coordinate of $\varphi_{\langle n\rangle}(q)$

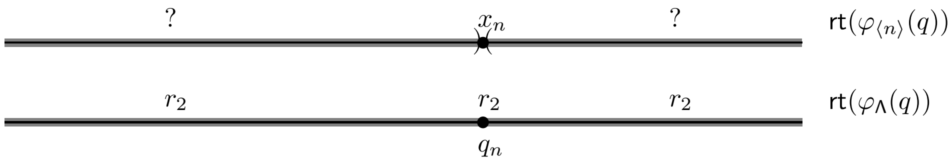


Figure 3. How to define $\varphi_{\langle n\rangle}$? We mark individual points on $\mathbb{Q}$ with dots and regions with grey lines. Above a point or a region, we indicate the right coordinate of $\varphi_{\wedge}(q)$ and $\varphi_{\langle n\rangle}(q)$ at that point or throughout that region. So far we have defined $\varphi_{\wedge}(q) = \langle\varphi(q), r_2\rangle$, for every $q \in \mathbb{Q}$ and we have defined $\varphi_{\langle n\rangle}(q_n) = \langle\varphi(q), x_n\rangle$.

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will simply be $\varphi(q)$, the hard question is how to fill in the question mark in $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), ?\rangle$. We will ensure that $\varphi_{\langle n\rangle}(q_n) = \langle\varphi(q_n), x_n\rangle$. But what about $\varphi_{\langle n\rangle}(q)$ when $q \neq q_n$? For some $q \in \mathbb{Q}$, it could be that $\langle\varphi(q), x_n\rangle \notin X$. So we will not want $\mathsf{rt}(\varphi_{\langle n\rangle}(q)) = x_n$ for every $q \in \mathbb{Q}$. What we *will* want is $\mathsf{rt}(\varphi_{\langle n\rangle}(q)) = x_n$ throughout some interval around $q_n$. Figure 3 illustrates our current predicament.

**Digression on intervals.** An *open* $\mathbb{Q}$*-interval* is any set of the form $\mathbb{Q} \cap (a, b)$, where $(a, b)$ is some open interval in the real line. We write $(a, b)_{\mathbb{Q}}$ for this open $\mathbb{Q}$-interval. Note that $a$ or $b$ could be irrational: if both $a$ and $b$ are irrational then we say that $(a, b)_{\mathbb{Q}}$ is an *irrational interval.* Every irrational interval is clopen, i.e., both an open and a closed subset of $\mathbb{Q}$. If $x \in X_1$, we say that an open $\mathbb{Q}$-interval $I$ is an $x$*-interval* iff the image of $I$ under $\varphi$ is $R_1(x) =_{\mathrm{df}} \{x' \in X_1 : xRx'\}$.

**Lemma 7.3.** *For every $q \in \mathbb{Q}$, there is an irrational $\varphi(q)$-interval $I$ such that $q \in I$.*

Proof. Choose any $q \in \mathbb{Q}$. Note that $R_1(\varphi(q))$ is open in the Alexandrov space $\langle X_1, \tau(R_1)\rangle$. So, since $\varphi$ is a continuous function from $\mathcal{Q}$ to $\langle X_1, \tau(R_1)\rangle$, there is a $\mathbb{Q}$-interval $J$ such that $q \in J$ and $\varphi(y) \in R_1(\varphi(q))$ for every $y \in J$.

Let $I$ be any irrational $\mathbb{Q}$-interval such that $q \in I \subseteq J$. It will suffice to show that $I$ is a $\varphi(q)$-interval. Note that $q \in I$ and $\varphi(y) \in R_1(\varphi(q))$ for every $y \in I$. Let $O = \varphi[I]$ be the image of $I$ under $\varphi$: then $\varphi(q) \in O$ and $O \subseteq R_1(\varphi(q))$. Since $\varphi$ is an open function, $O$ is open in $\langle X_1, \tau(R_1)\rangle$, so that $(\forall u \in O)(\forall v \in X_1)(uR_1v \Rightarrow v \in O)$. So, since $\varphi(q) \in O$, $R_1(\varphi(q)) \subseteq O$. Thus $\varphi[I] = O = R_1(\varphi(q))$. So $I$ a $\varphi(q)$-interval, as desired. ■

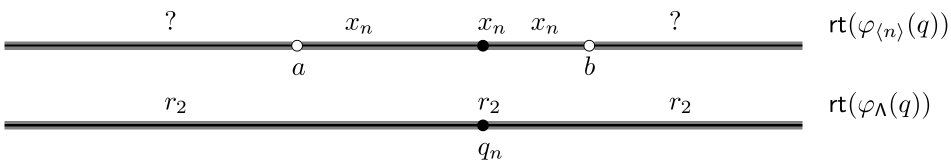


Figure 4. Continuing to define $\varphi_{\langle n\rangle}$. We mark $a$ and $b$ with open dots since they are irrational and thus not on the rational line: they are holes on the rational line. We define $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), x_n\rangle$, for every $q \in (a, b)_{\mathbb{Q}}$.

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**End of digression on intervals.** Now let us return to the construction of $\varphi_{\langle n\rangle}$. We have to fill in the question mark in $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), ?\rangle$ when $q \neq q_n$. Choose an irrational $\varphi(q_n)$-interval, $(a, b)_{\mathbb{Q}}$ with $q_n \in (a, b)_{\mathbb{Q}}$. For every $q \in (a, b)_{\mathbb{Q}}$, we will let $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), x_n\rangle$ as in Figure 4.

**Default successors.** We still have to fill in the question mark in $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), ?\rangle$ when $q \in \mathbb{Q} - (a, b)_{\mathbb{Q}}$. Bearing in mind that $\varphi_{\wedge}(q) = \langle\varphi(q), r_2\rangle$, it will be useful to find an $x \in X_2$ such that $r_2 R_2 x$ and $\langle\varphi(q), x\rangle \in X$, for every $q \in \mathbb{Q} - (a, b)_{\mathbb{Q}}$. Then we can simply fill in the question mark with $x$. For this, it will suffice to have a default $\langle r_1, x\rangle \in X$ such that $\langle r_1, r_2\rangle\ S_2\ \langle r_1, x\rangle$: since $X \in \mathsf{EX}$, we will also have $\langle\varphi(q), r_2\rangle\ S_2\ \langle\varphi(q), x\rangle$, for every $q \in \mathbb{Q} - (a, b)_{\mathbb{Q}}$. More generally, it will be useful as we proceed, to choose, for each $\langle u, v\rangle \in X$, a *default successor* (relative to the relation $S_2$), $\langle u, v'\rangle = \mathsf{defsucc}(\langle u, v\rangle) \in X$ such that $\langle u, v\rangle\ S_2\ \langle u, v'\rangle$. This can be done, since $X \in \mathsf{2SER}$. To fill in the question mark in $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), ?\rangle$, we take the right coordinate of $\mathsf{defsucc}(\langle r_1, r_2\rangle)$ as follows: for each $q \notin (a, b)_{\mathbb{Q}}$, set $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), \mathsf{rt}(\mathsf{defsucc}(\langle r_1, r_2\rangle))\rangle$. See Figure 5.

**Anchors.** An important point: To fill in the question mark in $\varphi_{\langle n\rangle}(q) = \langle\varphi(q), ?\rangle$ when $q \notin (a, b)_{\mathbb{Q}}$, we did not use the default successor of $\langle\varphi(q), r_2\rangle$: rather, we used the right coordinate of the default successor of $\langle r_1, r_2\rangle$. The reason is this: for every $q \notin (a, b)_{\mathbb{Q}}$, we want the right coordinate of $\varphi_{\langle n\rangle}(q)$ to be the same. This ensures that $\varphi_{\langle n\rangle}$ is continuous and open on $\mathbb{Q} - (a, b)_{\mathbb{Q}}$. We will think of the point $\langle r_1, r_2\rangle$ as an *anchor* for all the points in $\mathbb{Q} - (a, b)_{\mathbb{Q}}$ in the following sense: for every $q \in \mathbb{Q} - (a, b)_{\mathbb{Q}}$, we have $\langle r_1, r_2\rangle\ S_1\ \varphi_{\wedge}(q)$. Since $\mathsf{defsucc}(\langle r_1, r_2\rangle) \in X \in \mathsf{EX}$, we also have $\langle\varphi(q), \mathsf{rt}(\mathsf{defsucc}(\langle r_1, r_2\rangle))\rangle \in X$ when $q \in \mathbb{Q} - (a, b)_{\mathbb{Q}}$. Indeed, as we proceed

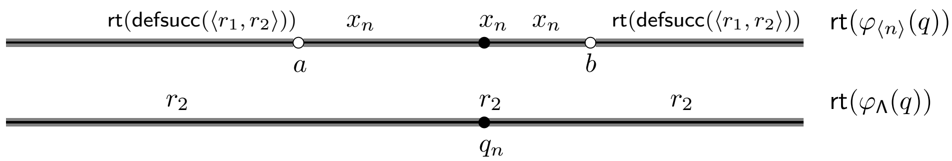


Figure 5. $\varphi_{\langle n\rangle}$ is defined.

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in the construction of each $\varphi_{\overline{a}}$, we will keep track of the anchors by defining $\mathsf{anchor}_{\overline{a}}(q) \in X$ for each $\overline{a} \in \mathbb{N}^*$ and each $q \in \mathbb{Q}$. So far, $\mathsf{anchor}_{\Lambda}(q) = \langle r_1, r_2\rangle$, for every $q \in \mathbb{Q}$. And

$$\mathsf{anchor}_{\langle n\rangle}(q) \quad = \quad \begin{cases} \langle \varphi(q_n), x_n\rangle \text{ if } q \in (a,b)_{\mathbb{Q}}; \\ \mathsf{defsucc}(\mathsf{anchor}_{\Lambda}(q)) \text{ otherwise.} \end{cases}$$

**Special clopen sets.** Note that, in moving from $\varphi_{\Lambda}$ to $\varphi_{\langle n\rangle}$, we cut an irrational interval $(a,b)_{\mathbb{Q}}$ out of $\mathbb{Q}$ for special attention. We will think of this as dividing $\mathbb{Q}$ into two clopen sets, $(a,b)_{\mathbb{Q}}$ and $\mathbb{Q} - (a,b)_{\mathbb{Q}}$. As the inductive construction of each $\varphi_{\overline{a}}$ proceeds, we will further subdivide $\mathbb{Q}$ into clopen sets. It will be useful to keep track, for each $q \in \mathbb{Q}$, which clopen set of which $q$ is a member at the $\overline{a}^{th}$ stage of the construction. Thus, we will define the clopen set $O_{\overline{a}}(q)$ for each $q \in \mathbb{Q}$. So far we have,

$$\begin{aligned} O_{\Lambda}(q) &= \mathbb{Q}, \text{ for every } q \in \mathbb{Q}. \\ O_{\langle n\rangle}(q) &= \begin{cases} (a,b)_{\mathbb{Q}}, \text{ for every } q \in (a,b)_{\mathbb{Q}}; \\ \mathbb{Q} - (a,b)_{\mathbb{Q}}, \text{ for every } q \in \mathbb{Q} - (a,b)_{\mathbb{Q}}. \end{cases} \end{aligned}$$

### 7.2. Constructing a surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathbf{L}}$ to $\mathcal{X}$: formal definitions

For each $\overline{a} \in \mathbb{N}^*$, we will simultaneously define three functions by induction on the construction of $\overline{a}$:

$$\begin{array}{rl} O_{\overline{a}} & \text{assigning a clopen subset of } \mathbb{Q} \text{ to each } q \in \mathbb{Q}; \\ \mathsf{anchor}_{\overline{a}} & \text{assigning a member of } X \text{ to each } q \in \mathbb{Q}; \text{ and} \\ \varphi_{\overline{a}} & \text{assigning a member of } X \text{ to each } q \in \mathbb{Q}. \end{array}$$

Before we give the definitions, recall the enumeration of $\mathbb{Q} \times X_2$ on page 17:

$$\langle q_0, x_0\rangle, \langle q_1, x_1\rangle, \langle q_2, x_2\rangle, \ldots, \langle q_n, x_n\rangle, \ldots$$

Also recall that on page 21 we chose, for each $\langle u,v\rangle \in X$, a default successor (relative to the relation $S_2$), $\langle u,v'\rangle = \mathsf{defsucc}(\langle u,v\rangle) \in X$ such that $\langle u,v\rangle\ S_2\ \langle u,v'\rangle$. For each $n\in\mathbb{N}$, choose an irrational $\varphi(q_n)$-interval $I_n$ such that $q_n \in I_n$: such intervals can be chosen by Lemma 7.3.

Now for the definitions. The base case, $O_\Lambda$, $\mathsf{anchor}_\Lambda$, and $\varphi_\Lambda$:

$$\begin{aligned} O_\Lambda(q) &= \mathbb{Q}, \text{ for every } q\in\mathbb{Q} \\ \mathsf{anchor}_\Lambda(q) &= \langle r_1, r_2\rangle, \text{ for every } q\in\mathbb{Q} \\ \varphi_\Lambda(q) &= \langle \varphi(q), r_2\rangle, \text{ for every } q\in\mathbb{Q} \end{aligned}$$

Assuming that $O_{\overline{a}}$, $\mathsf{anchor}_{\overline{a}}$, and $\varphi_{\overline{a}}$ have been defined, we must define $O_{\overline{a}n}$, $\mathsf{anchor}_{\overline{a}n}$, and $\varphi_{\overline{a}n}$. We consider two cases:

1. $\varphi_{\overline{a}}(q_n)\ S_2\ \langle\varphi(q_n), x_n\rangle \in X$; and
2. either $\langle\varphi(q_n), x_n\rangle \notin X$ or $\varphi_{\overline{a}}(q_n)\ \not{S}_2\ \langle\varphi(q_n), x_n\rangle$.

Our definition of $O_{\overline{a}n}$, $\mathsf{anchor}_{\overline{a}n}$, and $\varphi_{\overline{a}n}$ is as follows:

$$O_{\overline{a}n}(q) = \begin{cases} O_{\overline{a}}(q_n)\cap I_n, \text{ if } q\in O_{\overline{a}}(q_n)\cap I_n, \text{ in Case (1)} \\ O_{\overline{a}}(q_n) - I_n, \text{ if } q\in O_{\overline{a}}(q_n) - I_n, \text{ in Case (1)} \\ O_{\overline{a}}(q), \text{ if } q\notin O_{\overline{a}}(q_n), \text{ in Case (1)} \\ O_{\overline{a}}(q), \text{ in Case (2)} \end{cases}$$

$$\mathsf{anchor}_{\overline{a}n}(q) = \begin{cases} \langle\varphi(q_n), x_n\rangle, \text{ if } q\in O_{\overline{a}n}(q_n) \text{ in Case (1)} \\ \mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q)), \text{ if } q\notin O_{\overline{a}n}(q_n) \text{ in Case (1)} \\ \mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q)), \text{ in Case (2)} \end{cases}$$

$$\varphi_{\overline{a}n}(q) = \begin{cases} \langle\varphi(q), x_n\rangle, \text{ if } q\in O_{\overline{a}n}(q_n) \text{ in Case (1)} \\ \langle\varphi(q), \mathsf{rt}(\mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q)))\rangle, \text{ if } q\notin O_{\overline{a}n}(q_n) \text{ in Case (1)} \\ \langle\varphi(q), \mathsf{rt}(\mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q)))\rangle, \text{ in Case (2)} \end{cases}$$

**Lemma 7.4.** *For every $\overline{a}\in\mathbb{N}^*$, $n\in\mathbb{N}$, and $q,q'\in\mathbb{Q}$,*

1. $O_{\overline{a}n}(q)\subseteq O_{\overline{a}}(q)$*;*
2. $q\in O_{\overline{a}}(q)$*;*
3. $O_{\overline{a}}(q)$ *is clopen;*
4. $O_{\overline{a}}(q) = O_{\overline{a}}(q')$ *if* $q'\in O_{\overline{a}}(q)$*;*
5. $O_{\overline{a}}(q)\cap O_{\overline{a}}(q') = \emptyset$ *if* $q'\notin O_{\overline{a}}(q)$*;*
6. $\mathbb{Q} = \bigcup_{p\in\mathbb{Q}} O_{\overline{a}}(p)$*;*

7. *there are finitely many* $p_1, \ldots, p_k \in \mathbb{Q}$ *such that that* $\mathbb{Q} = \bigcup_{i=1,\ldots,k} O_{\overline{a}}(p_i)$ *and the* $O_{\overline{a}}(p_i)$*'s are nonempty and pairwise disjoint; incidentally,* $k \leq \mathsf{ln}(\overline{a}) + 1$*, where* $\mathsf{ln}(\overline{a})$ *is the length of* $\overline{a}$*;*
8. *if* $q' \in O_{\overline{a}}(q)$ *then* $\mathsf{anchor}_{\overline{a}}(q') = \mathsf{anchor}_{\overline{a}}(q)$*;*
9. $\mathsf{anchor}_{\overline{a}}(q) \in X$*;*
10. $\mathsf{lft}(\varphi_{\overline{a}}(q)) = \varphi(q)$*;*
11. $\mathsf{rt}(\varphi_{\overline{a}}(q)) = \mathsf{rt}(\mathsf{anchor}_{\overline{a}}(q))$*;*
12. *if* $q' \in O_{\overline{a}}(q)$ *then* $\mathsf{rt}(\varphi_{\overline{a}}(q')) = \mathsf{rt}(\varphi_{\overline{a}}(q))$*;*
13. $\mathsf{lft}(\mathsf{anchor}_{\overline{a}}(q))\ R_1\ \mathsf{lft}(\varphi_{\overline{a}}(q))$*;*
14. $\varphi_{\overline{a}}(q) \in X$*.*

Proof. Item (1) follows directly from the definition of $O_{\overline{a}n}$. Items (2) to (12) can be checked by a routine induction on the construction of $\overline{a}$.

**Item (13).** We prove this by induction on the construction of $\overline{a}$. The Base Case is the claim that

$$\mathsf{lft}(\mathsf{anchor}_{\wedge}(q))\ R_1\ \mathsf{lft}(\varphi_{\wedge}(q)), \text{ for every } q \in \mathbb{Q}.$$

This follows from the definitions of $\mathsf{anchor}_{\wedge}$ and $\varphi_{\wedge}$ and the facts that $r_1$ is a root of $X_1$ and $\varphi(q) \in X_1$. For the inductive step, assume that $\mathsf{lft}(\mathsf{anchor}_{\overline{a}}(q))\ R_1\ \mathsf{lft}(\varphi_{\overline{a}}(q))$, for every $q \in \mathbb{Q}$. We want to show that $\mathsf{lft}(\mathsf{anchor}_{\overline{a}n}(q))\ R_1\ \mathsf{lft}(\varphi_{\overline{a}n}(q))$, for every $q \in \mathbb{Q}$. Choose $q \in \mathbb{Q}$. We consider two cases:

(1) $\varphi_{\overline{a}}(q_n)\ S_2\ \langle \varphi(q_n), x_n\rangle \in X$ and $q \in O_{\overline{a}n}(q_n)$; and
(2) either $\langle \varphi(q_n), x_n\rangle \notin X$ or $\varphi_{\overline{a}}(q_n)\ \not S_2\ \langle \varphi(q_n), x_n\rangle$ or $q \notin O_{\overline{a}n}(q_n)$.

**Case (1).** In this case, $\mathsf{anchor}_{\overline{a}n}(q) = \langle \varphi(q_n), x_n\rangle$. Note that $q_n \in O_{\overline{a}}(q_n)$, by Item (2). So $O_{\overline{a}n}(q_n) = O_{\overline{a}}(q_n) \cap I_n$, by the definition of $O_{\overline{a}n}(q_n)$. So $q \in I_n$, which is a $\varphi(q_n)$-interval. So $\varphi(q_n)\ R_1\ \varphi(q)$. So

$$\mathsf{lft}(\mathsf{anchor}_{\overline{a}n}(q))\ R_1\ \mathsf{lft}(\varphi_{\overline{a}n}(q)),$$

as desired.

**Case (2).** In this case, $\mathsf{anchor}_{\overline{a}n}(q) = \mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q))$. So,

$$\mathsf{lft}(\mathsf{anchor}_{\overline{a}n}(q)) = \mathsf{lft}(\mathsf{anchor}_{\overline{a}}(q)).$$

By the inductive hypothesis, $\mathsf{lft}(\mathsf{anchor}_{\overline{a}}(q))\ R_1\ \mathsf{lft}(\varphi_{\overline{a}}(q))$. So

$$\mathsf{lft}(\mathsf{anchor}_{\overline{a}n}(q)) = \mathsf{lft}(\mathsf{anchor}_{\overline{a}}(q))\ R_1\ \mathsf{lft}(\varphi_{\overline{a}}(q)) = \varphi(q) = \mathsf{lft}(\varphi_{\overline{a}n}(q)),$$

as desired.

**Item (14)** follows from Items (11) and (13), and the fact that $X \in \mathsf{EX}$. ■

Our next step is to prove that each $\varphi_{\overline{a}}$ is a continuous and open function from $\mathcal{Q}$ to the reflexive, transitive frame $\langle X, S_1\rangle$. But first a useful general lemma.

**Lemma 7.5.** *Suppose that*

1. $\mathcal{Z} = \langle Z, \tau\rangle$ *and* $\mathcal{Z}' = \langle Z', \tau'\rangle$ *are topological spaces;*
2. $Z = \bigcup_{i\in I} O_i$*, for some index set* $I$ *and nonempty, pairwise disjoint, open sets* $O_i$*; let* $\mathcal{O}_i$ *be the subspace of* $\mathcal{Z}$ *whose underlying set is* $O_i$*;*
3. $\sigma_i : O_i \to Z'$ *is a continuous [open] function from* $\mathcal{O}_i$ *to* $\mathcal{Z}'$*, for each* $i \in I$*;*
4. $\sigma : Z \to Z'$*; and*
5. *for every* $z \in Z$ *and* $i \in I$*, if* $z \in O_i$ *then* $\sigma(z) = \sigma_i(z)$*.*

*Then* $\sigma$ *is a continuous [open] function from* $\mathcal{Z}$ *to* $\mathcal{Z}'$*.*

**Lemma 7.6.** *For each* $\overline{a} \in \mathbb{N}^*$*,* $\varphi_{\overline{a}}$ *is a continuous and open function from* $\mathcal{Q}$ *to the reflexive, transitive frame* $\langle X, S_1\rangle$ *– alternatively, to the Alexandrov space* $\langle X, \tau(S_1)\rangle$*.*

Proof. Fix $\overline{a} \in \mathbb{N}^*$. By Lemma 7.4, Item (7), there are finitely many $p_1, \dots, p_k \in \mathbb{Q}$ such that that $\mathbb{Q} = \bigcup_{i=1,\dots,k} O_{\overline{a}}(p_i)$ and the $O_{\overline{a}}(p_i)$'s are nonempty and pairwise disjoint. The $O_{\overline{a}}(p_i)$ are open, by Lemma 7.4, Item (3). Let $O_i = O_{\overline{a}}(p_i)$ and let $\mathcal{O}_i$ be the subspace of $\mathcal{Q}$ whose underlying set is $O_i$. Let $\rho_i$ be $\varphi$ restricted to $O_i$. Note that $\rho_i$ is a continuous and open function from $\mathcal{O}_i$ to the reflexive-transitive frame $\mathcal{X}_1 = \langle X, R_1\rangle$, since $\varphi$ is a continuous and open function from $\mathcal{Q}$ to $\mathcal{X}_1$.

For each $i = 1, \dots, k$, let $v_i = \mathsf{rt}(\mathsf{anchor}_{\overline{a}}(p_i))$ and let $\sigma_i$ be $\varphi_{\overline{a}}$ restricted to $O_i$. Then, by Lemma 7.4, Items (8), (10), and (11),

$$\sigma_i(q) = \langle \varphi(q), v_i\rangle = \langle \rho_i(q), v_i\rangle, \text{ for every } q \in O_i.$$

Since

- $\rho_i$ is a continuous and open function from $\mathcal{O}_i$ to $\mathcal{X}_1$,
- $\mathsf{lft}(\sigma_i(q)) = \rho_i(q)$ for every $q \in O_i$, and
- $\mathsf{rt}(\sigma_i(q)) = \mathsf{rt}(\sigma_i(q'))$ for every $q, q' \in O_i$,

$\sigma_i$ is a continuous and open function from $\mathcal{O}_i$ to the reflexive-transitive frame $\langle X, S_1\rangle$ and thus to the Alexandrov space $\langle X, \tau(S_1)\rangle$.

Note that $\varphi_{\overline{a}} : \mathbb{Q} \to X$ and that, for every $q \in \mathbb{Q}$ and $i = 1, \ldots, k$, if $q \in O_i$ then $\varphi_{\overline{a}}(q) = \sigma_i(q)$. Thus, by Lemma 7.5, $\varphi_{\overline{a}}$ is a continuous and open function from $\mathcal{Q}$ to $\langle X, \tau(S_1)\rangle$ and thus to the reflexive-transitive frame $\langle X, S_1\rangle$. ■

**We now define our surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathbf{L}}$ to $\mathcal{X}$:**

$$\psi(q, \overline{a}) = \varphi_{\overline{a}}(q).$$

By Lemma 7.4, Item (14), $\varphi_{\overline{a}}(q) \in X$ for every $q \in \mathbb{Q}$ and $\overline{a} \in \mathbb{N}^*$. Thus $\psi(q, \overline{a}) \in X$ for every $q \in \mathbb{Q}$ and $\overline{a} \in \mathbb{N}^*$. It remains to show that $\psi$ is a surjective p-morphism from $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$ to $X$, i.e., that $\psi$ is 1-continuous, 1-open, 2-monotone, and 2-lifting, and surjective. The 1-continuity and 1-openness of $\psi$ follow from the continuity and openness of each $\varphi_{\overline{a}}$. It remains to show that $\psi$ is 2-monotone, and 2-lifting, and surjective.

**$\psi$ is 2-monotone.** We want to show that

$$(\forall q, q' \in \mathbb{Q})(\forall \overline{a}, \overline{a}' \in \mathbb{N}^*)(\text{if } \langle q, \overline{a}\rangle {R_{\mathrm{L}}}' \langle q', \overline{a}'\rangle \text{ then } \psi(q, \overline{a})\ S_2\ \psi(q', \overline{a}')). \quad (1)$$

(1) is equivalent to

$$(\forall q \in \mathbb{Q})(\forall \overline{a}, \overline{a}' \in \mathbb{N}^*)(\text{if } \overline{a} R_{\mathrm{L}} \overline{a}' \text{ then } \varphi_{\overline{a}}(q)\ S_2\ \varphi_{\overline{a}'}(q)). \quad (2)$$

For each $q \in \mathbb{Q}$, define the function $\chi_q : \mathbb{N}^* \to X_2$ as follows:

$$\chi_q(\overline{a}) = \mathsf{rt}(\varphi_{\overline{a}}(q)) \quad (3)$$

For (2), it suffices to show

$$(\forall q \in \mathbb{Q})(\forall \overline{a}, \overline{a}' \in \mathbb{N}^*)(\text{if } \overline{a} R_{\mathrm{L}} \overline{a}' \text{ then } \chi_q(\overline{a})\ R_2\ \chi_q(\overline{a}')). \quad (4)$$

Note that (4) is equivalent to the claim that each $\chi_q$ is a monotone function from $\mathcal{N}_{\mathrm{L}}$ to $\mathcal{X}_2$. By Lemma 7.1, it suffices to show that each $\chi_q$ is a monotone function from $\mathcal{N}^*$ to $\mathcal{X}_2$. Thus, it suffices to show that

$$(\forall q \in \mathbb{Q})(\forall \overline{a} \in \mathbb{N}^*)(\forall n \in \mathbb{N})(\chi_q(\overline{a})\ R_2\ \chi_q(\overline{a}n)), \quad (5)$$

Equivalently,

$$(\forall q \in \mathbb{Q})(\forall \overline{a} \in \mathbb{N}^*)(\forall n \in \mathbb{N})(\mathsf{rt}(\varphi_{\overline{a}}(q)) \ R_2 \ \mathsf{rt}(\varphi_{\overline{a}n}(q))). \tag{6}$$

To show (6), choose $q \in \mathbb{Q}$, $\overline{a} \in \mathbb{N}^*$, and $\forall n \in \mathbb{N}$. We want to show that

$$\mathsf{rt}(\varphi_{\overline{a}}(q)) \ R_2 \ \mathsf{rt}(\varphi_{\overline{a}n}(q)). \tag{7}$$

It will help to recall the enumeration of $\mathbb{Q} \times X_2$ on page 17 and the definitions of $O_{\overline{a}n}$, $\mathsf{anchor}_{\overline{a}n}$, and $\varphi_{\overline{a}n}$ on page 21. We consider two cases:

(1) $\varphi_{\overline{a}}(q_n) \ S_2 \ \langle \varphi(q_n), x_n \rangle \in X$ and $q \in O_{\overline{a}n}(q_n)$; and
(2) either $\langle \varphi(q_n), x_n \rangle \notin X$ or $\varphi_{\overline{a}}(q_n) \ \not S_2 \ \langle \varphi(q_n), x_n \rangle$ or $q \notin O_{\overline{a}n}(q_n)$.

**Case (1).** In this case, $\varphi_{\overline{a}n}(q_n) = \langle \varphi(q_n), x_n \rangle$, by Lemma 7.4, Item 2, and the definition of $\varphi_{\overline{a}n}$. So $\varphi_{\overline{a}}(q_n) \ S_2 \ \varphi_{\overline{a}n}(q_n)$. So

$$\mathsf{rt}(\varphi_{\overline{a}}(q_n)) \ R_2 \ \mathsf{rt}(\varphi_{\overline{a}n}(q_n)) \tag{8}$$

Not only do we have $q \in O_{\overline{a}n}(q_n)$; we also have $q \in O_{\overline{a}}(q_n)$, by Lemma 7.4, Item (1). So by Lemma 7.4, Item (12), we have both $\mathsf{rt}(\varphi_{\overline{a}n}(q)) = \mathsf{rt}(\varphi_{\overline{a}n}(q_n))$ and $\mathsf{rt}(\varphi_{\overline{a}}(q)) = \mathsf{rt}(\varphi_{\overline{a}}(q_n))$. So our desired (7) follows from (8).

**Case (2).** In this case, by the definition of $\varphi_{\overline{a}n}(q)$,

$$\mathsf{rt}(\varphi_{\overline{a}n}(q)) = \mathsf{rt}(\mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q))). \tag{9}$$

And by Lemma 7.4, Item (11),

$$\mathsf{rt}(\varphi_{\overline{a}}(q)) = \mathsf{rt}(\mathsf{anchor}_{\overline{a}}(q)). \tag{10}$$

Given the way each $\mathsf{defsucc}(\langle u, v \rangle)$ was chosen on page 21,

$$\mathsf{anchor}_{\overline{a}}(q) \ S_2 \ \mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q)). \tag{11}$$

So

$$\mathsf{rt}(\mathsf{anchor}_{\overline{a}}(q)) \ R_2 \ \mathsf{rt}(\mathsf{defsucc}(\mathsf{anchor}_{\overline{a}}(q))). \tag{12}$$

So, by (9), (10) and (12)

$$\mathsf{rt}(\varphi_{\overline{a}}(q)) \ R_2 \ \mathsf{rt}(\varphi_{\overline{a}n}(q)) \tag{13}$$

as desired.

**$\psi$ is 2-lifting.** It suffices to show that

$$(\forall q \in \mathbb{Q})(\forall \overline{a} \in \mathbb{N}^*)(\forall \langle u, v\rangle \in X)$$
$$(\text{if } \psi(q, \overline{a})\ S_2\ \langle u, v\rangle \text{ then } (\exists n \in \mathbb{N})(\psi(q, \overline{a}n) = \langle u, v\rangle).$$

The reason this suffices is that $\langle q, \overline{a}\rangle R_{\mathrm{L}}{}'\langle q, \overline{a}n\rangle$, for every $n \in \mathbb{N}$. So suppose that $\psi(q, \overline{a})\ S_2\ \langle u, v\rangle$. Then $\varphi_{\overline{a}}(q)\ S_2\ \langle u, v\rangle$. So $\mathsf{lft}(\varphi_{\overline{a}}(q)) = u$. Also, by Lemma 7.4, Item (10), $\mathsf{lft}(\varphi_{\overline{a}}(q)) = \varphi(q)$. So $u = \varphi(q)$. Now choose $n \in \mathbb{N}$ such that $\langle q_n, x_n\rangle = \langle q, v\rangle$.

$\varphi_{\overline{a}n}(q) = \langle \varphi(q), v\rangle$ by the definition of $\varphi_{\overline{a}n}$. So $\psi(q, \overline{a}n) = \langle u, v\rangle$, as desired.

$\psi$ **is surjective.** First, we claim that

$$\text{the root } \langle r_1, r_2\rangle \text{ is in the image of } \mathbb{Q} \times \mathcal{N}_{\mathrm{L}} \text{ under } \psi. \tag{14}$$

Note that there is some $q \in \mathbb{Q}$ with $\varphi(q) = r_1$, since $\varphi$ is a surjective p-morphism from $\mathbb{Q}$ to $\mathcal{X}_1$. So $\psi(\langle q, \Lambda\rangle) = \varphi_\Lambda(q) = \langle r_1, r_2\rangle$, which suffices for (14). Note that $\psi$, $\mathbb{Q} \times \mathcal{N}_{\mathrm{L}}$ and $\mathcal{X} = \langle X, S_1, S_2\rangle$ satisfy the conditions for an application of Lemma 4.3: $\mathbb{Q} \times \mathcal{N}_{\mathrm{L}}$ is a topologized frame and $\mathcal{X} = \langle X, S_1, S_2\rangle$ is a rooted 2-frame where $S_1$ is reflexive and transitive, by observation 1 on page 16; $\langle r_1, r_2\rangle$ is a root of $\mathcal{X}$; and $\psi$ is a p-morphism from $\mathbb{Q} \times \mathcal{N}_{\mathrm{L}}$ to $\mathcal{X} = \langle X, S_1, S_2\rangle$ with $\langle r_1, r_2\rangle \in \psi[\mathbb{Q} \times \mathcal{N}_{\mathrm{L}}]$. So $\psi$ is surjective, by Lemma 4.3. And thus ends the proof of Lemma 7.2.

### 7.3. Pulling it all together

Now that we have proved Lemma 7.2, we prove Theorem 3.4, when L is a consistent Kripke complete and Horn axiomatizable 1-modal logic. As noted in Remark 3.5, it is easy to check that $[\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}} \subseteq \mathrm{S4} \times_{tf} \mathrm{L}$. To see that $\mathrm{S4} \times_{tf} \mathrm{L} \subseteq [\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$, suppose that $A \notin [\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$. By Lemma 5.4, there is a countable rooted $[\mathrm{S4}, \mathrm{L}]^{\mathsf{EX}}$-frame $\mathcal{X}$ such that $A \notin \mathsf{Log}(\mathcal{X})$. By Corollary 5.3, $\mathcal{X}$ is a p-morphic image of some

$$\mathcal{X}' \in \mathsf{EX} \cap \mathsf{CT} \cap \mathsf{SR} \cap \mathsf{SF}(\mathsf{Fr}(\mathrm{L}_1) \times \mathsf{Fr}(\mathrm{L}_2)).$$

So $A \notin \mathsf{Log}(\mathcal{X}')$, by Lemma 4.2. Also, $\mathcal{X}'$ is a p-morphic image of $\mathcal{Q} \times \mathcal{N}_{\mathrm{L}}$, by Lemma 7.2. So $A \notin \mathsf{Log}(\mathcal{Q} \times \mathcal{N}_{\mathrm{L}})$, by Lemma 4.2. So $A \notin \mathrm{S4} \times_{tf} \mathrm{L}$, since $\mathcal{Q} \in \mathsf{Top}(\mathrm{S4})$ and $\mathcal{N}_{\mathrm{L}} \in \mathsf{Fr}(\mathrm{L})$. QED

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