COMPARING FIXED POINT AND REVISION THEORIES OF TRUTH

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Abstract. In response to the liar's paradox, Kripke developed the *fixed point* semantics for languages expressing their own truth concepts. (Martin and Woodruff independently developed this semantics, but not to the same extent as Kripke.) Kripke's work suggests a number of related theories of truth for such languages. Gupta and Belnap develop their *revision theory* of truth in contrast to the fixed point theories. The current paper considers three natural ways to compare the various resulting theories of truth, and establishes the resulting relationships among these theories. The point is to get a sense of the lay of the land amid a variety of options. Our results will also provide technical fodder for the methodological remarks of the companion paper to this one.

§1. Introduction. Given a first order language *L*, a *classical model for L* is an ordered pair M = ⟨D, I⟩, where D, the *domain of discourse*, is a nonempty set; and where I is a function assigning to each name of *L* a member of D, to each n-place function symbol of *L* an n-place function on D, and to each n-place relation symbol a function from D^n to {**t**, **f**}. Suppose that *L* and L^+ are first order languages, where L^+ is *L* expanded with a distinguished predicate ***T***, and where *L* has a quote name '*A*' for each sentence *A* of L^+. A *ground model for L* is classical model M = ⟨D, I⟩ for *L* such that I('*A*') = *A* ∈ D for each sentence *A* of L^+.

Given a ground model M for *L*, we can think of I(*X*) as the *interpretation* or, to borrow an expression from Gupta and Belnap [3], the *signification* of *X* where *X* is a name, function symbol or relation symbol. Gupta and Belnap characterize an expression's or concept's *signification* in a world w as "an abstract something that carries all the information about all the expression's [or concept's] extensional relations in w". If we want to interpret ***T****x* as "*x* is true", then, given a ground model, we would like to find an appropriate signification, or an appropriate range of significations, for ***T***.

We might try to expand M to a classical model M′ = ⟨D, I′⟩ for L^+. For ***T*** to mean truth, M′ should assign the same truth value to the sentences ***T***'*A*' and *A*, for every sentence *A* of L^+. Unfortunately, not every ground model M = ⟨D, I⟩ can thus be expanded: if λ is a name of *L* and if I(λ) = ¬***T***λ, then I′(λ) = I′('¬***T***λ') so that ***T***'¬***T***λ' and ***T***λ are assigned the same truth value by M′; thus ***T***'¬***T***λ' and ¬***T***λ are assigned different truth values by M′. This is a formalization of the liar's paradox, with the sentence ¬***T***λ as the offending liar's sentence.

In a semantics for languages capable of expressing their own truth concepts, ***T*** will not, in general, have a classical signification. Kripke [8] and Martin and Woodruff [10] present the *fixed point* semantics for such languages. Kripke suggests a whole host of related approaches to the problem of assigning, given a ground model M, a signification to ***T***. Gupta and Belnap [3] present their *revision theories* in contrast to the various options presented by Kripke.

In the current paper, we motivate three different ways of comparing fixed point and revision theories of truth, and we establish the various relationships the theories have to one another in these three different senses. The general point of this is to help us get the lay of the land amid the variety of choices. There is a more specific use we make of these comparisons: in the companion paper to this one, Kremer [7], we use the current results to critique one of Gupta and Belnap's motivations for their revision theoretic approach, i.e. their claim that the revision theory has the advantage of treating truth like a classical concept when there is no vicious reference.

In the course of our investigation, we close two problems left open by Gupta and Belnap [3]. We also give a simplified proof of their "Main Lemma".

§2. Fixed point semantics.[1] The intuition behind the fixed point semantics is that pathological sentences such as the liar sentence are neither true nor false. In general a *three-valued model* for a language *L* is just like a classical model, except that the function I assigns, to each n-place predicate, a function from D^n to {**t**, **f**, **n**}. A classical model is a special case of a three-valued model. Officially **t**(rue), **f**(alse) and **n**(either) are three truth values, but **n** can be thought of as the absence of a truth value.[2] We order the truth values as follows: $\mathbf{n} \leq \mathbf{n} \leq \mathbf{t} \leq \mathbf{t}$ and $\mathbf{n} \leq \mathbf{n} \leq \mathbf{f} \leq \mathbf{f}$. We say that $M = \langle D, I\rangle \leq M' = \langle D, I'\rangle$ iff $I(X) = I'(X)$ for each name or

[1]We will follow Gupta and Belnap's presentation of the fixed point semantics and of the revision theory of truth. Much of this material is culled from [3] and elsewhere. Among the numbered definitions, theorems, and lemmas, those not explicitly attributed to a source are original to the current paper.

[2]We will not consider four-valued models, with the additional truth value **b**(oth). See Visser [13] and [14] and Woodruff [15].

function symbol X, and $I(G)(d_1, ..., d_n) \leq I'(G)(d_1, ..., d_n)$ for each n-place predicate symbol G and each $d_1, ..., d_n \in D$.

Given a three-valued model $M = \langle D, I \rangle$ and an assignment s of values to the variables, the value, $Val_{M, s}(t) \in D$ of each term t is defined in the standard way. The atomic formula $Gt_1...t_n$ is assigned the value $I(G)(Val_{M, s}(t_1), ..., Val_{M, s}(t_n))$. To evaluate composite expressions, we must have some *evaluation scheme*: for example, if A is **f**(alse) and B is **n**(either), we must decide whether (A & B) is **f** or **n**. For classical models, we will just use the standard classical evaluation scheme, τ. For nonclassical models, we will consider the *weak Kleene scheme*, μ, and the *strong Kleene scheme*, κ. These both agree with τ on classical truth values. According to both μ and κ, ¬**n** = **n**. According to μ, (**t** & **n**) = (**n** & **t**) = (**f** & **n**) = (**n** & **f**) = **n**. And according to κ, (**t** & **n**) = (**n** & **t**) = **n** and (**f** & **n**) = (**n** & **f**) = **f**. If we treat universal quantification analogously to conjunction, then for each sentence A and each evaluation scheme ρ = τ, μ, or κ, we can define $Val_{M, \rho}(A)$: the truth value of A in M according to ρ. ($Val_{M, \tau}(A)$ is defined only when M is classical.) We also consider a fourth scheme, van Fraassen's *supervaluation* scheme, σ:

$$Val_{M, \sigma}(A) =_{df} \begin{cases} \mathbf{t}\ [\mathbf{f}], \text{ if } Val_{M', \tau}(A) = \mathbf{t}\ [\mathbf{f}] \text{ for every classical } M' \geq M. \\ \mathbf{n}, \text{ otherwise.} \end{cases}$$

Note: if $Val_{M, \rho}(A) = \mathbf{n}$, then $Val_{M, \rho}(A \vee \neg A) = \mathbf{n}$ if ρ = κ or μ, and $Val_{M, \rho}(A \vee \neg A) = \mathbf{t}$ if ρ = σ.

For the fixed point semantics, suppose, as in §1, that L and L^+ are first order languages, where L^+ is L expanded with a distinguished predicate ***T***, and where L has a quote name 'A' for each sentence A of L^+. And suppose that $M = \langle D, I \rangle$ is a (classical) ground model for L, as defined in §1. We want to expand M to a three-valued model by adding a signification for the predicate ***T***. Let an *hypothesis* be a function $h:D \rightarrow \{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$, and a *classical* hypothesis, a function $h:D \rightarrow \{\mathbf{t}, \mathbf{f}\}$. Hypotheses are potential significations of ***T***. Let M + h be the model $M' = \langle D, I' \rangle$ for L^+, where I′ and I agree on the constants of L and where $I'(\boldsymbol{T}) = h$. Models of the form M + h are *expanded* models. If we want $\boldsymbol{T}x$ to mean "x is true", then we want to expand a ground model M to a model M + h so that $\text{Val}_{\text{M + h, }\rho}(A) = \text{Val}_{\text{M + h, }\rho}(\boldsymbol{T}\text{‘}A\text{’})$ for every sentence A of L^+, where we are working with some evaluation scheme ρ. This is equivalent to the condition, $\text{Val}_{\text{M + h, }\rho}(A) = \text{h}(A)$. We will also insist that if $\text{d} \in \text{D}$ is not a sentence of L^+, then $\text{I}'(\boldsymbol{T})(\text{d}) = \text{h}(\text{d}) = \mathbf{f}$. For $\rho = \tau, \mu, \kappa$, or σ, define the *jump operator* ρ_{M} on the set of hypotheses as follows, restricting this definition to classical hypotheses for $\rho = \tau$:

$$\rho_{\text{M}}(\text{h})(A) = \text{Val}_{\text{M + h, }\rho}(A), \text{ if } A \in \text{S} = \{A\text{: } A \text{ is a sentence of } L^+\}$$

$$\rho_{\text{M}}(\text{h})(\text{d}) = \mathbf{f} \text{ if d} \in \text{D - S.}$$

The hypotheses meeting our conditions, above, under which $\boldsymbol{T}x$ means "x is true", are the *fixed points* of ρ_{M}: the hypotheses h such that $\rho_{\text{M}}(\text{h}) = \text{h}$. The fixed points deliver, for the language L^+, models M + h satisfying what M. Kremer [6] calls "the fixed point conception of truth", according to which, as Kripke [8] puts it, "we are entitled to assert (or deny) of a sentence that it is true precisely under the circumstances when we can assert (or deny) the sentence itself."

Kripke [8] proves that μ_{M}, $[\kappa_{\text{M}}, \sigma_{\text{M}}]$ has a fixed point, for every ground model M. In fact, Kripke's results are stronger. Say that $\text{h} \le \text{h}'$ iff $\text{h}(\text{d}) \le \text{h}'(\text{d})$ for every $\text{d} \in \text{D}$. And say that a function ρ on hypotheses is *monotone* iff, for all hypotheses h and h′, if $\text{h} \le \text{h}'$ then $\rho(\text{h}) \le \rho(\text{h}')$. μ_{M}, κ_{M}, and σ_{M} are monotone, for every ground model M. Each monotone function ρ not only has a fixed point, but a *least* fixed point, lfp(ρ). Say that h and h′ are *compatible* iff $\text{h} \le \text{h}''$ and $\text{h}' \le \text{h}''$ for some hypothesis h″, and that h is *intrinsic* iff h is compatible with every fixed point. For example, lfp(ρ) is intrinsic. Each monotone function ρ not only has a *least* fixed point, but a *greatest* intrinsic fixed point, gifp(ρ), which is not in general identical to lfp(ρ). Say that a sentence A is ρ-*grounded* iff lfp(ρ) = **t** or **f**, and ρ-*intrinsic* iff gifp(ρ) = **t** or **f**. The liar sentence is neither κ-grounded nor κ-intrinsic since it gets the value **n** at every fixed point h. The truth-teller is neither κ-grounded nor κ-intrinsic since it gets the value **t** at some fixed points and the value **f** at others. If $\text{I}(b) = \boldsymbol{T}b \vee \neg\boldsymbol{T}b$, then $\boldsymbol{T}b \vee \neg\boldsymbol{T}b$ is κ-intrinsic and σ-grounded, but not κ-grounded: $\text{gifp}(\kappa_{\text{M}})(\boldsymbol{T}b \vee \neg\boldsymbol{T}b) = \text{lfp}(\sigma_{\text{M}})(\boldsymbol{T}b \vee \neg\boldsymbol{T}b) = \mathbf{t}$, while $\text{lfp}(\kappa_{\text{M}})(\boldsymbol{T}b \vee \neg\boldsymbol{T}b) = \mathbf{n}$.

The fixed point semantics yields a number of plausible significations of ***T***: the fixed points generated by your favourite evaluation scheme. Many have considered the proposal that the least fixed point yields the correct signification of ***T***.[3] M. Kremer [6] decisively argues that Kripke [8] does not endorse this proposal, and that this proposal misinterprets the fixed point semantics: the fixed point conception of truth, mentioned above, favours no particular fixed point. M. Kremer emphasizes a tension between the fixed point conception of truth and another intuition, the "supervenience of semantics": the intuition that the interpretation of ***T*** should be determined by the interpretation of the nonsemantic names, function symbols and predicates.

Fix some evaluation scheme. The dispute between a supervenience fixed point theorist—for specificity, say a least fixed point theorist—and a nonsupervenience fixed point theorist can be brought out as follows. Fix some uninterpreted language L, and let L^+ be L expanded with a privileged predicate ***T***. Suppose that, other than their use of ***T***, the discourse of two communities X and Y is represented by the same ground model M, while X's use of ***T*** is represented by the *least* fixed point h_X and Y's use of ***T*** is represented by the fixed point $h_Y \neq h_X$. Let $L_X = \langle L^+, M + h_X \rangle$ and $L_Y = \langle L^+, M + h_Y \rangle$ be the interpreted languages spoken by X and Y. According to the least fixed point theorist, X uses ***T*** to express truth in L_X but Y does not use ***T*** to express truth in L_Y, despite the fact that, in L_Y, A and ***T***'*A*' have the same truth value for each sentence A. According to the nonsupervience theorist, on the other hand, the fact that X and Y use ***T*** to express *truth* in L_X and L_Y, respectively, is given by the fact that h_X and h_Y are fixed points: each community's use of ***T*** satisfies the necessary and, for the nonsupervenience theorist, sufficient conditions for ***T*** to express truth in the community's language.

We have on board two proposals for interpreting the fixed point semantics. On the *supervenience* proposal, the language spoken by a community is determined by its use of nonsemantic vocabulary—represented by a ground model—and the interpretation of ***T*** as *truth* is given by some particular fixed point, usually assumed to be the least fixed point. The greatest

[3] See Haack [4], Grover [2], Davis [1], Kroon [9], Parsons [11], Kirkham [5], and Read [12].

intrinsic fixed point might also seem natural: "The largest intrinsic fixed point is the unique 'largest' interpretation of ***T****x* which is consistent with our intuitive idea of truth and makes no arbitrary choices in truth assignments. It is thus an object of special theoretical interest." (Kripke [8].) On the *nonsupervenience* proposal, the language spoken by the community is not determined by its use of nonsemantic vocabulary: the communities X and Y, in the preceding paragraph, speak distinct languages in which ***T*** expresses truth, despite a shared ground model. If we fix an evaluation scheme and a ground model, all the fixed points provide acceptable significations of *truth*.

We will not adjudicate between these two proposals. Rather, we will introduce a number of *supervenience* theories of truth, which depend on which evaluation scheme we use, and on whether we privilege the least fixed point or the greatest intrinsic fixed point. One reasons to restrict ourselves to the supervenience approach is that Gupta and Belnap's revision theories depend on the supervenience of semantics, and so it is the supervenience fixed point theories that are most readily comparable to the revision theories.

Definition 2.1. Let $\rho = \mu$, κ, or σ. The sentence A of L^+ *is valid in the ground model* M *according to* (the theory) $\mathbf{T}^{\mathrm{lfp},\,\rho}$ iff $\mathrm{lfp}(\rho_M)(A) = \mathbf{t}$. The sentence A of L^+ *is valid in the ground model* M *according to* $\mathbf{T}^{\mathrm{gifp},\,\rho}$ iff $\mathrm{gifp}(\rho_M)(A) = \mathbf{t}$.[4] We define the set of sentences valid in M according to such and such a theory as follows:

$$\mathbf{V}_M^{\mathrm{lfp},\,\rho} =_{\mathrm{df}} \{A: \ \mathrm{lfp}(\rho_M)(A) = \mathbf{t}\} = \{A: \ A \text{ is valid in M according to } \mathbf{T}^{\mathrm{lfp},\,\rho}\}, \text{ and}$$

$$\mathbf{V}_M^{\mathrm{gifp},\,\rho} =_{\mathrm{df}} \{A: \ \mathrm{gifp}(\rho_M)(A) = \mathbf{t}\} = \{A: \ A \text{ is valid in M according to } \mathbf{T}^{\mathrm{gifp},\,\rho}\}.$$

Before we consider revision theories, we define two variants, defined by Kripke [8], of the supervaluation jump operator σ_M. Say that h is *weakly consistent* iff the set of sentences $\{A \in S: \ h(A) = \mathbf{t}\}$ is consistent. Say that h is *strongly consistent* iff $\{A \in S: \ h(A) = \mathbf{t}\} \cup \{\neg A: \ A$

[4]Note that we have not *defined* the theories of truth, $\mathbf{T}^{\mathrm{lfp},\,\rho}$ and such: we have specified each theory's verdict regarding which sentences are valid in which ground models but not, for example, each theory's verdict regarding what the valid inferences are.

$\in$ S and h(A) = **f**} is consistent. Note: a classical hypothesis h is strongly consistent iff {$A \in$ S: h(A) = **t**} is complete and consistent. $\sigma1_M$(h) [$\sigma2_M$(h)] is defined only for weakly [strongly] consistent h, as follows:

$\sigma1_M(h)(A)$ = **t** [**f**] iff $\tau_M(h')(A)$ = **t** [**f**] for all weakly consistent classical $h' \geq h$.
n, otherwise, for sentences $A \in$ S.

$\sigma1_M(h)(d)$ = **f**, for $d \in$ (D - S).

$\sigma2_M(h)(A)$ = **t** [**f**] iff $\tau_M(h')(A)$ = **t** [**f**] for all strongly consistent classical $h' \geq h$.
n, otherwise, for sentences $A \in$ S.

$\sigma2_M(h)(d)$ = **f**, for $d \in$ (D - S).[5]

$\sigma1_M$ [$\sigma2_M$] is a monotone operator on the weakly [strongly] consistent hypotheses. This suffices for $\sigma1_M$ [$\sigma2_M$] to have both a least fixed point and a greatest intrinsic fixed point. We will treat $\sigma1$ and $\sigma2$ as two new three-valued evaluation schemes. Theories $\mathbf{T}^{lfp,\ \sigma1}$, $\mathbf{T}^{gifp,\ \sigma2}$, etc., and sets $\mathbf{V}_M^{lfp,\ \sigma1}$, $\mathbf{V}_M^{gifp,\ \sigma2}$, etc. are introduced as in Definition 2.1, above.

§3. Revision theories of truth. Gupta and Belnap's most interesting objection to the fixed point semantics stems from an uncommon take on a common observation: the observation that there are connectives that fixed point languages cannot express, for example, *exclusion negation*, ¬**n** = **t**; and the Lukasiewicz biconditional, (**n** ≡ **n**) = **t**. Their objection is *not* that there is a gap between the resources of object language and metalanguage, but that "there is a gap between the resources of the language that is the original object of investigation and those of the languages that are amenable to fixed point theories". (p. 101) The language that is the original object of investigation can express genuinely paradoxical sentences, whose behaviour is unstable. And one *source* of the language's ability to express such paradoxicalities is the fact that it *can* express

[5]An equivalent definition of $\sigma2_M(h)(A)$ is $\sigma2_M(h)(A)$ = **t** [**f**] iff A is true [false] in all classical models $M' \geq M + h$ such that the extension of ***T*** in M′ is complete and consistent. Gupta and Belnap [3] define a jump operator $\sigma_M^c(h)$ in this way, but for weakly rather than strongly consistent h. Unfortunately, the weak consistency of h does not guarantee the existence of a model $M' \geq M + h$ such that the extension of ***T*** in M′ is complete and consistent. In fact, the existence of such a model M′ is equivalent to the strong consistency of h. $\sigma2_M$ is identical to Gupta and Belnap's σ_M^c, with the definition in [3] corrected so that it is restricted to strongly consistent h.

exclusion negation. A fixed point language cannot, in the end, express genuinely paradoxical sentences: even the liar behaves stably. So fixed point theories do not deliver an analysis of the unstable phenomenon that we are trying to understand. "There are appearances of the Liar here, but they deceive." (p. 96)

Working with a purely two-valued object language, Gupta and Belnap imagine beginning with a classical hypothesis h regarding the extension of ***T***, and then revising h by using the jump operator, or *rule of revision*, τ_M. As the revision procedure proceeds (h, τ_M(h), τ_M^2(h), ...) a liar sentence will flip back and forth between true and false. A truth-teller will keep whatever value it had to begin with. Other sentences might display unstable behaviour to begin with, but eventually settle down to a particular truth value. Some sentences will be very well behaved: they will settle down to a truth value that is independent of the initial hypothesis h. Gupta and Belnap formalize the carrying out of such procedures into the transfinite with their notion of a *revision sequence*.

Given any function ρ on hypotheses, a ρ-*sequence*, or a *revision sequence for* ρ, is an ordinal-length sequence ***S*** of hypotheses such that $S_{\alpha+1} = \rho(S_\alpha)$ for every ordinal α; and such that for every limit ordinal λ, every truth value **x** and every d ∈ D, S_λ(d) = **x** if there is a β < λ such that S_α(d) = **x** for every ordinal α between β and λ. This second clause is the *limit* rule for ρ-sequences. Note that if ***S*** is a ρ-sequence then ρ is defined on S_α for every ordinal α; so, if ***S*** is a τ_M-sequence then S_α is classical for every ordinal α. ***S*** *culminates in* h iff there is a β such that S_α = h for every α ≥ β. For the purposes of the revision theory of truth, we are primarily interested in τ_M-sequences, but other revision sequences are of interest. Note that if ρ = μ, κ, σ, or σ1 or σ2 and if M = ⟨D, I⟩ is a ground model, then there is a unique ρ_M-sequence ***S*** such that S_0(d) = **n** for every d ∈ D. Furthermore, that ρ_M-sequence culminates in lfp(ρ_M).

As mentioned, Gupta and Belnap want to formalize the behaviour of truth, instabilities and all. Relative to a ground model M, this behaviour is arguably represented by the class of τ_M-sequences. Given a ground model M, the class of τ_M-sequences delivers a verdict about which sentences are well-behaved or ill-behaved, as well as a representation of *how* various sentences are ill-behaved. For this reason, Gupta and Belnap propose that the signification of truth is the revision rule τ_M, since this rule arguably fits the Gupta-Belnap characterization (see §1, above) of an expression's or concept's signification. The *most* well-behaved sentences are those that are stably **t** in every τ_M-sequence. Accordingly, Gupta and Belnap introduce the revision theory $\mathbf{T}^*$.

Definition 3.1. ([3]) The sentence A of L^+ *is valid in* M *according to* (the theory) $\mathbf{T}^*$ iff A is stably **t** in all τ_M-sequences. $\mathbf{V}_M^* =_{df} \{A\colon\ A$ is stably **t** in every τ_M-sequence$\}$.

We might want to weaken this condition on the validity of a sentence A in a ground model M. In some ground models, there are sentences that are *nearly* stably **t** in the following sense: they are stably true except possibly at limit ordinals and for a finite number of steps after limit ordinals. Formally, a sentence A of L^+ is *nearly stably* **t** [**f**] in the τ_M-revision sequence $\boldsymbol{S}$ iff there is an ordinal β such that for all $\gamma \geq \beta$, there is a natural number m such that for all $n \geq m$, $\boldsymbol{S}_{\gamma+n}(A) = \mathbf{t}$ [$\mathbf{f}$]. Gupta and Belnap's theory $\mathbf{T}^{\#}$ is based on near stability.

Definition 3.2. ([3]) The sentence A of L^+ *is valid in* M *according to* (the theory) $\mathbf{T}^{\#}$ iff A is nearly stably **t** in all τ_M-sequences. $\mathbf{V}_M^{\#} =_{df} \{A\colon\ A$ is nearly stably **t** in every τ_M-sequence$\}$.

Finally, we might put constraints on which hypotheses are legitimate hypotheses concerning the extension of $\boldsymbol{T}$, and hence on which τ_M-sequences are legitimate revision sequences. A natural condition to put on the legitimacy of a classical hypothesis h is that the resulting extension of $\boldsymbol{T}$ be consistent and complete, i.e. that h be strongly consistent. A τ_M-sequence $\boldsymbol{S}$ is *maximally consistent* iff $\boldsymbol{S}_\alpha$ is *strongly consistent* for every ordinal α. Gupta and Belnap's theory $\mathbf{T}^c$ is based on maximally consistent τ_M-sequences.

Definition 3.3. ([3]) The sentence A of L^+ *is valid in* M *according to* (the theory) $\mathbf{T}^c$ iff A is stably **t** in all maximally consistent τ_M-sequences. $\mathbf{V}_M^c =_{df} \{A\colon\ A$ is stably **t** in every maximally consistent τ_M-sequence$\}$.

All three revision theories are *supervenience* theories in the sense of §2: the behaviour of truth and the status of various sentences is determined by the nonsemantic vocabulary, whose use is represented by the ground model. There is no other way to go in the revision-theoretic setting: for most ground models M there is no class H of privileged hypotheses, like the fixed points, such that for distinct h, h′ ∈ H we could take the expanded models M + h and M + h′ to represent distinct *languages* in which ***T*** represents truth. On the revision theories, each language is represented by a ground model, and the behaviour of truth is represented by the various ways in which one hypothesis leads to another as we carry out the revision process.

§4. Three ways to compare theories of truth. The harder parts of the proofs of the theorems in this section are reserved for §5. The first relation that we define, to compare theories of truth, is the most obvious.

Definition 4.1. Given any two supervenience theories $\mathbf{T}$ and $\mathbf{T}'$, we say that $\mathbf{T} \leq_1 \mathbf{T}'$ iff for every language L every ground model M and every sentence A of L^+, if A is valid in M according to $\mathbf{T}$ then A is valid in M according to $\mathbf{T}'$. We say that $\mathbf{T} <_1 \mathbf{T}'$ iff $\mathbf{T} \leq_1 \mathbf{T}'$ and $\mathbf{T} \not\leq_1 \mathbf{T}'$. Note that $\leq_1$ is reflexive and transitive.

Theorem 4.2. $<_1$ behaves as in the following diagram, i.e. it is the smallest transitive relation satisfying the conditions given in the diagram. Since $\leq_1$ is reflexive, the diagram completely determines $\leq_1$. The subscripted 1 has been dropped from the diagram.

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{T}^{\#} & & & & \\
 & & & & \vee & & & & \\
 & & & & \mathbf{T}^{*} & < & & & \mathbf{T}^{c} \\
 & & & & \vee & & & & \vee \\
\mathbf{T}^{\mathrm{lfp},\,\mu} & < & \mathbf{T}^{\mathrm{lfp},\,\kappa} & < & \mathbf{T}^{\mathrm{lfp},\,\sigma} & < & \mathbf{T}^{\mathrm{lfp},\,\sigma1} & < & \mathbf{T}^{\mathrm{lfp},\,\sigma2} \\
\wedge & & \wedge & & \wedge & & \wedge & & \wedge \\
\mathbf{T}^{\mathrm{gifp},\,\mu} & & \mathbf{T}^{\mathrm{gifp},\,\kappa} & & \mathbf{T}^{\mathrm{gifp},\,\sigma} & & \mathbf{T}^{\mathrm{gifp},\,\sigma1} & & \mathbf{T}^{\mathrm{gifp},\,\sigma2}
\end{array}
$$

Proof. For $\mathbf{T}^{\mathrm{lfp},\,\mu} \leq_1 \mathbf{T}^{\mathrm{lfp},\,\kappa} \leq_1 \mathbf{T}^{\mathrm{lfp},\,\sigma} \leq_1 \mathbf{T}^{\mathrm{lfp},\,\sigma1} \leq_1 \mathbf{T}^{\mathrm{lfp},\,\sigma2}$, it suffices to show that $\mathrm{lfp}(\mu_M) \leq \mathrm{lfp}(\kappa_M) \leq \mathrm{lfp}(\sigma_M) \leq \mathrm{lfp}(\sigma1_M) \leq \mathrm{lfp}(\sigma2_M)$ for any ground model M. For $\rho = \mu, \kappa, \sigma, \sigma1$, and $\sigma2$, let $\boldsymbol{S}(\rho)$ be the unique ρ_M-sequence such that $\boldsymbol{S}(\rho)_0(d) = \mathbf{n}$ for every $d \in D$. By transfinite induction, $\boldsymbol{S}(\mu)_\alpha \leq \boldsymbol{S}(\kappa)_\alpha \leq \boldsymbol{S}(\sigma)_\alpha \leq \boldsymbol{S}(\sigma1)_\alpha \leq \boldsymbol{S}(\sigma2)_\alpha$ for every ordinal α. So $\mathrm{lfp}(\mu_M) \leq \mathrm{lfp}(\kappa_M) \leq \mathrm{lfp}(\sigma_M) \leq \mathrm{lfp}(\sigma1_M) \leq \mathrm{lfp}(\sigma2_M)$, since each $\boldsymbol{S}(\rho)$ culminates in $\mathrm{lfp}(\rho_M)$.

For $\mathbf{T}^{\text{lfp}, \rho} \leq_1 \mathbf{T}^{\text{gifp}, \rho}$ ($\rho = \mu, \kappa, \sigma, \sigma1$, or $\sigma2$), note that $\text{lfp}(\rho_M) \leq \text{gifp}(\rho_M)$ since $\text{lfp}(\rho_M)$ is intrinsic.

$\mathbf{T}^* \leq_1 \mathbf{T}^\#$ and $\mathbf{T}^* \leq_1 \mathbf{T}^c$ can be proved directly from the definitions.

To see that $\mathbf{T}^{\text{lfp}, \sigma} \leq_1 \mathbf{T}^*$, fix a ground model $M = \langle D, I \rangle$ and let $\boldsymbol{S}$ be the unique σ_M-sequence such that $\boldsymbol{S}_0(d) = \mathbf{n}$ for every $d \in D$. Then $\boldsymbol{S}$ culminates in $\text{lfp}(\sigma)$. And let $\boldsymbol{S}'$ be any τ_M-revision sequence. By transfinite induction, it can be proved that $\boldsymbol{S}_\alpha \leq \boldsymbol{S}'_\alpha$ for every ordinal α. So if $\text{lfp}(\sigma)(A) = \mathbf{t}$, then A is stably $\mathbf{t}$ in $\boldsymbol{S}'$. Since $\boldsymbol{S}'$ was arbitrary, if $\text{lfp}(\sigma)(A) = \mathbf{t}$ then A is valid in M according to $\mathbf{T}^*$. Thus $\mathbf{T}^{\text{lfp}, \sigma} \leq_1 \mathbf{T}^*$. Similarly $\mathbf{T}^{\text{lfp}, \sigma2} \leq_1 \mathbf{T}^c$.

This establishes all of the positive claims of the form $\mathbf{T} \leq_1 \mathbf{T}'$ in Theorem 4.2. The counterexamples in §5, below, establish the negative claims of the form $\mathbf{T} \not\leq_1 \mathbf{T}'$. ⊣

Of particular interest are ground models in which truth behaves like a classical concept. Suppose, for example, that one is devising a semantics for languages that contain their own truth predicates. All else being equal, one might want a semantics that delivers, whenever possible, something approaching a classical theory: we know that truth behaves paradoxically, but it seems an advantage to minimalize this paradoxicality. Consider, for example, a classical ground model $M = \langle D, I \rangle$ that makes no distinctions, other than with quote names, among the sentences of L^+: for an extreme case, suppose that L has no nonquote names, no function symbols and no nonlogical predicates. There is no circular reference in the ground model, and there seems to be no vicious reference of any kind. And yet $\text{lfp}(\mu_M)$ and $\text{lfp}(\kappa_M)$ are nonclassical (see the proof of Theorem 4.5): this suggests that truth does not behave like a classical concept in M, at least not according to the least fixed point theories $\mathbf{T}^{\text{lfp}, \mu}$ and $\mathbf{T}^{\text{lfp}, \kappa}$. On the other hand, $\text{gifp}(\mu_M)$ and $\text{gifp}(\kappa_M)$ are both classical, as is $\text{lfp}(\sigma_M)$ (this follows from Corollary 4.24, below). So, at least relative to this particular ground model, the theories $\mathbf{T}^{\text{gifp}, \mu}$, $\mathbf{T}^{\text{gifp}, \kappa}$ and $\mathbf{T}^{\text{lfp}, \sigma}$ have an advantage over $\mathbf{T}^{\text{lfp}, \mu}$ and $\mathbf{T}^{\text{lfp}, \kappa}$. This motivates our definition of $\leq_2$, below (Definition 4.4).

Definition 4.3. Let $\rho = \mu, \kappa, \sigma$, or $\sigma1$ or $\sigma2$. $\mathbf{T}^{\text{lfp}, \rho}$ [$\mathbf{T}^{\text{gifp}, \rho}$] *dictates that truth behaves like a classical concept in the ground model* M iff $A \in \mathbf{V}_M^{\text{lfp}, \rho}$ [$\mathbf{V}_M^{\text{gifp}, \rho}$] or $\neg A \in \mathbf{V}_M^{\text{lfp}, \rho}$ [$\mathbf{V}_M^{\text{gifp}, \rho}$] for every sentence A of L^+. Similarly, $\mathbf{T}^*$ [$\mathbf{T}^\#$, $\mathbf{T}^c$] *dictates that truth behaves like a classical concept in the ground model* M iff $A \in \mathbf{V}^*_M$ [$\mathbf{V}^\#_M$, $\mathbf{V}^c_M$] or $\neg A \in \mathbf{V}^*_M$ [$\mathbf{V}^\#_M$, $\mathbf{V}^c_M$] for every sentence A of L^+.

Definition 4.4. Given any two supervenience fixed point or revision theories $\mathbf{T}$ and $\mathbf{T}'$, we say that $\mathbf{T} \leq_2 \mathbf{T}'$ iff for every language L and every ground model M, if $\mathbf{T}$ dictates that truth behaves like a classical concept in M then so does $\mathbf{T}'$. Note that $\mathbf{T} \leq_2 \mathbf{T}'$ iff, for every language L and every ground model M, if $\mathbf{T}$ dictates that truth behaves like a classical concept in M, then every sentence valid in M according to $\mathbf{T}$ is also valid in M according to $\mathbf{T}'$. We say that $\mathbf{T} \equiv_2 \mathbf{T}'$ iff $\mathbf{T} \leq_2 \mathbf{T}'$ and $\mathbf{T}' \leq_2 \mathbf{T}$. We say that $\mathbf{T} <_2 \mathbf{T}'$ iff $\mathbf{T} \leq_2 \mathbf{T}'$ and $\mathbf{T} \not\equiv_2 \mathbf{T}'$. Note that $\leq_2$ is reflexive and transitive. Note also that if $\mathbf{T} \leq_1 \mathbf{T}'$ then $\mathbf{T} \leq_2 \mathbf{T}'$.

Theorem 4.5. $<_2$ behaves as in the following diagram, i.e. it is the smallest transitive relation satisfying the conditions given in the diagram. Since $\leq_2$ is reflexive, the diagram completely determines $\leq_2$. The subscripted 2 has been dropped from the diagram.

$$\begin{array}{ccccccccccccccccc}
 & & & & \mathbf{T}^\# & & & & & & & & & & & & \\
 & & & & \vee & & & & & & & & & & & & \\
 & & & & \mathbf{T}^* & & < & & \mathbf{T}^c & < & \mathbf{T}^{\text{gifp},\,\sigma 2} & < & \mathbf{T}^{\text{gifp},\,\sigma 1} & < & \mathbf{T}^{\text{gifp},\,\sigma} & < \mathbf{T}^{\text{gifp},\,\kappa} < & \mathbf{T}^{\text{gifp},\,\mu} \\
 & & & & \vee & & & & \vee & & & & & & & & \\
\mathbf{T}^{\text{lfp},\,\mu} & \equiv & \mathbf{T}^{\text{lfp},\,\kappa} & < & \mathbf{T}^{\text{lfp},\,\sigma} & < & \mathbf{T}^{\text{lfp},\,\sigma 1} & < & \mathbf{T}^{\text{lfp},\,\sigma 2} & & & & & & & &
\end{array}$$

Proof. The fact that $\mathbf{T}^{\text{lfp},\,\mu} \equiv_2 \mathbf{T}^{\text{lfp},\,\kappa}$ follows from the fact that, in no ground model does $\mathbf{T}^{\text{lfp},\,\mu}$ or $\mathbf{T}^{\text{lfp},\,\kappa}$ dictate that truth behaves like a classical concept. To see this, choose a ground model $M = \langle D, I \rangle$ and let $\boldsymbol{S}$ be the unique μ_M-sequence such that $\boldsymbol{S}_0(d) = \mathbf{n}$ for every $d \in D$. By transfinite induction, it can be shown that $\boldsymbol{S}_\alpha(\forall x(\boldsymbol{T}x \vee \neg \boldsymbol{T}x)) = \mathbf{n}$ for every ordinal α. But then $\text{lfp}(\mu_M)(\forall x(\boldsymbol{T}x \vee \neg \boldsymbol{T}x)) = \mathbf{n}$ since $\boldsymbol{S}$ culminates in $\text{lfp}(\mu_M)$. Similarly $\text{lfp}(\kappa_M)(\forall x(\boldsymbol{T}x \vee \neg \boldsymbol{T}x)) = \mathbf{n}$.

The following follow from the already proven positive part of Theorem 4.2: $\mathbf{T}^{\text{lfp},\,\kappa} \leq_2 \mathbf{T}^{\text{lfp},\,\sigma} \leq_2 \mathbf{T}^{\text{lfp},\,\sigma 1} \leq_2 \mathbf{T}^{\text{lfp},\,\sigma 2} \leq_2 \mathbf{T}^c$ and $\mathbf{T}^{\text{lfp},\,\sigma} \leq_2 \mathbf{T}^* \leq_2 \mathbf{T}^\#$ and $\mathbf{T}^* \leq_2 \mathbf{T}^c$.

To see that $\mathbf{T}^c \leq_2 \mathbf{T}^{\text{gifp},\,\sigma 2}$, suppose that M is a ground model in which $\mathbf{T}^c$ dictates that truth behaves like a classical concept. So there is a classical hypothesis h in which all maximally consistent τ_M-sequences culminate. It suffices to show that h is the greatest fixed point of $\sigma 2_M$, in which case $\text{gifp}(\sigma 2_M) = h$ is classical, in which case $\mathbf{T}^{\text{gifp},\,\sigma 2}$ dictates that truth behaves like a classical concept in M. Let h′ be any fixed point of $\sigma 2_M$. Since h′ is strongly consistent, we can choose a strongly consistent classical h″ ≥ h′. Let $\boldsymbol{S}$ be any maximally consistent τ_M-sequence with $\boldsymbol{S}_0$ = h″ ≥ h′. By the monotonicity of $\sigma 2_M$ together with the fact that $\sigma 2_M$ agrees with τ_M on all classical hypotheses, we can show by transfinite induction that $\boldsymbol{S}_\alpha$ ≥ h′ for every ordinal α. So h ≥ h′, since $\boldsymbol{S}$ culminates in h. Thus, h is the greatest fixed point of $\sigma 2_M$, as desired.

To see that $\mathbf{T}^{\text{gifp}, \sigma 2} \leq_2 \mathbf{T}^{\text{gifp}, \sigma 1} \leq_2 \mathbf{T}^{\text{gifp}, \sigma} \leq_2 \mathbf{T}^{\text{gifp}, \kappa} \leq_2 \mathbf{T}^{\text{gifp}, \mu}$, order the evaluation schemes transitively as follows, μ ≤ κ ≤ σ ≤ σ1 ≤ σ2; and choose ρ and ρ′ where ρ ≤ ρ′. It suffices to show that if gifp(ρ'_M) is classical then gifp(ρ_M) = gifp(ρ'_M). So suppose that gifp(ρ'_M) is classical. Then it is a fixed point of τ_M, and hence of both ρ_M and ρ'_M. To show that gifp(ρ_M) = gifp(ρ'_M), it suffices to show that h ≤ gifp(ρ'_M) for every fixed point h of ρ_M. Choose a fixed point h of ρ_M. ρ'_M is defined on h—in case ρ'_M is σ1 or σ2, h is strongly consistent since h is a fixed point of ρ_M. Furthermore, h = ρ_M(h) ≤ ρ'_M(h). Thus there is exactly one ρ'_M-sequence $\boldsymbol{S}$ such that $\boldsymbol{S}_0$ = h, and $\boldsymbol{S}$ culminates in some fixed point h′ of ρ'_M, in fact in the least fixed point of ρ'_M such that h ≤ h′. Since gifp(ρ'_M) is classical, gifp(ρ'_M) is the greatest fixed point of ρ'_M. Thus h ≤ h′ ≤ gifp(ρ'_M) as desired.

This establishes all of the positive claims of the form $\mathbf{T} \leq_2 \mathbf{T}'$ in Theorem 4.5. The counterexamples in §5, below, establish the negative claims of the form $\mathbf{T} \nleq_2 \mathbf{T}'$. ⊣

Remark 4.6. Theorem 4.5 answers a question of Gupta and Belnap [3] (Problem 6B.12): "Does the condition ‘lfp($\sigma 2_M$) is classical’ imply ‘M is Thomason’ [we define *Thomason* models below]?" The answer is no, since $\mathbf{T}^{\text{lfp}, \sigma 2} \nleq_2 \mathbf{T}^*$ (see Example 5.11, below) and since, by Theorem 4.8, below, a ground model is Thomason iff $\mathbf{T}^*$ dictates that truth behaves like a classical concept in it.

The next comparative relation, $\leq_3$, is trickier to motivate, and is best understood in the context of investigating whether this or that theory dictates that truth behaves like a classical concept in M.

For starters, it is not always easy to tell whether some theory dictates that truth behaves like a classical concept in M. Gupta and Belnap devote some time to investigating the circumstances under which, in effect, $\mathbf{T}^*$ dictates that truth behaves like a classical concept in a ground model, though they do not put it in these terms. As we shall see, their investigation can be broadened to theories other than $\mathbf{T}^*$. Gupta and Belnap proceed by introducing the notion of a *Thomason* ground model, and by investigating the circumstances under which a ground model is Thomason.

Definition 4.7. ([3]) A ground model M is *Thomason* iff all τ_M-sequences culminate in one and the same fixed point.

Theorem 4.8. A ground model is Thomason iff $\mathbf{T}^*$ dictates that truth behaves like a classical concept in it.

Proof. This follows immediately from the definitions. ⊣

Gupta and Belnap's principal results concerning Thomason models all have the same general character, and all make it relatively easy to show that a wide range of ground models are, in fact, Thomason. The simplest example concerns any ground model M for the language *L* described above: a language with no nonquote names, no function symbols and no nonlogical predicates. Any such model is Thomason. This might be expected since, other than with quote names, there is no way to distinguish in the language among the sentences of the language.

This is a special case of Gupta and Belnap's result, Theorem 4.11, below. Essentially, Theorem 4.11 states that any ground model that cannot distinguish among the sentences, other than with quote names, is Thomason. First we need to make the notion of "distinguishing among sentences" precise.

Definition 4.9. ([3], Definitions 2D.2) Suppose that $M = \langle D, I \rangle$ is a model for *L* and $X \subseteq D$.

(i) The interpretation of a name *c* is X-*neutral* in M iff $I(c) \notin X$.

(ii) The interpretation of an n-place predicate *F* is X-*neutral* in M, iff for all $d_1, ..., d_n, d'_i \in D$, if $d_i, d'_i \in X$ then $I(F)(d_1, ..., d_i, ..., d_n) = I(F)(d_1, ..., d'_i, ..., d_n)$.

(iii) The interpretation of an n-place function symbol *f* is X-*neutral* in M, iff both the range of I(*f*) is disjoint from X and for all $d_1, ..., d_n, d'_i \in D$, if $d_i, d'_i \in X$ then $I(f)(d_1, ..., d_i, ..., d_n) = I(f)(d_1, ..., d'_i, ..., d_n)$.

Definition 4.10. ([3], Definition 6A.2) A model M = ⟨D, I⟩ is X-*neutral* iff the interpretations in M of all the nonquote names, nonlogical predicates, and function symbols are X-neutral.

Theorem 4.11. ([3], Theorem 6A.5) If the ground model M is S-neutral then M is Thomason.

Proof. This is a special case of Corollary 4.24, below. ⊣

Gupta and Belnap strengthen this theorem: Suppose that the ground model can in fact distinguish among sentences, but only among sentences that are in some sense unproblematic, for example among sentences with no occurrences of ***T*** or among μ-grounded sentences. Then M is still Thomason.

Theorem 4.12. ([3], Theorem 6B.4, Convergence to a fixed point I) If M is X-neutral then M is Thomason, provided that X contains either (i) all sentences that have occurrences of ***T***, or (ii) all sentences that are μ-ungrounded in M, or (iii) all sentences that are κ-ungrounded in M, or (iv) all sentences that are σ-ungrounded in M.

Proof. (i) is a special case of Corollary 4.24, below. (ii), (iii) and (iv) are special cases of Theorem 4.21, below. ⊣

Note that (ii), (iii) and (iv) of Theorem 4.12 can be reworded as follows.

Theorem 4.13. Let $\mathbf{V}_M = \mathbf{V}_M^{lfp,\,\mu}$ or $\mathbf{V}_M^{lfp,\,\kappa}$ or $\mathbf{V}_M^{lfp,\,\sigma}$, and suppose that $Y \subseteq \{A\colon A \in \mathbf{V}_M \text{ or } \neg A \in \mathbf{V}_M\}$. Then if the ground model M is (S - Y)-neutral then M is Thomason.

Gupta and Belnap present the following example as an application of Theorem 4.12. This shows how easy it can be, equipped with Theorem 4.12 or 4.13, to show that a ground model is Thomason.

Example 4.14. ([3], Example 6B.6) Suppose that the ground model M = ⟨D, I⟩ is S-neutral except for the name *a*. Furthermore suppose that *Hb* is true in M. Then M is Thomason if (i) I(*a*) = *Hb*, (ii) I(*a*) = ***T***'*Hb*', (iii) I(*a*) = *Hb* ∨ ¬***T****a*, or (iv) I(*a*) = ***T****a* ∨ ¬***T****a*.

Gupta and Belnap's other main theorem concerning Thomason models is as follows.

Theorem 4.15. ([3], Theorem 6B.8, Convergence to a fixed point II) Suppose that M is an (S - Y)-neutral model and that Y ⊆ {*A*: $A \in \mathbf{V}^{*}_{M}$ or $\neg A \in \mathbf{V}^{*}_{M}$}. Then M is Thomason.

Proof. This is a special case of Theorem 4.21, below. ⊣

Gupta and Belnap then go on to ask a related question.

Question 4.16. ([3], Problem 6B.15) Suppose that M is (S - Y)-neutral and that Y ⊆ {*A*: $A \in \mathbf{V}^{c}_{M}$ or $\neg A \in \mathbf{V}^{c}_{M}$}. Is M Thomason?

As pointed out above, an investigation into the conditions under which a ground model M is Thomason is, in effect, an investigation into the conditions under which $\mathbf{T}^{*}$ dictates that truth behaves like a classical concept in M. It turns out that, for a wide range of our theories **T**, if M is (S - Y)-neutral where Y ⊆ {*A*: $A \in \mathbf{V}_{M}$ or $\neg A \in \mathbf{V}_{M}$} and where $\mathbf{V}_{M}$ = {*A*: *A* is valid in the ground model M according to **T**}, then $\mathbf{T}^{*}$ does, in fact, dictate that truth behaves like a classical concept in M. To help generalize this investigation, we define a third relation $\leq_3$ between theories.

Definition 4.17. Suppose that **T** and **T′** are supervenience theories and that, for any ground model M, $\mathbf{V}_{M}$ = {*A*: *A* is valid in the ground model M according to **T**}. We say that $\mathbf{T} \leq_3 \mathbf{T}'$ iff for every language *L* every ground model M and every Y ⊆ {*A*: $A \in \mathbf{V}_{M}$ or $\neg A \in \mathbf{V}_{M}$}, if M is (S - Y)-neutral then **T′** dictates that truth behaves like a classical concept in M. We say that $\mathbf{T} <_3 \mathbf{T}'$ iff $\mathbf{T} \leq_3 \mathbf{T}'$ and $\mathbf{T} \not\leq_3 \mathbf{T}'$. We will see that $\leq_3$ is transitive but not reflexive.

Remark 4.18. Theorem 4.13 (ii), (iii) and (iv) and Theorem 4.15 can be summarized as follows: $\mathbf{T}^{\text{lfp}, \mu} \leq_3 \mathbf{T}^{*}$, $\mathbf{T}^{\text{lfp}, \kappa} \leq_3 \mathbf{T}^{*}$, $\mathbf{T}^{\text{lfp}, \sigma} \leq_3 \mathbf{T}^{*}$ and $\mathbf{T}^{*} \leq_3 \mathbf{T}^{*}$. Question 4.16 amounts to this: $\mathbf{T}^{c} \leq_3 \mathbf{T}^{*}$? Theorem 4.21, below, delivers a negative answer to this question.

Lemma 4.19. $\leq_3$ is transitive.

Proof. Suppose that $\mathbf{T} \leq_3 \mathbf{T}'$ and $\mathbf{T}' \leq_3 \mathbf{T}''$, and that M is an (S - Y)-neutral ground model where $Y \subseteq \{A\colon A \in \mathbf{V}_M \text{ or } \neg A \in \mathbf{V}_M\}$ and where $\mathbf{V}_M = \{A\colon A$ is valid in the ground model M according to $\mathbf{T}\}$. Let $\mathbf{V}'_M = \{A\colon A$ is valid in the ground model M according to $\mathbf{T}'\}$. Note that $S = \{A\colon A \in \mathbf{V}'_M \text{ or } \neg A \in \mathbf{V}'_M\}$, since $\mathbf{T} \leq_3 \mathbf{T}'$. So $Y \subseteq \{A\colon A \in \mathbf{V}'_M \text{ or } \neg A \in \mathbf{V}'_M\}$. So $\mathbf{T}''$ dictates that truth behaves like a classical concept in M, as desired. ⊣

Lemma 4.20. (1) If $\mathbf{T} \leq_3 \mathbf{T}'$ and $\mathbf{T}' \leq_2 \mathbf{T}''$ then $\mathbf{T} \leq_3 \mathbf{T}''$. (2) If $\mathbf{T} \leq_3 \mathbf{T}'$ then $\mathbf{T} \leq_2 \mathbf{T}'$. (3) If $\mathbf{T} \leq_1 \mathbf{T}'$ and $\mathbf{T}' \leq_3 \mathbf{T}''$ then $\mathbf{T} \leq_3 \mathbf{T}''$.

Proof. (1) follows immediately from the definitions. For (2) Suppose that $\mathbf{T} \leq_3 \mathbf{T}'$ and that $\mathbf{T}$ dictates that truth behaves like a classical concept in M. Then M is (S - S)-neutral where $S \subseteq \{A\colon A \in \mathbf{V}_M \text{ or } \neg A \in \mathbf{V}_M\}$. So $\mathbf{T}'$ dictates that truth behaves like a classical concept in M, since $\mathbf{T} \leq_3 \mathbf{T}'$. For (3), assume that $\mathbf{T} \leq_1 \mathbf{T}'$ and $\mathbf{T}' \leq_3 \mathbf{T}''$ and that M is (S - Y)-neutral where $Y \subseteq \{A\colon A \in \mathbf{V}_M \text{ or } \neg A \in \mathbf{V}_M\}$. Since $\mathbf{T} \leq_1 \mathbf{T}'$, M is (S - Y)-neutral where $Y \subseteq \{A\colon A \in \mathbf{V}'_M \text{ or } \neg A \in \mathbf{V}'_M\}$. So, since $\mathbf{T}' \leq_3 \mathbf{T}''$, $\mathbf{T}''$ dictates that truth behaves like a classical concept in M, as desired. ⊣

Theorem 4.21. (1) $<_3$ behaves as in the following diagram, i.e. it is the smallest transitive relation satisfying the conditions given in the diagram. Since $\leq_3$ is not reflexive, we need parts (2) and (3) to completely determine $\leq_3$. The subscripted 3 has been dropped from the diagram.

$$\begin{array}{ccccccccccccccccc}
 & & \mathbf{T}^{\#} \\
 & & \vee \\
 & & \mathbf{T}^{*} & & < & & \mathbf{T}^{c} & < & \mathbf{T}^{gifp,\,\sigma2} & < & \mathbf{T}^{gifp,\,\sigma1} & < & \mathbf{T}^{gifp,\,\sigma} & < & \mathbf{T}^{gifp,\,\kappa} & < & \mathbf{T}^{gifp,\,\mu} \\
 & & \vee & & & & \vee \\
\mathbf{T}^{lfp,\,\kappa} & < & \mathbf{T}^{lfp,\,\sigma} & < & \mathbf{T}^{lfp,\,\sigma1} & < & \mathbf{T}^{lfp,\,\sigma2} \\
 & & \vee \\
 & & \mathbf{T}^{lfp,\,\mu}
\end{array}$$

(2) $\mathbf{T}^{*} \leq_3 \mathbf{T}^{*}$ and $\mathbf{T}^{c} \leq_3 \mathbf{T}^{c}$ and $\mathbf{T}^{lfp,\,\sigma2} \leq_3 \mathbf{T}^{lfp,\,\sigma2}$ and $\mathbf{T}^{gifp,\,\rho} \leq_3 \mathbf{T}^{gifp,\,\rho}$ for $\rho = \mu, \kappa, \sigma, \sigma1$ or $\sigma2$.

(3) $\mathbf{T}^{\#} \not\leq_3 \mathbf{T}^{\#}$ and $\mathbf{T}^{lfp,\,\rho} \not\leq_3 \mathbf{T}^{lfp,\,\rho}$ for $\rho = \mu, \kappa, \sigma$ or $\sigma1$.

Proof. The proofs of (2) and (3) are tricky and left until §5. Given (2) and (3), and Lemma 4.20, and Theorems 4.2 and 4.5, much of the information contained in (1) can be straightforwardly proved. First, every claim of the form $\mathbf{T} \not\leq_2 \mathbf{T}'$ given in Theorem 4.5 implies, given Lemma 4.20 (2), that $\mathbf{T} \not\leq_3 \mathbf{T}'$. Furthermore, the facts that $\mathbf{T}^{\text{lfp}, \mu} \not\leq_3 \mathbf{T}^{\text{lfp}, \kappa}$ and that $\mathbf{T}^{\text{lfp}, \kappa} \not\leq_3 \mathbf{T}^{\text{lfp}, \mu}$ follow from the fact that neither $\mathbf{T}^{\text{lfp}, \mu}$ nor $\mathbf{T}^{\text{lfp}, \kappa}$ ever dictates that truth behaves like a classical concept, even when the ground model is S-neutral, as shown in the proof of Theorem 4.5. The fact that $\mathbf{T}^{*} \leq_3 \mathbf{T}^{c} \leq_3 \mathbf{T}^{\text{gifp}, \sigma 2} \leq_3 \mathbf{T}^{\text{gifp}, \sigma 1} \leq_3 \mathbf{T}^{\text{gifp}, \sigma} \leq_3 \mathbf{T}^{\text{gifp}, \kappa} \leq_3 \mathbf{T}^{\text{gifp}, \mu}$ follows from the fact that $\mathbf{T}^{*} \leq_2 \mathbf{T}^{c} \leq_2 \mathbf{T}^{\text{gifp}, \sigma 2} \leq_2 \mathbf{T}^{\text{gifp}, \sigma 1} \leq_2 \mathbf{T}^{\text{gifp}, \sigma} \leq_2 \mathbf{T}^{\text{gifp}, \kappa} \leq_2 \mathbf{T}^{\text{gifp}, \mu}$ and from (2) and Lemma 4.20 (1). Similarly for the fact that $\mathbf{T}^{\text{lfp}, \sigma 2} \leq_3 \mathbf{T}^{c}$. The fact that $\mathbf{T}^{\text{lfp}, \sigma 1} \leq_3 \mathbf{T}^{\text{lfp}, \sigma 2}$ follows from the fact that $\mathbf{T}^{\text{lfp}, \sigma 1} \leq_1 \mathbf{T}^{\text{lfp}, \sigma 2}$ (Theorem 4.2) and that $\mathbf{T}^{\text{lfp}, \sigma 2} \leq_3 \mathbf{T}^{\text{lfp}, \sigma 2}$ (Theorem 4.21 (2)) and from Lemma 4.20 (3).

So, for Theorem 4.21, it suffices to show (2) and (3), as well as $\mathbf{T}^{\text{lfp}, \mu} \leq_3 \mathbf{T}^{\text{lfp}, \sigma} \leq_3 \mathbf{T}^{\text{lfp}, \sigma 1}$ and $\mathbf{T}^{\text{lfp}, \kappa} \leq_3 \mathbf{T}^{\text{lfp}, \sigma}$. For (2) and (3) see §5. For the rest, see Corollary 4.26. ⊣

Remark 4.22. The positive part of Theorem 4.21 generalizes Gupta and Belnap's Theorems 4.13 (ii), (iii) and (iv), and 4.15, stated above. The negative parts generalize the negative answer to Gupta and Belnap's Question 17, asked above.

The fact that $\mathbf{T}^{\text{lfp}, \sigma} \not\leq_3 \mathbf{T}^{\text{lfp}, \sigma}$ means that the following conjecture is false: If the ground model M is (S - Y)-neutral and Y ⊆ {A: $\text{lfp}(\sigma_M)(A) = \mathbf{t}$ or $\text{lfp}(\sigma_M)(A) = \mathbf{f}$}, then $\text{lfp}(\sigma_M)$ is classical. Similarly for σ1. But we have something almost as good.

Theorem 4.23. (The Proviso Theorem) Let ρ = σ or σ1. If the ground model M is (S - Y)-neutral and Y ⊆ {A: $\text{lfp}(\rho_M)(A) = \mathbf{t}$ or $\text{lfp}(\rho_M)(A) = \mathbf{f}$}, then $\text{lfp}(\rho_M)$ is classical, subject to the following proviso: for every n, there is a sentence $B \notin$ Y of degree > n such that $\text{lfp}(\rho_M)(B) = \mathbf{t}$, and a sentence $B \notin$ Y of degree > n such that $\text{lfp}(\rho_M)(B) = \mathbf{f}$.

Proof. See §5, below. ⊣

Corollary 4.24. If the ground model M is X-neutral, where X contains all sentences that have occurrences of ***T***, then the following theories dictate that truth behaves like a classical concept in M: $\mathbf{T}^{\text{lfp}, \sigma}$, $\mathbf{T}^{\text{lfp}, \sigma 1}$, $\mathbf{T}^{\text{lfp}, \sigma 2}$, $\mathbf{T}^{*}$, $\mathbf{T}^{\#}$, $\mathbf{T}^{c}$, and $\mathbf{T}^{\text{gifp}, \rho}$ for ρ = μ, κ, σ, σ1, or σ2. In particular, if the ground model M is S-neutral, then those theories dictate that truth behaves like a classical concept in M.

Proof. Here we rely on the positive part of Theorem 4.5, which we have already proved. Assume that the ground model M is M is X-neutral, where X contains all sentences that have occurrences of $\boldsymbol{T}$. Let Y = {A: A is a sentence in which $\boldsymbol{T}$ does not occur}. So M is (S - Y)-neutral and Y $\subseteq$ {A: lfp(σ_M)(A) = **t** or lfp(σ_M)(A) = **f**}. Also, we claim that the proviso in Theorem 4.23 is satisfied for $\rho = \sigma$. In particular, for any sentence A, define the sentence $\boldsymbol{T}^0(A)$ = A and $\boldsymbol{T}^{n+1}(A) = \boldsymbol{T}$‘$\boldsymbol{T}^n(A)$’. Then, for every n, the sentence $\boldsymbol{T}^n(\forall x(\boldsymbol{T}x \vee \neg\boldsymbol{T}x))$ is a sentence B $\notin$ Y of degree > n such that lfp(σ_M)(B) = **t** and the sentence $\boldsymbol{T}^n(\neg\forall x(\boldsymbol{T}x \vee \neg\boldsymbol{T}x))$ is a sentence $B \notin$ Y of degree > n such that lfp(σ_M)(B) = **f**. So, by Theorem 4.23, $\mathbf{T}^{\mathrm{lfp},\,\sigma}$ dictates that truth behaves like a classical concept in M. For the other theories $\mathbf{T}^{\mathrm{lfp},\,\sigma1}$, $\mathbf{T}^{\mathrm{lfp},\,\sigma2}$, $\mathbf{T}^{*}$, $\mathbf{T}^{\#}$, $\mathbf{T}^{c}$, and the $\mathbf{T}^{\mathrm{gifp},\,\rho}$, the result follows from this and Theorem 4.5, above. ⊣

Remark 4.25. Theorem 4.24 generalizes Gupta and Belnap’s Theorem 4.13 (i), stated above.

Corollary 4.26. $\mathbf{T}^{\mathrm{lfp},\,\kappa} \leq_3 \mathbf{T}^{\mathrm{lfp},\,\sigma} \leq_3 \mathbf{T}^{\mathrm{lfp},\,\sigma1}$ and $\mathbf{T}^{\mathrm{lfp},\,\mu} \leq_3 \mathbf{T}^{\mathrm{lfp},\,\sigma}$.

Proof. To see that $\mathbf{T}^{\mathrm{lfp},\,\sigma} \leq_3 \mathbf{T}^{\mathrm{lfp},\,\sigma1}$, suppose that M is (S - Y)-neutral and that Y $\subseteq$ {A: lfp(σ_M)(A) = **t** or lfp(σ_M)(A) = **f**}. If $\mathbf{T}^{\mathrm{lfp},\,\sigma}$ dictates that truth behaves like a classical concept in M, then so does $\mathbf{T}^{\mathrm{lfp},\,\sigma1}$. So suppose that $\mathbf{T}^{\mathrm{lfp},\,\sigma}$ does not dictate that truth behaves like a classical concept in M. First notice that Y $\subseteq$ {A: lfp($\sigma1_M$)(A) = **t** or lfp($\sigma1_M$)(A) = **f**}. Also, we claim that the proviso in Theorem 4.23 is satisfied for $\rho = \sigma1$. In particular, choose some sentence C such that lfp(σ_M)(C) = **n**. Then, for every n, the sentence $\boldsymbol{T}^n(\neg(\boldsymbol{T}$‘$C$’& $\boldsymbol{T}$‘$\neg C$’)) is a sentence B $\notin$ Y of degree > n such that lfp($\sigma1_M$)(B) = **t** and the sentence $\boldsymbol{T}^n(\boldsymbol{T}$‘$C$’& $\boldsymbol{T}$‘$\neg C$’) is a sentence $B \notin$ Y of degree > n such that lfp($\sigma1_M$)(B) = **f**. Thus lfp($\sigma1_M$) is classical, as desired.

The proof that $\mathbf{T}^{\mathrm{lfp},\,\kappa} \leq_3 \mathbf{T}^{\mathrm{lfp},\,\sigma}$ is similar. If M is (S - Y)-neutral and Y $\subseteq$ {A: lfp(κ_M)(A) = **t** or lfp(κ_M)(A) = **f**}, then M is (S - Y)-neutral where Y $\subseteq$ {A: lfp(σ_M)(A) = **t** or lfp(σ_M)(A) = **f**}. Furthermore the proviso in Theorem 4.23 is satisfied for $\rho = \sigma$, since for every n, the sentence $\boldsymbol{T}^n(\forall x(\boldsymbol{T}x \vee \neg\boldsymbol{T}x))$ is a sentence $B \notin$ Y of degree > n such that lfp(σ_M)(B) = **t** and the sentence $\boldsymbol{T}^n(\neg\forall x(\boldsymbol{T}x \vee \neg\boldsymbol{T}x))$ is a sentence $B \notin$ Y of degree > n such that lfp(σ_M)(B) = **f**. This suffices. Similarly, $\mathbf{T}^{\mathrm{lfp},\,\mu} \leq_3 \mathbf{T}^{\mathrm{lfp},\,\sigma}$. ⊣

5. Proofs and counterexamples. Each of our main theorems, Theorems 4.2, 4.5 and 4.21, makes positive claims of the form $\mathbf{T} \leq_n \mathbf{T}'$ and negative claims of the form $\mathbf{T} \nleq_n \mathbf{T}'$, for n = 1, 2 or 3. We also want to show Theorem 4.23 (the Proviso Theorem). Given the work already done in §4, it suffices to show Theorem 4.21 (2) and (3); to show Theorem 4.23 (the Proviso Theorem); and to show the negative claims of Theorems 4.2 and 4.5.

We begin with some preliminary notions. Then we prove our *Major Lemma* (Lemma 5.5) and *Major Corollary* (Corollary 5.6), which we will use to help establish our results from §4. Before that we will use the Major Corollary to give a simplified proof of Gupta and Belnap's *Main Lemma* (Lemma 5.7), the lemma they use to study the conditions under which a model is Thomason: our new proof avoids their double transfinite induction, and their consideration, at one point, of six cases and subcases.

Definition 5.1. Suppose that M = ⟨D, I⟩ and M′ = ⟨D′, I′⟩ are models of a first order language *L*, that *N* is a set of names from *L*, and that Ψ:D → D′ is a bijection. Ψ is *an N-restricted isomorphism from* M *to* M′ iff $I(H)(d_1, ..., d_n) = I'(H)(\Psi(d_1), ..., \Psi(d_n))$ for every n-place predicate letter *H* and every n-tuple $\langle d_1, ..., d_n\rangle$; $\Psi(I(h)(d_1, ..., d_n)) = I'(h)(\Psi(d_1), ..., \Psi(d_n))$ for every n-place function symbol *h* (n > 0) and every n-tuple $\langle d_1, ..., d_n\rangle$; and $\Psi(I(c)) = I'(c)$ for every $c \in N$.

Lemma 5.2. Suppose that M and M′ are models of a first order language *L*, that *N* is a set of names from *L*, and that Ψ is an *N*-restricted isomorphism from M to M′. Suppose that ρ = τ, μ, κ or σ. Suppose that every name occurring in the sentence *A* is in *N*. Then $\mathrm{Val}_{M,\rho}(A) = \mathrm{Val}_{M',\rho}(A)$.

Definition 5.3. ([3], Definition 6A.2) The *degree* of a term or formula *X* of L^+, denoted deg(*X*), is defined as follows. (i) If *X* is a variable or nonquote name then deg(*X*) = 0 = deg(⊥). (ii) If *A* is a sentence of degree n, then the deg(‘*A*’) = n + 1. (iii) If $t_1, ..., t_n$ are terms of degrees $i_1, ..., i_n$, respectively, and if *f* [*F*] is an n-place function symbol [predicate], then $\deg(ft_1...t_n)$

$[\deg(Ft_1...t_n)] = \max(i_1, ..., i_n)$. (iv) If x is a variable, A and B are formulas, and $\deg(A) = m$ and $\deg(B) = n$, then $\deg(\forall xA) = \deg(\neg A) = m$ and $\deg(A \,\&\, B) = \deg(A \vee B) = \max(m, n)$.

Definition 5.4. Suppose that $M = \langle D, I\rangle$ is a ground model and that $Y \subseteq S$. Say that $h =_Y h'$ iff $h(A) = h'(A)$ for every $A \in Y$. If n is a natural number, say that $h =_n h'$ iff $h(A) = h'(A)$ for every sentence A of degree < n. Note that $h =_0 h'$ for any h and h′. If h is a classical hypothesis, define $\tau^0_M(h) = h$, and $\tau^{n+1}_M(h) = \tau_M(\tau^n_M(h))$. Finally, define $\tau^\omega_M(h)\text{:}D \rightarrow D$ as follows:

$\tau^\omega_M(h)(d) \quad = \mathbf{t}$, if, for some m, $\tau^n_M(h)(d) = \mathbf{t}$ for every $n \geq m$.

$\tau^\omega_M(h)(d) \quad = \mathbf{f}$, if, for some m, $\tau^n_M(h)(d) = \mathbf{f}$ for every $n \geq m$.

$\tau^\omega_M(h)(d) \quad = \mathbf{n}$ otherwise.

Note that if h is classical, then $\tau^n_M(h)$ is always classical but $\tau^\omega_M(h)$ might not be.

Lemma 5.5. (The Major Lemma) Suppose that the ground model $M = \langle D, I\rangle$ is (S - Y)-neutral, where $Y \subseteq S$. Suppose that h and h′ are strongly consistent classical hypotheses, with $h =_n h'$ and $h =_Y h'$. Then $\tau_M(h) =_{n+1} \tau_M(h')$.

Proof. Let $Y' = \{A\text{: } h(A) = h'(A)\}$. Note that $Y \subseteq Y'$, and that $h =_{Y'} h'$. Also note that $A \in Y'$ iff $\neg A \in Y'$ iff $\neg\neg A \in Y'$ iff $\neg\neg\neg A \in Y'$, etc., since h and h′ are strongly consistent. Thus we have

(*) $(A \notin Y'$ and $h(A) = \mathbf{t})$ iff $(\neg A \notin Y'$ and $h(\neg A) = \mathbf{f})$ iff $(\neg\neg A \notin Y'$ and $h(\neg\neg A) = \mathbf{t})$ iff $(\neg\neg\neg A \notin Y'$ and $h(\neg\neg\neg A) = \mathbf{f})$, etc.

Let $U = \{A\text{: } A$ is of degree $\geq n$ and $A \notin Y$ and $h(A) = \mathbf{t}\}$ and $V = \{A\text{: } A$ is of degree $\geq n$ and $A \notin Y$ and $h(A) = \mathbf{f}\}$. Similarly, let $U' = \{A\text{: } A$ is of degree $\geq n$ and $A \notin Y$ and $h'(A) = \mathbf{t}\}$ and $V' = \{A\text{: } A$ is of degree $\geq n$ and $A \notin Y$ and $h'(A) = \mathbf{f}\}$. Note that $U \cup V = U' \cup V'$.

If $(U \cup V) \cap (S - Y') = \emptyset$, then every sentence of degree $\geq n$ is in Y′. In that case, h = h′ and we are done. So assume that $(U \cup V) \cap (S - Y') \neq \emptyset$. Given (*), $A \in U \cap (S - Y')$ iff $\neg A \in V \cap (S - Y')$ iff $\neg\neg A \in U \cap (S - Y')$ iff $\neg\neg\neg A \in V \cap (S - Y')$, etc., for every sentence A. So U and V are countably infinite (we are assuming that the language is countable). Similarly, U′ and V′ are countably infinite. Let Φ be a bijection from U ∪ V to U′ ∪ V′ such that Φ maps U onto U′ and V onto V′.

Define a function Ψ:D → D as follows:

If *A* is a sentence of degree < n or *A* ∈ Y, then Ψ(*A*) = *A*.

If *A* is a sentence of degree ≥ n, then Ψ(*A*) = Φ(*A*).

If d ∈ (D - S), then Ψ(d) = d.

Note that Ψ is an *N*-restricted isomorphism from M + h to M + h′, where *N* is the set of names of degree ≤ n. So $\mathrm{Val}_{M + h, \tau}(A) = \mathrm{Val}_{M + h', \tau}(A)$, for every sentence *A* of degree < n + 1. So $\tau_M(h) =_{n+1} \tau_M(h')$. ⊣

Corollary 5.6. (The Major Corollary) Suppose that the ground model M = ⟨D, I⟩ is (S - Y)-neutral, where Y ⊆ S. Suppose that h and h′ are strongly consistent classical hypotheses such that $\tau_M^n(h) =_Y \tau_M^{n+1}(h) =_Y \tau_M^n(h') =_Y \tau_M^{n+1}(h')$ for every n. Then $\tau_M^\omega(h) = \tau_M^\omega(h')$ is classical and is a fixed point of τ_M.

Proof. By induction, we can show that $\tau_M^n(h) =_n \tau_M^{n+1}(h) =_n \tau_M^n(h') =_n \tau_M^{n+1}(h')$ for every n. The base case is vacuously true. The induction step is simply an application of the Major Lemma. But from this it follows that $\tau_M^\omega(h) = \tau_M^\omega(h')$ and $\tau_M^\omega(h)$ is classical. It remains to show that $\tau_M^\omega(h)$ is a fixed point of τ_M. Note that $\tau_M^\omega(h) =_n \tau_M^n(h)$ for every n. So, by the Major Lemma, $\tau_M(\tau_M^\omega(h)) =_{n+1} \tau_M^{n+1}(h)$ for every n. So $\tau_M(\tau_M^\omega(h)) =_{n+1} \tau_M^\omega(h)$ for every n. So $\tau_M(\tau_M^\omega(h)) = \tau_M^\omega(h)$, as desired. ⊣

Lemma 5.7. (Gupta and Belnap's Main Lemma, [3], Lemma 6A.4) Let M = ⟨D, I⟩ be X-neutral (X ⊆ D). Let ***S*** and ***S′*** be τ_M-sequences, and let Y be the set of those sentences that are either stably **t** in both ***S*** and ***S′*** or stably **f** in both. If (S - Y) ⊆ X, then there is some ordinal α such that for all β ≥ α, $\boldsymbol{S}_\alpha = \boldsymbol{S'}_\beta$.

Proof. This proof differs from Gupta and Belnap's. Choose an ordinal γ such that, by the γth stage both in ***S*** and in ***S′***, all of the sentences in Y have stabilized: i.e., for every *A* ∈ Y and every β ≥ γ, $\boldsymbol{S}_\beta(A) = \boldsymbol{S'}_\beta(A) = \boldsymbol{S}_\gamma(A) = \boldsymbol{S'}_\gamma(A)$. In other words, for every β ≥ γ, $\boldsymbol{S}_\beta =_Y \boldsymbol{S'}_\beta =_Y \boldsymbol{S}_\gamma =_Y$

$\boldsymbol{S}'_\gamma$. γ can be chosen to be a successor ordinal. So $\boldsymbol{S}_\gamma$ and $\boldsymbol{S}'_\gamma$ are strongly consistent. By our Major Corollary, $\tau^\omega_M(\boldsymbol{S}_\gamma) = \tau^\omega_M(\boldsymbol{S}'_\gamma)$ is classical and is a fixed point of τ_M. But notice that, since $\tau^\omega_M(\boldsymbol{S}_\gamma) = \tau^\omega_M(\boldsymbol{S}'_\gamma)$ is classical, we have $\boldsymbol{S}_{\gamma+\omega} = \tau^\omega_M(\boldsymbol{S}_\gamma)$ and $\boldsymbol{S}'_{\gamma+\omega} = \tau^\omega_M(\boldsymbol{S}'_\gamma)$ by the limit rule for τ_M-sequences. Let $\alpha = \gamma + \omega$. Since $\boldsymbol{S}_\alpha = \boldsymbol{S}'_\alpha$ is a fixed point of τ_M, we conclude that for all $\beta \geq \alpha$, $\boldsymbol{S}_\alpha = \boldsymbol{S}'_\beta$, as desired. ⊣

Now we can start proving our positive results from §4.

Theorem 4.21 (2). (i) $\mathbf{T}^* \leq_3 \mathbf{T}^*$. (ii) $\mathbf{T}^c \leq_3 \mathbf{T}^c$. (iii) $\mathbf{T}^{lfp,\,\sigma2} \leq_3 \mathbf{T}^{lfp,\,\sigma2}$. (iv) $\mathbf{T}^{gifp,\,\rho} \leq_3 \mathbf{T}^{gifp,\,\rho}$ for $\rho = \mu, \kappa, \sigma, \sigma1$ or $\sigma2$.

Proof. (i) (The proof of (i) is from [3].) Suppose that M is an (S - Y)-neutral model and that $Y \subseteq \{A\colon\ A \in \mathbf{V}^*_M \text{ or } \neg A \in \mathbf{V}^*_M\}$. To show that all τ_M-sequences culminate in one and the same fixed point, choose any two τ_M-sequences, $\boldsymbol{S}$ and $\boldsymbol{S}'$. Let X = (S - Y), and let Y′ be the set of those sentences that are either stably **t** in both $\boldsymbol{S}$ and $\boldsymbol{S}'$ or stably **f** in both. Clearly (S - Y′) ⊆ X. So, by Gupta and Belnap's Main Lemma (Lemma 10.5), there is some ordinal α such that for all $\beta \geq \alpha$, $\boldsymbol{S}_\alpha = \boldsymbol{S}'_\beta$. It follows that $\boldsymbol{S}_\alpha = \boldsymbol{S}'_\alpha$ is a fixed point in which both $\boldsymbol{S}$ and $\boldsymbol{S}'$ culminate.

(ii) is proved analogously to (i), since it suffices to show that if M is an (S - Y)-neutral model where $Y \subseteq \{A\colon\ A \in \mathbf{V}^c_M \text{ or } \neg A \in \mathbf{V}^c_M\}$, then all maximally consistent τ_M-sequences culminate in one and the same fixed point.

(iii) Suppose that M is (S - Y)-neutral for some Y ⊆ {*A*: lfp($\sigma2_M$)(*A*) = **t** or lfp($\sigma2_M$)(*A*) = **f**}. To show that h = lfp($\sigma2_M$) is classical, suppose not. Let *C* be a sentence of the least possible degree, say k, such that h(*C*) = **n**. Note that $C \notin Y$. We will get a contradiction by showing that h(*C*) = **t** or **f**. Recall the definition of $\sigma2_M(A)$ for sentences *A*:

$\sigma2_M$(h)(*A*) = **t** [**f**] iff τ_M(h′)(*A*) = **t** [**f**] for all classical and strongly consistent h′ ≥ h.
n, otherwise.

To show that h(*C*) = **t** or **f** it suffices to show that $\sigma2_M$(h)(*C*) = **t** or **f**, since h is a fixed point of $\sigma2_M$. For the latter, it suffices to show that τ_M(h′)(*C*) = τ_M(h″)(*C*) for any classical and strongly consistent hypotheses h′ ≥ h and h″ ≥ h. Choose such hypotheses h′ and h″. Note that

$h' =_k h''$ since $h(A) = \mathbf{t}$ or $\mathbf{f}$, for any sentence A of degree $< k$. Note also that $h' =_Y h''$. So by our Major Lemma 5.5, $\tau_M(h') =_{k+1} \tau_M(h'')$. Thus $\tau_M(h')(C) = \tau_M(h'')(C)$, as desired.

(iv) We will show something more general. Fix a ground model M. If ρ is a partial function on the set of hypotheses, we say that ρ is *normal* iff ρ satisfies the following conditions: ρ is monotone; if h is classical and ρ is defined on h, then $\rho(h) = \tau_M(h)$; for every fixed point h of ρ, there is a classical hypothesis h′ such that $h \le h'$ and ρ is defined on h′; if ρ is defined on the classical hypothesis h, then ρ is also defined on $\tau_M(h)$; and ρ is defined on every fixed point of τ_M. Note that μ_M, κ_M, σ_M, $\sigma1_M$, and $\sigma2_M$ are all normal.

Suppose that ρ is a normal operator on hypotheses, and that i is an intrinsic fixed point of ρ. Suppose that M is (S - Y)-neutral where $i(A) = \mathbf{t}$ or $i(A) = \mathbf{f}$ for every sentence $A \in Y$. We will show that gifp(ρ) is classical. This will suffice for our claim that $\mathbf{T}^{gifp,\rho} \le_3 \mathbf{T}^{gifp,\rho}$ for ρ = μ, κ, σ, σ1 or σ2.

To show that gifp(ρ) is classical, it will suffice to show that ρ has a greatest fixed point which is classical: any classical greatest fixed point is also the greatest intrinsic fixed point. For this it suffices to show that for any fixed points f and g, there is a classical fixed point h such that $f \le h$ and $g \le h$. So choose any fixed points f and g. Since i is intrinsic, there exist fixed points f′ and g′ such that $f \le f'$ and $i \le f'$ and $g \le g'$ and $i \le g'$. Choose classical hypotheses, not necessarily fixed points, $f'' \ge f'$ and $g'' \ge g'$, so that ρ is defined on both f″ and g″. Here is a picture.

```
   f″       g″            classical hypotheses, maybe not fixed points
    \      /
     f′  g′               fixed points, maybe not classical
    / \ / \
   /   i   \              intrinsic fixed point, maybe not classical
  /         \
 f           g            fixed points, maybe not classical
```

Observe: $\tau_M^n(f'') = \rho^n(f'') \ge \rho^n(f') = f' \ge i$ and $\tau_M^n(g'') = \rho^n(g'') \ge \rho^n(g') = f' \ge i$ for every n. Recall that $Y \subseteq \{A\colon\ i(A) = \mathbf{t}$ or $i(A) = \mathbf{f}\}$. So $\tau_M^n(f'') =_Y i =_Y \tau_M^n(g'')$ for every n. Thus $\tau_M^n(f'') =_Y \tau_M^{n+1}(f'') =_Y \tau_M^n(g'') =_Y \tau_M^{n+1}(g'')$, for every n. Let $h = \tau_M^\omega(f'')$. By our Major Corollary 5.6, h =

$\tau^{\omega}_{M}(f'') = \tau^{\omega}_{M}(g'')$ is classical and is a fixed point of τ_M and hence of ρ. It now suffices to show that h ≥ f and h ≥ g. For this it suffices to show that h ≥ f′ and h ≥ g′. Note that if f′(*A*) = **t** then $\tau^{n}_{M}(f'')(A)$ = **t** for every n, since $\tau^{n}_{M}(f'') \geq f'$. So h(*A*) = $\tau^{\omega}_{M}(f'')(A)$ = **t**. Similarly, if f′(*A*) = **f** then h(*A*) = **f**. So h ≥ f′. Similarly, h ≥ g′, as desired. In the picture below, the arrow pointing from f″ to h indicates that any revision sequence that begins with f″ culminates in h. Similarly for the arrow pointing from g″ to h.

```
    f″→ h ← g″
     \ / \ /
      f′  g′
     / \ / \
    /   i   \
   /         \
  f           g
```

⊣

Theorem 4.23 will be a corollary to Lemma 5.8, a reworking of the Major Lemma.

Lemma 5.8. Suppose that the ground model M = 〈D, I〉 is (S - Y)-neutral, where Y ⊆ S. Suppose that h and h′ are classical hypotheses, with h $=_n$ h′ and h $=_Y$ h′. Suppose furthermore that all four of the following sets U, U′, V, and V′ are countably infinite: U = {*A*: *A* is of degree ≥ n and *A* ∉ Y and h(*A*) = **t**} and V = {*A*: *A* is of degree ≥ n and *A* ∉ Y and h(*A*) = **f**} and U′ = {*A*: *A* is of degree ≥ n and *A* ∉ Y and h′(*A*) = **t**} and V′ = {*A*: *A* is of degree ≥ n and *A* ∉ Y and h′(*A*) = **f**}. Then $\tau_M(h) =_{n+1} \tau_M(h')$.

Proof. The proof follows the proof of Lemma 5.5, with a simplification: there is no need to define Y′ or to mention its properties, since there is no need to prove that U, U′, V and V′ are countably infinite, since that is given by hypothesis. ⊣

Theorem 4.23. Let ρ = σ or σ1. If the ground model M is (S - Y)-neutral and Y ⊆ {*A*: lfp(ρ_M)(*A*) = **t** or lfp(ρ_M)(*A*) = **f**}, then lfp(ρ_M) is classical, subject to the following proviso: for every n, there is a sentence *B* ∉ Y of degree > n such that lfp(ρ_M)(*B*) = **t**, and a sentence *B* ∉ Y of degree > n such that lfp(ρ_M)(*B*) = **f**.

Proof. We will run the proof for ρ = σ. The proof is exactly the same for ρ = σ1. The proof closely follows the proof of Theorem 4.21 (2)(iii), with h = lfp(σ_M). So suppose that the ground model M is (S - Y)-neutral; that $Y \subseteq \{A\text{: } \mathrm{lfp}(\sigma_M)(A) = \mathbf{t} \text{ or } \mathrm{lfp}(\sigma_M)(A) = \mathbf{f}\}$; and that, for every n, there is a sentence $B \notin Y$ of degree > n such that $\mathrm{lfp}(\rho_M)(B) = \mathbf{t}$, and a sentence $B \notin Y$ of degree > n such that $\mathrm{lfp}(\rho_M)(B) = \mathbf{f}$. For a reductio, suppose that $h = \mathrm{lfp}(\sigma_M)$ is not classical.

Let C be a sentence of the least possible degree, say k, such that $h(C) = \mathbf{n}$. Note that $C \notin Y$. We will get a contradiction by showing that $h(C) = \mathbf{t}$ or $\mathbf{f}$. Recall that, for any sentence A,

$$\sigma_M(h)(A) = \begin{cases} \mathbf{t}\ [\mathbf{f}] \text{ iff } \tau_M(h')(A) = \mathbf{t}\ [\mathbf{f}] \text{ for all classical } h' \geq h\text{; and} \\ \mathbf{n}\text{, otherwise.} \end{cases}$$

To show that $h(C) = \mathbf{t}$ or $\mathbf{f}$ it suffices to show that $\sigma_M(h)(C) = \mathbf{t}$ or $\mathbf{f}$, since h is a fixed point. For the latter, it suffices to show that $\tau_M(h')(C) = \tau_M(h'')(C)$ for any classical hypotheses $h' \geq h$ and $h'' \geq h$. Choose such hypotheses h′ and h″. Note that $h' =_k h''$ since $h(A) = \mathbf{t}$ or $\mathbf{f}$, for any sentence A of degree < k. Note also that $h' =_Y h''$.

Define four sets U′, U″, V′, and V″ as follows: $U' = \{A\text{: } A \text{ is of degree} \geq k \text{ and } A \notin Y \text{ and } h'(A) = \mathbf{t}\}$ and $V' = \{A\text{: } A \text{ is of degree} \geq k \text{ and } A \notin Y \text{ and } h'(A) = \mathbf{f}\}$ and $U'' = \{A\text{: } A \text{ is of degree} \geq k \text{ and } A \notin Y \text{ and } h''(A) = \mathbf{t}\}$ and $V'' = \{A\text{: } A \text{ is of degree} \geq k \text{ and } A \notin Y \text{ and } h''(A) = \mathbf{f}\}$. We claim that U′ is countably infinite (assuming the language is countable). Recall that for every n, there is a sentence $B \notin Y$ of degree > n such that $\mathrm{lfp}(\rho_M)(B) = \mathbf{t}$. So for every n, there is a sentence $B \notin Y$ of degree > n such that $h(B) = \mathbf{t}$. So U′ is countably infinite. Similarly, U″, V′ and V″ are countably infinite. So $\tau_M(h') =_{k+1} \tau_M(h'')$, by Lemma 5.8. Thus $\tau_M(h')(C) = \tau_M(h'')(C)$, as desired. ⊣

It remains to prove Theorem 4.21 (3), and the negative claims in Theorems 4.2 and 4.5. We do this with a series of counterexamples. We will bring it all together after presenting the examples.

Example 5.9. ([3], Example 6B.9) This example will show that $\mathbf{T}^\# \not\leq_3 \mathbf{T}^\#$. Consider a language L with no nonquote names, with no function symbols, with a one-place predicate G, and no other nonlogical predicates. Let L^+ be L extended with a new one-place predicate $\boldsymbol{T}$. We will also suppose that L has a quote name 'C' for every sentence C of L^+. Let $A = \exists x(Gx \,\&\, \neg \boldsymbol{T}x)$ and let $Y = \{\boldsymbol{T}^n A\text{: } n \geq 0\}$. Let $M = \langle D, I\rangle$ be the ground model where D is the set of sentences of L^+ and where $I(G)(d) = \mathbf{t}$ iff $d \in Y$. Note that every sentence in Y is nearly stably **t** in every τ_M-sequence, though no sentence in Y is stably **t** in any τ_M-sequence. So $C \in \mathbf{V}^{\#}_M$, for all $C \in Y$. So M is (S - Y)-neutral where $Y \subseteq \{A\text{: } A \in \mathbf{V}^{\#}_M \text{ or } \neg A \in \mathbf{V}^{\#}_M\}$. We will now show that there is a τ_M-sequences $\boldsymbol{S}$ such that the sentence $B = \exists x \exists y(Gx \,\&\, Gy \,\&\, \neg \boldsymbol{T}x \,\&\, \neg \boldsymbol{T}y \,\&\, x \neq y)$ is neither nearly stably **t** in $\boldsymbol{S}$ nor nearly stably **f** in $\boldsymbol{S}$. Thus $\mathbf{T}^{\#}$ does not dictate that truth behaves like a classical concept in M. Incidentally, this falsifies the claim in [3] that "all sentences are nearly stable in all τ-sequences for M" (p. 214).

Define sets $X_0 = Y$ and $X_{n+1} = Y - \{\boldsymbol{T}^n A\}$ for $n \geq 0$. Also define $Z_n = Y - \{\boldsymbol{T}^n A, \boldsymbol{T}^{n+1} A\}$. There is a τ_M-sequence $\boldsymbol{S}$ such that, for each $C \in Y$, each limit ordinal λ and each $n \geq 0$,

$\boldsymbol{S}_n(C) = \mathbf{t}$ iff $C \in X_n$

$\boldsymbol{S}_{\lambda+\omega^2+n}(C) = \mathbf{t}$ iff $C \in Z_n$

$\boldsymbol{S}_{\lambda+n}(C) = \mathbf{t}$ iff $C \in X_n$, if λ is a limit ordinal not of the form $\alpha + \omega^2$.

Note that $\boldsymbol{S}_{\lambda+\omega^2+n+1}(B) = \mathbf{t}$ and $\boldsymbol{S}_{\lambda+\omega+n+1}(B) = \mathbf{f}$, for every limit ordinal λ and every natural number n. So B is not nearly stable in $\boldsymbol{S}$. ⊣

Example 5.10. (Gupta) This example will show that $\mathbf{T}^{\#} \not\leq_2 \mathbf{T}^{*}$ and that $\mathbf{T}^{\#} \not\leq_2 \mathbf{T}^{\text{gifp}, \mu}$. Modify Example 5.9 as follows. Let Y be the smallest set containing each $\boldsymbol{T}^n A$, and such that if $C \in Y$ then $C \vee C \in Y$. Note that every sentence in Y is nearly stable **t** in every revision sequence, but no sentence in Y is stably **t** or stably **f** in any revision sequence. So τ_M has no classical fixed point. So neither $\mathbf{T}^{*}$ nor $\mathbf{T}^{\text{gifp}, \mu}$ dictates that truth behaves like a classical concept in M. But it follows from Claim 2, below, that $\mathbf{T}^{\#}$ does dictate that truth behaves like a classical concept in M.

Notice that, for any classical hypothesis h and any $n \geq 0$, we have the following: for countably many $C \in Y$ of degree $\geq n$, $\tau^2_M(h)(C) = \mathbf{t}$ and for countably many $C \in Y$ of degree $\geq$

n, $\tau^2_M(h)(C) = \mathbf{f}$. Similarly, for countably many $C \notin Y$ of degree ≥ n, $\tau^2_M(h)(C) = \mathbf{t}$ and for countably many $C \notin Y$ of degree ≥ n, $\tau^2_M(h)(C) = \mathbf{f}$.

Claim 1. For any two classical hypotheses h and h′ and any n ≥ 0, $\tau^{n+2}_M(h) =_n \tau^{n+2}_M(h')$. Fix h and h′. Our result is proved by induction on n. The base case is vacuously true. For the inductive step, assume that $\tau^{n+2}_M(h) =_n \tau^{n+2}_M(h')$. To show that $\tau^{n+3}_M(h) =_{n+1} \tau^{n+3}_M(h')$, we will construct an *N*-restricted isomorphism Ψ from $M + \tau^{n+2}_M(h)$ to $M + \tau^{n+2}_M(h')$, where $N = \{$'*A*': $\deg(A) < n\}$. Define U, U′, V, V′, W, W′, X and X′ as follows:

$$U =_{df} \{A: \deg(A) \geq n \text{ and } A \in Y \text{ and } \tau^{n+2}_M(h) = \mathbf{t}\}$$
$$U' =_{df} \{A: \deg(A) \geq n \text{ and } A \in Y \text{ and } \tau^{n+2}_M(h') = \mathbf{t}\}$$
$$V =_{df} \{A: \deg(A) \geq n \text{ and } A \in Y \text{ and } \tau^{n+2}_M(h) = \mathbf{f}\}$$
$$V' =_{df} \{A: \deg(A) \geq n \text{ and } A \in Y \text{ and } \tau^{n+2}_M(h') = \mathbf{f}\}$$
$$W =_{df} \{A: \deg(A) \geq n \text{ and } A \notin Y \text{ and } \tau^{n+2}_M(h) = \mathbf{t}\}$$
$$W' =_{df} \{A: \deg(A) \geq n \text{ and } A \notin Y \text{ and } \tau^{n+2}_M(h') = \mathbf{t}\}$$
$$X =_{df} \{A: \deg(A) \geq n \text{ and } A \notin Y \text{ and } \tau^{n+2}_M(h) = \mathbf{f}\}$$
$$X' =_{df} \{A: \deg(A) \geq n \text{ and } A \notin Y \text{ and } \tau^{n+2}_M(h') = \mathbf{f}\}.$$

Each of these sets is countably infinite. Define Ψ by patching together the identity function on the sentences of degree < n, and bijections from U to U′, V to V′, W to W′ and X to X′.

Claim 2. For any sentence *A* of degree < n, either (i) $\tau^m_M(h)(A) = \mathbf{t}$ for every classical hypothesis h and every m ≥ n + 2; or (ii) $\tau^m_M(h)(A) = \mathbf{f}$ for every classical hypothesis h and every m ≥ n + 2. To see this, consider any classical hypotheses h and h′ and any m, m′ ≥ n + 2. Note that if we apply Claim 1 to $\tau^{m-(n+2)}_M(h)$ and $\tau^{m'-(n+2)}_M(h')$, we get $\tau^m_M(h)(A) = \tau^{m'}_M(h')(A)$. This suffices. ⊣

Example 5.11. This example will be of a ground model M such that $lfp(\sigma1_M)$ and $lfp(\sigma2_M)$ are classical, and furthermore such that M is (S - Y)-neutral where $Y \subseteq \{B: B \in \mathbf{V}^c_M \text{ or } \neg B \in \mathbf{V}^c_M\}$. On the negative side, neither $\mathbf{T}^*$ nor $\mathbf{T}^\#$ dictates that truth behaves like a classical concept in M. Thus, $\mathbf{T}^{lfp,\,\sigma1} \not\lesssim_2 \mathbf{T}^*$ and $\mathbf{T}^{lfp,\,\sigma1} \not\lesssim_2 \mathbf{T}^\#$; $\mathbf{T}^{lfp,\,\sigma2} \not\lesssim_2 \mathbf{T}^*$ and $\mathbf{T}^{lfp,\,\sigma2} \not\lesssim_2 \mathbf{T}^\#$; and $\mathbf{T}^c \not\lesssim_3 \mathbf{T}^*$ and $\mathbf{T}^c$

$\not\leq_3 \mathbf{T}^{\#}$, from which it follows—given Theorem 4.21 (2) and Lemma 4.20—that $\mathbf{T}^{c} \not\leq_2 \mathbf{T}^{*}$ and $\mathbf{T}^{c} \not\leq_2 \mathbf{T}^{\#}$. The fact that $\mathbf{T}^{c} \not\leq_3 \mathbf{T}^{*}$ negatively answers Gupta and Belnap's Question 4.16, above.

Consider a language L with exactly one nonquote name b, with no function symbols, and with a one-place predicate G, and no other nonlogical predicates. Let L^{+} be L extended with a new one-place predicate $\boldsymbol{T}$. We will also suppose that L has a quote name 'B' for every sentence B of L^{+}. For any sentence B of L^{+}, we define $\boldsymbol{T}^{n}B$ as follows: $\boldsymbol{T}^{0}B = B$ and $\boldsymbol{T}^{n+1}B = \boldsymbol{T}`\boldsymbol{T}^{n}B'$. For any formula $B$ of $L^{+}$, we define $\neg^{n}B$ as $B$ when n is even and as $\neg B$ when n is odd. Let $A$ be the sentence $\boldsymbol{T}`\boldsymbol{T}b' \,\&\, \boldsymbol{T}`\neg\boldsymbol{T}b'$. Let $\mathrm{Z} = \{\boldsymbol{T}^{n}A\text{: } n \geq 0\}$. Let $\mathrm{Y} = \mathrm{Z} \cup \{\exists x(Gx \,\&\, \boldsymbol{T}x) \,\&\, \neg\boldsymbol{T}b\}$. Let $\mathrm{M} = \langle \mathrm{D}, \mathrm{I}\rangle$ be a ground model, where D is the set of sentences of L^{+}, and where $\mathrm{I}(b) = \exists x(Gx \,\&\, \boldsymbol{T}x) \,\&\, \neg\boldsymbol{T}b$, and $\mathrm{I}(G)(\mathrm{d}) = \mathbf{t}$ iff $\mathrm{d} \in \mathrm{Z}$. Note that M is (S - Y)-neutral.

Claim 1. Neither $\mathbf{T}^{*}$ nor $\mathbf{T}^{\#}$ dictates that truth behaves like a classical concept in M. Proof: Say that the classical hypothesis h is *interesting* iff $\mathrm{h}(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \mathrm{h}(\boldsymbol{T}b) = \mathrm{h}(\neg\boldsymbol{T}b) = \mathbf{t}$ and $\mathrm{h}(B) = \mathbf{f}$, for every $B \in \mathrm{Z}$. Then, for any interesting hypothesis h, if $k \geq 2$ then $\tau^{k}_{\mathrm{M}}(\mathrm{h})(\boldsymbol{T}^{k-1}A) = \tau^{k}_{\mathrm{M}}(\mathrm{h})(\neg^{k-1}\boldsymbol{T}b) = \tau^{k}_{\mathrm{M}}(\mathrm{h})(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \mathbf{t}$ and $\tau^{k}_{\mathrm{M}}(\mathrm{h})(\neg^{k}\boldsymbol{T}b) = \tau^{k}_{\mathrm{M}}(\mathrm{h})(\boldsymbol{T}^{n}A) = \mathbf{f}$, where $n \neq k - 1$. So we can construct a τ-sequence $\boldsymbol{S}$ for M such that $\boldsymbol{S}_{\lambda}$ is interesting for every limit ordinal λ and such that the value of $\boldsymbol{T}b$ never stabilizes. In fact, we can assure that $\boldsymbol{T}b$ is not even nearly stable.

Claim 2. For every $B \in \mathrm{Y}$, either $B \in \mathbf{V}^{c}_{\mathrm{M}}$ or $\neg B \in \mathbf{V}^{c}_{\mathrm{M}}$. Proof: It suffices to show that every sentence in the set Y is stably $\mathbf{f}$ in any maximally consistent τ-sequence $\boldsymbol{S}$. So suppose that $\boldsymbol{S}$ is a maximally consistent τ-sequence. Then $\boldsymbol{S}_{n}(A) = \mathbf{f}$, for each n, by the strong consistency of $\boldsymbol{S}_{n}$. So $\boldsymbol{S}_{k}(\boldsymbol{T}^{n}A) = \mathbf{f}$ for $k \geq 0$ and $n \leq k$. So $\boldsymbol{S}_{\omega}(\boldsymbol{T}^{n}A) = \mathbf{f}$ for every n. So $\boldsymbol{S}_{\omega+1}(\exists x(Gx \,\&\, \boldsymbol{T}x) \,\&\, \neg\boldsymbol{T}b) = \boldsymbol{S}_{\omega+1}(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \boldsymbol{S}_{\omega+1}(\boldsymbol{T}^{n}A) = \mathbf{f}$, for every n. So $\boldsymbol{S}_{\omega+2}(\boldsymbol{T}b) = \boldsymbol{S}_{\omega+2}(\exists x(Gx \,\&\, \boldsymbol{T}x) \,\&\, \neg\boldsymbol{T}b) = \boldsymbol{S}_{\omega+2}(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \boldsymbol{S}_{\omega+2}(\boldsymbol{T}^{n}A) = \mathbf{f}$, for every n. So for every $\alpha \geq \omega + 2$ and every n, $\boldsymbol{S}_{\alpha}(\boldsymbol{T}b) = \boldsymbol{S}_{\alpha}(\exists x(Gx \,\&\, \boldsymbol{T}x) \,\&\, \neg\boldsymbol{T}b) = \boldsymbol{S}_{\alpha}(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \boldsymbol{S}_{\alpha}(\boldsymbol{T}^{n}A) = \mathbf{f}$. So every sentence in Y is stably $\mathbf{f}$ in $\boldsymbol{S}$.

Claim 3. lfp($\sigma 1_M$) is classical. Proof: It suffices, given Theorem 4.23, to prove that lfp($\sigma 1_M$)(B) = **f** for every sentence $B \in$ Y. Let $\boldsymbol{S}$ be the $\sigma 1_M$-sequence that iteratively builds lfp($\sigma 1_M$) from the null hypothesis: $\boldsymbol{S}_0$(d) = **n** for each d $\in$ D. Note that $\boldsymbol{S}_{k+1}(A)$ = **f**, for natural numbers k. The reason is that in calculating $\boldsymbol{S}_{k+1}(A)$, we consider weakly consistent classical h $\geq \boldsymbol{S}_k$. So $\boldsymbol{S}_{k+1}(\boldsymbol{T}^n A)$ = **f** for k $\geq$ 0 and n $\leq$ k. So $\boldsymbol{S}_\omega(\boldsymbol{T}^n A)$ = **f** for every n. Thus, as in the proof of Claim 2, for every $\alpha \geq \omega + 2$, $\boldsymbol{S}_\alpha(\boldsymbol{T}b) = \boldsymbol{S}_\alpha(\exists x(Gx \;\&\; \boldsymbol{T}x) \;\&\; \neg\boldsymbol{T}b) = \boldsymbol{S}_\alpha(\exists x(Gx \;\&\; \boldsymbol{T}x)) = \boldsymbol{S}_\alpha(\boldsymbol{T}^n A)$ = **f**, for every n. Thus, lfp($\sigma 1_M$)(B) = **f** for every sentence $B \in$ Y, as desired.

Claim 4. lfp($\sigma 2_M$) is classical. Proof: Note that $\sigma 1_M(h) \leq \sigma 2_M(h)$ for any strongly consistent hypothesis h. So lfp($\sigma 1_M$) $\leq$ lfp($\sigma 2_M$). So lfp($\sigma 2_M$) is classical, given Claim 3. ⊣

Example 5.12. (Gupta) This example will show that $\mathbf{T}^{\text{lfp}, \sigma} \not\leq_3 \mathbf{T}^{\text{lfp}, \sigma}$. Consider a language L with no nonquote names, with no function symbols, with a one-place predicate G and no other nonlogical predicates. Let L^+ be L extended with a new one-place predicate $\boldsymbol{T}$. We will also suppose that L has a quote name 'B' for every sentence B of L^+. Let D = S $\cup$ $\mathbb{N}$. For each Y $\subseteq$ S, let Y* = {A: $\neg A \in$ Y}. For each Y $\subseteq$ D, we will use the notation [Y] for the ground model $\langle$D, $I_Y\rangle$, where

$I_Y(G)$(d) = **t** if d $\in$ Y, and

$I_Y(G)$(d) = **f** if d $\notin$ Y.

For nonintersecting U, V $\subseteq$ D, we will use the notation (U, V) for the hypothesis h such that

h(d) = **t** if d $\in$ U,

h(d) = **f** if d $\in$ V,

h(d) = **n** otherwise.

We define a jump operator, ϕ, not on hypotheses but rather on subsets of S. For each Y $\subseteq$ S, $\phi(Y) =_{df} \{A\text{: } \mathrm{Val}_{[Y \cup \mathbb{N}] + (Y, Y^* \cup \mathbb{N}), \sigma}(A) = \mathbf{t}\}$. Though ϕ is not in any sense monotone, it will come in handy, as we shall see. Let $Y_0 = \varnothing$. Let $Y_{n+1} = \phi(Y_n)$. Let Y_ω = {A: there is an n such that $A \in Y_m$ for every m $\geq$ n} = $\cup_n \cap_{m \geq n} Y_m$.

Below, we will prove that the hypothesis $(Y_\omega, Y_\omega^* \cup \mathbb{N})$ is not classical, and is the least fixed point of $\sigma_{[Y_\omega \cup \mathbb{N}]}$. But note that the ground model $[Y_\omega \cup \mathbb{N}]$ is $(S - Y_\omega)$-neutral and $\text{lfp}(\sigma_{[Y_\omega \cup \mathbb{N}]})(A) = \mathbf{t}$ for every $A \in Y_\omega$. Thus $\mathbf{T}^{\text{lfp}, \sigma} \not\leq_3 \mathbf{T}^{\text{lfp}, \sigma}$, as desired.

Our argument that $(Y_\omega, Y_\omega^* \cup \mathbb{N}) = \text{lfp}(\sigma_{[Y_\omega \cup \mathbb{N}]})$ proceeds in numbered claims.

Claim 1. $\forall x(\boldsymbol{T}x \supset Gx) \notin Y_n$ and $\neg\forall x(\boldsymbol{T}x \supset Gx) \notin Y_n$. The proof is by induction. It is vacuously true for n = 0. For the inductive step, assume that $\forall x(\boldsymbol{T}x \supset Gx) \notin Y_n$ and $\neg\forall x(\boldsymbol{T}x \supset Gx) \notin Y_n$. To show that $\forall x(\boldsymbol{T}x \supset Gx) \notin Y_{n+1}$ and $\neg\forall x(\boldsymbol{T}x \supset Gx) \notin Y_{n+1}$, it suffices to show that $\text{Val}_{[Y_n \cup \mathbb{N}] + (Y_n, Y_n^* \cup \mathbb{N}), \sigma}(\forall x(\boldsymbol{T}x \supset Gx)) = \mathbf{n}$. Consider the classical hypotheses, h = $(Y_n, D - Y_n)$ and $h' = (Y_n \cup \{\forall x(\boldsymbol{T}x \supset Gx)\}, (D - Y_n) - \{\forall x(\boldsymbol{T}x \supset Gx)\})$. By the inductive hypothesis, we have $(Y_n, Y_n^* \cup \mathbb{N}) \leq h, h'$. Furthermore, $\text{Val}_{[Y_n \cup \mathbb{N}] + h, \tau}(\forall x(\boldsymbol{T}x \supset Gx)) = \mathbf{t}$ and $\text{Val}_{[Y_n \cup \mathbb{N}] + h', \tau}(\forall x(\boldsymbol{T}x \supset Gx)) = \mathbf{f}$. So $\text{Val}_{[Y_n \cup \mathbb{N}] + (Y_n, Y_n^* \cup \mathbb{N}), \sigma}(\forall x(\boldsymbol{T}x \supset Gx)) = \mathbf{n}$, as desired.

Claim 2. $(Y_\omega, Y_\omega^* \cup \mathbb{N})$ is not classical. Proof: Given Claim 1, $\forall x(\boldsymbol{T}x \supset Gx) \notin Y_\omega$ and $\forall x(\boldsymbol{T}x \supset Gx) \notin Y_\omega^*$.

Before we state Claim 3, we define $X_n =_{df} S - (Y_n \cup Y_n^*)$ and $X_\omega =_{df} (S - (Y_\omega \cup Y_\omega^*))$.

Claim 3. For each $n \geq 1$ and for each m, there is some sentence of degree m in Y_n and some sentence of degree m in X_n. Proof: Note that $(\boldsymbol{T}^m A \vee \neg\boldsymbol{T}^m A) \in Y_n$ and that $(\boldsymbol{T}^m A \vee \neg\boldsymbol{T}^m A)$ & $\forall x(\boldsymbol{T}x \supset Gx) \in X_n$, for any sentence A.

Before we state Claim 4, we introduce some notation. For $U, V \subseteq S$, say that $U =_n V$ iff for every A of degree < n, $A \in U$ iff $A \in V$.

Claim 4. For every n and every $m \geq n + 1$, $Y_{n+1} =_n Y_m$. The proof is by induction on n. It is vacuously true for n = 0. For the induction step assume that $Y_{n+1} =_n Y_m$. We want to show that $Y_{n+2} =_{n+1} Y_{m+1}$. It suffices to construct an N-restricted isomorphism Ψ from $[Y_{n+1} \cup \mathbb{N}]$ to $[Y_m \cup \mathbb{N}]$, where $N = \{`A\text{'}: \deg(A) < n\}$. Define seven subsets of S as follows.

U	$=_{df}$	$\{A: \deg(A) < n\}$
V	$=_{df}$	$\{A: \deg(A) \geq n \;\&\; A \in Y_{n+1}\}$
W	$=_{df}$	$\{A: \deg(A) \geq n \;\&\; A \in Y_{n+1}^*\}$

$Z =_{df} \{A\colon\ deg(A) \geq n\ \&\ A \in X_{n+1}\}$

$V' =_{df} \{A\colon\ deg(A) \geq n\ \&\ A \in Y_m\}$

$W' =_{df} \{A\colon\ deg(A) \geq n\ \&\ A \in Y_m{}^*\}$

$Z' =_{df} \{A\colon\ deg(A) \geq n\ \&\ A \in X_m\}$.

Note that each of V, W, Z, V′, W′ and Z′ is countably infinite, by Claim 3. Also note that

$(S - U) = V \cup W \cup Z = V' \cup W' \cup Z'$,

$Y_{n+1} \cap U = Y_m \cap U$, and

$Y_{n+1}{}^* \cap U = Y_m{}^* \cap U$.

Let Ψ:D $\rightarrow$ D be a bijection such that $\Psi(d) = d$ for every $d \in D - S$ and $\Psi(A) = A$ for every $A \in U$; and such that Ψ maps V onto V′ and W onto W′ and Z onto Z′. Then Ψ is an *N*-restricted isomorphism, as desired.

Claim 5. $(Y_\omega, Y_\omega{}^* \cup \mathbb{N})$ is a fixed point of $\sigma_{[Y_\omega \cup \mathbb{N}]}$. For this, it suffices to show that Y_ω is a fixed point of ϕ. For this, it suffices to show that $\phi(Y_\omega) =_{n+1} Y_\omega$ for every n. Given Claim 4, $Y_\omega =_{n+1} Y_{n+2}$ for every n. So it suffices to show that $\phi(Y_\omega) =_{n+1} Y_{n+2}$ for every n. Choose any n. Note that $Y_\omega =_n Y_{n+1}$, by Claim 4. To show that $\phi(Y_\omega) =_{n+1} Y_{n+2}$, it suffices to construct an *N*-restricted isomorphism from $[Y_\omega \cup \mathbb{N}]$ to $[Y_{n+1} \cup \mathbb{N}]$, where $N = \{$'A': $deg(A) < n\}$. The construction follows the lines of the construction in the proof of Claim 4.

Claim 6. $(Y_\omega, Y_\omega{}^* \cup \mathbb{N}) = lfp(\sigma_{[Y_\omega \cup \mathbb{N}]})$. Let $(Z, Z^* \cup \mathbb{N}) = lfp(\sigma_{[Y_\omega \cup \mathbb{N}]})$. For Claim 6, it suffices to show by induction on n that $Y_\omega =_n Z$, for each n. The base case is obvious. So suppose that $Y_\omega =_n Z$. We want to show that $Y_\omega =_{n+1} Z$. Note, incidentally, that $Y_\omega{}^* =_n Z^*$.

$Z \subseteq Y_\omega$, since $(Z, Z^* \cup \mathbb{N}) = lfp(\sigma_{[Y_\omega \cup \mathbb{N}]}) \leq (Y_\omega, Y_\omega{}^* \cup \mathbb{N})$. So it suffices to show that for every sentence A of degree $< n + 1$, if $A \notin Z$ then $A \notin Y_\omega$. So suppose that $deg(A) < n + 1$ and $A \notin Z$. Then there is some classical hypothesis $(X, D - X) \geq (Z, Z^* \cup \mathbb{N})$ such that A is false in the classical model $[Y_\omega \cup \mathbb{N}] + (X, D - X)$. To show that $A \notin Y_\omega$, we will construct a classical hypothesis $(W, D - W) \geq (Y_\omega, Y_\omega{}^* \cup \mathbb{N})$ such that A is false in the classical model $[Y_\omega \cup \mathbb{N}] + (W, D - W)$. After we construct (W, D - W), it will suffice to define an *N*-restricted isomorphism Ψ from $[Y_\omega \cup \mathbb{N}] + (X, D - X)$ to $[Y_\omega \cup \mathbb{N}] + (W, D - W)$, where $N = \{$'B': $\deg(B) < n\}$.

Define seven disjoint subsets of S, as follows:

U	$=_{df}$	$\{A\colon \deg(A) < n\}$
A	$=_{df}$	$(X \cap Y_\omega) - U$
B	$=_{df}$	$(X \cap Y_\omega^*) - U$
C	$=_{df}$	$X - (Y_\omega \cup Y_\omega^* \cup U)$
F	$=_{df}$	$((S - X) \cap Y_\omega) - U$
G	$=_{df}$	$((S - X) \cap Y_\omega^*) - U$
H	$=_{df}$	$(S - X) - (Y_\omega \cup Y_\omega^* \cup U)$

Note the following:

X	$=$	$A \dot\cup B \dot\cup C \dot\cup (U \cap X)$
$(S - X)$	$=$	$F \dot\cup G \dot\cup H \dot\cup (U \cap (S - X))$
$C \dot\cup H$	$=$	$S - (Y_\omega \cup Y_\omega^* \cup U)$
$Y_\omega - U$	$=$	$A \dot\cup F$
$Y_\omega^* - U$	$=$	$B \dot\cup G$
$Y_\omega \cap U$	$\subseteq$	$X \cap U$, since $Y_\omega =_n Z$
$Y_\omega^* \cap U$	$\subseteq$	$(S - X) \cap U$, since $Y_\omega^* =_n Z^*$
$Z - U$	$\subseteq$	A
$Z^* - U$	$\subseteq$	G

Note also that each of the following sets contains sentences of arbitrarily large degree: Z, Z^*, and $S - (Y_\omega \cup Y_\omega^*)$. So each of the following sets is countably infinite: A, G, and $C \dot\cup H$.

Choose $P \subseteq C$ and $Q \subseteq H$ so that $P \dot\cup Q$ is of the same cardinality as $B \dot\cup C$. And let $R_1 = C - P$ and $R_2 = H - Q$. Finally, let J be a set of even numbers of the same cardinality as F. And let $K = \mathbb{N} - J$. K is countably infinite.

Let $W = (X \cap U) \dot\cup A \dot\cup F \dot\cup P \dot\cup Q$. Then $S - W = ((S - X) \cap U) \dot\cup B \dot\cup G \dot\cup R_1 \dot\cup R_2$. So $Y_\omega = (Y_\omega \cap U) \dot\cup A \dot\cup F \subseteq W$, and $Y_\omega^* = ((S - X) \cap U) \dot\cup B \dot\cup G \subseteq S - W$. So $(W, D - W) \geq (Y_\omega, Y_\omega^* \cup \mathbb{N})$.

Construct an *N*-restricted isomorphism Ψ from $M = [Y_\omega \cup \mathbb{N}] + (X, D - X)$ to $M' = [Y_\omega \cup \mathbb{N}] + (W, D - W)$ by patching together

the identity function on U,

a bijection from A onto $Y_\omega - U = A \dot\cup F$,

a bijection from $B \dot\cup C$ onto $P \dot\cup Q$,

a bijection from $G \dot\cup R_1 \dot\cup R_2$ onto $B \dot\cup G \dot\cup R_1 \dot\cup R_2$,

a bijection from F onto J, and

a bijection from $\mathbb{N} = J \dot\cup K$ onto K.

To see that Ψ is an *N*-restricted isomorphism from M to M′, first note that Ψ maps the extension of G in M onto the extension of G in M′. The reason is that $Y_\omega \cup \mathbb{N} = (U \cap Y_\omega) \dot\cup A \dot\cup F \dot\cup J \dot\cup K$ and Ψ maps A to $A \dot\cup F$, and F to J, and $J \dot\cup K$ to K. Also, Ψ maps $X = (U \cap X) \dot\cup A \dot\cup B \dot\cup C$ to $W = (U \cap X) \dot\cup A \dot\cup F \dot\cup P \dot\cup Q$, since Ψ maps A onto $A \dot\cup F$, and $B \dot\cup C$ onto $P \dot\cup Q$. So Ψ maps the extension of ***T*** in M onto the extension of ***T*** in M′. Finally note that for every name ‘*A*’ in *N*, Ψ maps the denotation of ‘*A*’ in M to the denotation of ‘*A*’ in M′, since $\Psi(B) = B$ if $B \in U$. Thus Ψ is an *N*-restricted homomorphism and Claim 6 is proved. ⊣

Example 5.13. (Gupta) Here we modify Example 5.12 to get a proof that $\mathbf{T}^{\text{lfp}, \sigma 1} \not\leq_3 \mathbf{T}^{\text{lfp}, \sigma 1}$. As we shall see, our modified example will also show that $\mathbf{T}^{\text{lfp}, \sigma 2} \not\leq_2 \mathbf{T}^{\text{lfp}, \sigma 1}$.

Example 5.13 is like Example 5.12, except that the definition of the jump operator ϕ must now be $\phi(Y) =_{df} \{A\text{: } \text{Val}_{[Y \cup \mathbb{N}] + (Y, Y^* \cup \mathbb{N}), \sigma 1}(A) = \mathbf{t}\}$. For the proof of Claim 1, we have to check that the two hypotheses, $h = (Y_n, D - Y_n)$ and $h' = (Y_n \cup \{\forall x(\boldsymbol{T}x \supset Gx)\}, (D - Y_n) - \{\forall x(\boldsymbol{T}x \supset Gx)\})$, are not only classical but also weakly consistent. It suffices to check that $Y_n \cup \{\forall x(\boldsymbol{T}x \supset Gx)\}$ is consistent for every n. If n = 0, then it is obvious. If n = k + 1, then note that every sentence in $Y_n \cup \{\forall x(\boldsymbol{T}x \supset Gx)\}$ is true in the classical model $Y_k \cup \mathbb{N} + (Y_k, D - Y_k)$. So $Y_n \cup \{\forall x(\boldsymbol{T}x \supset Gx)\}$. The proofs of the analogues Claims 2, 3, 4 and 5 go through unmodified, so that $(Y_\omega, Y_\omega^* \cup \mathbb{N})$ is a nonclassical fixed point of $\sigma1_{[Y_\omega \cup \mathbb{N}]}$.

We have to modify the construction in the proof of the analogue of Claim 6 as follows. In the fourth sentence of the second paragraph, we start with some *weakly consistent* classical hypothesis $(X, D - X) \geq (Z, Z^* \cup \mathbb{N})$ such that A is false in the classical model $[Y_\omega \cup \mathbb{N}] + (X, D - X)$. To show that $A \notin Y_\omega$, we will construct a *weakly consistent* classical hypothesis $(W, D - W) \geq (Y_\omega, Y_\omega^* \cup \mathbb{N})$ such that A is false in the classical model $[Y_\omega \cup \mathbb{N}] + (W, D - W)$.

Up until the choice of $P \subseteq C$ and $Q \subseteq H$, the construction proceeds exactly as above. Before we choose P and Q, we will prove that $(X \cap U) \cup Y_\omega = (X \cap U) \dot\cup A \dot\cup F$ is consistent. Suppose not. Then, by compactness, $Y_\omega \cup \{B_1, ..., B_k\}$ is inconsistent for some $B_1, ..., B_k \in (X \cap U)$. So Y_ω logically implies $B =_{df} \neg(B_1 \& ... \& B_k)$. So $B \in Y_\omega$. B is of degree < n, since each $B_i \in U$. So $B \in Z$, since $Y_\omega =_n Z$. But $Z \subseteq X$ and $\{B_1, ..., B_k\} \subseteq X$. So X is inconsistent. So $(X, D - X)$ is not weakly consistent, a reductio. So $(X \cap U) \cup Y_\omega$ is consistent.

Now we will choose $P \subseteq C$ and $Q \subseteq H$, but more carefully than above. Note that $C \dot\cup H$ contains infinitely many sentences and is closed under negation. Also $(X \cap U) \cup Y_\omega$ is consistent. So there are countably infinitely many sentences in $C \dot\cup H$ that are consistent with $(X \cap U) \cup Y_\omega$. So we can choose $P \subseteq C$ and $Q \subseteq H$ so that $(X \cap U) \dot\cup A \dot\cup F \dot\cup P \dot\cup Q = (X \cap U) \cup Y_\omega \dot\cup P \dot\cup Q$ is consistent and so that $P \dot\cup Q$ has the same cardinality as $B \dot\cup C$.

Let $W = (X \cap U) \dot\cup A \dot\cup F \dot\cup P \dot\cup Q$, as above. W is consistent. So the hypothesis $(W, D - W)$ is weakly consistent. The construction of the restricted isomorphism goes through as above. So A is false in the classical model $[Y_\omega \cup \mathbb{N}] + (W, D - W)$, as desired.

Thus $(Y_\omega, Y_\omega^* \cup \mathbb{N}) = \mathrm{lfp}(\sigma1_{[Y_\omega \cup \mathbb{N}]})$ and is nonclassical. But note that the ground model $[Y_\omega \cup \mathbb{N}]$ is $(S - Y_\omega)$-neutral and $\mathrm{lfp}(\sigma1_{[Y_\omega \cup \mathbb{N}]})(A) = \mathbf{t}$ for every $A \in Y_\omega$. Thus $\mathbf{T}^{\mathrm{lfp},\, \sigma1} \not\leq_3 \mathbf{T}^{\mathrm{lfp},\, \sigma1}$, as desired.

We furthermore claim that lfp($\sigma2_{[Y_\omega \cup \mathbb{N}]}$) is classical. Firstly, lfp($\sigma1_{[Y_\omega \cup \mathbb{N}]}$) ≤ lfp($\sigma2_{[Y_\omega \cup \mathbb{N}]}$). So the ground model $[Y_\omega \cup \mathbb{N}]$ is (S - Y_ω)-neutral and lfp($\sigma2_{[Y_\omega \cup \mathbb{N}]}$)(*A*) = **t** for every $A \in Y_\omega$. So lfp($\sigma2_{[Y_\omega \cup \mathbb{N}]}$)(*A*) is classical, since $\mathbf{T}^{\text{lfp, }\sigma2} \leq_3 \mathbf{T}^{\text{lfp, }\sigma2}$, as proved above. Thus $\mathbf{T}^{\text{lfp, }\sigma2} \nleq_2 \mathbf{T}^{\text{lfp, }\sigma1}$. ⊣

Example 5.14. This example will show that $\mathbf{T}^{\text{gifp, }\mu} \nleq_2 \mathbf{T}^{\text{gifp, }\kappa}$. Consider a language *L* with exactly two nonquote names, *b* and *c*, no function symbols and no nonlogical predicates. Let M = ⟨S, I⟩ be that ground model such that I(*b*) = *B* = ***T**b* & ***T**c*, and I(*c*) = *C* = ***T**b* ∨ ¬***T**c*. The facts in the following table can easily be established by calculating:

If ⟨h(*B*), h(*C*)⟩	=	tt	tf	tn	ft	ff	fn	nt	nf	nn
then ⟨μ_M(h)(*B*), μ_M(h)(*C*)⟩	=	tt	ft	nn	ff	ft	nn	nn	nn	nn
and ⟨κ_M(h)(*B*), κ_M(h)(*C*)⟩	=	tt	ft	nt	ff	ft	fn	nn	ft	nn

Given this table, we can argue as in Gupta and Belnap's Transfer Theorem ([3], Theorem 2D.4) to the following conclusion: μ_M has three fixed points, which are completely determined by the ordered triple ⟨h(*B*), h(*C*), h(∀*x*(***T**x* ∨ ¬***T**x*))⟩ and κ_M has three fixed points, which are completely determined by the ordered triple ⟨h(*B*), h(*C*), h(∀*x*(***T**x* ∨ ¬***T**x*))⟩. Furthermore τ_M has exactly one fixed point, and in that fixed point *B* and *C* are both **t**. Also, that unique fixed point of τ_M is also a fixed point of μ_M and κ_M. The fixed points of μ_M and κ_M line up as follows:

fixed points of μ_M fixed points of κ_M

```
      ttt                        ttt
       |                          |
      ttn                fnn     ttn
       |                    \   /
      nnn                    nnn
```

Thus gifp(μ_M) is classical but gifp(κ_M) is not. ⊣

Example 5.15. This example will show that $\mathbf{T}^{\text{gifp, }\kappa} \nleq_2 \mathbf{T}^{\text{gifp, }\sigma}$. Consider a language *L* with exactly two nonquote names, *b* and *c*, no function symbols and no nonlogical predicates. Let M = ⟨S, I⟩ be that ground model such that I(*b*) = *B* = ***T**b* ∨ (***T**c* & ¬***T**c*), and I(*c*) = *C* = (***T**b* & (***T**c*

$\vee \neg \boldsymbol{T}c)) \vee (\neg \boldsymbol{T}b \;\&\; \neg \boldsymbol{T}c)$. The facts in the following table can easily be established by calculating:

If $\langle h(B), H(C)\rangle$	=	tt	tf	tn	ft	ff	fn	nt	nf	nn
then $\langle \kappa_M(h)(B), \kappa_M(h)(C)\rangle$	=	tt	tt	tn	ff	ft	nn	nn	nn	nn
and $\langle \sigma_M(h)(B), \sigma_M(h)(C)\rangle$	=	tt	tt	tt	ff	ft	fn	nn	nt	nn

Given this table, we can argue as in Gupta and Belnap's Transfer Theorem ([3], Theorem 2D.4) to the following conclusion: κ_M has four fixed points, which are completely determined by the ordered triple $\langle h(B), h(C), h(\forall x(\boldsymbol{T}x \vee \neg \boldsymbol{T}x))\rangle$ and σ_M has three fixed points, which are completely determined by the ordered pair $\langle h(B), h(C)\rangle$. (The reason we only need look at these pairs of truth values is that the proviso in Gupta and Belnap's Transfer Theorem can be dropped for σ.) Furthermore τ_M has exactly one fixed point, and in that fixed point B and C are both **t**. Also, that unique fixed point of τ_M is also a fixed point of κ_M and σ_M. The fixed points of κ_M and σ_M line up as follows:

fixed points of κ_M fixed points of σ_M

```
ttt              fn    tt
 |                 \  /
ttn                 nn
 |
tnn
 |
nnn
```

Thus gifp(κ_M) is classical, but gifp(σ_M) is not. ⊣

Example 5.16. This example will show that $\mathbf{T}^{\text{gifp}, \sigma} \not\leq_2 \mathbf{T}^{\text{gifp}, \sigma 1}$. Consider a language L with exactly four nonquote names, b, c, d and e, no function symbols and no nonlogical predicates. Let $M = \langle S, I\rangle$ be that ground model such that $I(b) = B = \boldsymbol{T}b \vee (\boldsymbol{T}d \;\&\; \boldsymbol{T}e)$, $I(c) = C = \boldsymbol{T}b \vee \neg \boldsymbol{T}c$, $I(d) = D = \boldsymbol{T}c$ and $I(e) = E = \neg \boldsymbol{T}c$. The facts in the following table can be established by calculating. The asterisks are classical wildcards, either **t** or **f**, and the question marks can vary with the wildcards:

If $\langle h(B), h(C), h(D), h(E)\rangle$	=	tt**	ft**	*f**
then $\langle \tau_M(h)(B), \tau_M(h)(C), \tau_M(h)(B), \tau_M(h)(C)\rangle$	=	tttf	?ftf	?tft

From Gupta and Belnap's Transfer Theorem, we can conclude that τ_M has a unique fixed point, say h_0, where $h_0(B) = h_0(C) = h_0(D) = \mathbf{t}$ and $h_0(E) = \mathbf{f}$. Since h_0 is a fixed point of τ_M, it is also a fixed point of σ_M and of $\sigma 1_M$.

Furthermore, by an argument similar to that given for the Transfer Theorem, we can conclude that the fixed points of σ are completely determined by the values $\langle h(B), h(C), h(D), h(E)\rangle$. (The reason we only need look at these quartuples of truth values is that the proviso in Gupta and Belnap's Transfer Theorem can be dropped for σ.) Thus, we can conclude that h_0 is the only classical fixed point of σ_M, and the only fixed point h of σ_M for which $h(B) = h(C) = h(D) = \mathbf{t}$ and $h(E) = \mathbf{f}$. In fact, h_0 is the only fixed point h of σ such that $h(B) = h(C) = \mathbf{t}$, since any fixed point satisfying this also satisfies $h(D) = \mathbf{t}$ and $h(E) = \mathbf{f}$.

Claim 1. σ has no fixed points h such that $h(B) = \mathbf{f}$ or $h(C) = \mathbf{f}$. To see this, let h be any fixed point of σ. Suppose that $h(C) = \mathbf{f}$. Then, since h is a fixed point of σ_M, $h(\boldsymbol{T}c) = h(\boldsymbol{T}`C')$ $= h(C) = \mathbf{f}$, so that $h(C) = h(\boldsymbol{T}b \vee \neg\boldsymbol{T}c) = \mathrm{Val}_{M + h, \sigma}(\boldsymbol{T}b \vee \neg\boldsymbol{T}c) = \mathbf{t}$, a contradiction. On the other hand, suppose that $h(B) = \mathbf{f}$. Then $h(C) = \mathbf{t}$ or $\mathbf{n}$. If $h(C) = \mathbf{t}$ then, since h is a fixed point of σ_M, $h(\boldsymbol{T}c) = h(\boldsymbol{T}`C') = h(C) = \mathbf{t}$, so that $h(C) = h(\boldsymbol{T}b \vee \neg\boldsymbol{T}c) = \mathrm{Val}_{M + h, \sigma}(\boldsymbol{T}b \vee \neg\boldsymbol{T}c) = \mathbf{f}$, a contradiction. So $h(C) = \mathbf{n}$. So $h(D) = h(\boldsymbol{T}c) = \mathbf{n} = h(\neg\boldsymbol{T}c) = h(E)$. Let h′ be a classical hypothesis such that $h' \geq h$ and $h'(\boldsymbol{T}c) = h'(\neg\boldsymbol{T}c) = \mathbf{t}$, and let h″ be a classical hypothesis such that $h'' \geq h$ and $h''(\boldsymbol{T}c) = h''(\neg\boldsymbol{T}c) = \mathbf{f}$. Then $\mathrm{Val}_{M + h', \tau}(\boldsymbol{T}d) = \mathrm{Val}_{M + h'', \tau}(\boldsymbol{T}e) = \mathbf{t}$ and $\mathrm{Val}_{M + h'', \tau}(\boldsymbol{T}d)$ $= \mathrm{Val}_{M + h'', \tau}(\boldsymbol{T}e) = \mathbf{f}$. Thus $\tau_M(h')(B) = \mathrm{Val}_{M + h', \tau}(\boldsymbol{T}b \vee (\boldsymbol{T}d \,\&\, \boldsymbol{T}e)) = \mathbf{t}$ and $\tau_M(h'')(B) =$ $\mathrm{Val}_{M + h'', \tau}(\boldsymbol{T}b \vee (\boldsymbol{T}d \,\&\, \boldsymbol{T}e)) = \mathbf{f}$. So $\sigma_M(h)(B) = \mathbf{n}$. This contradicts h's being a fixed point of σ_M. This proves Claim 1.

Given Claim 1, σ has no fixed point that are incompatible with h_0. Thus h_0 is σ-intrinsic. Thus, since h_0 is classical, $h_0 = \mathrm{gifp}(\sigma_M)$.

As for σ1, let g be the (weakly classical) hypothesis such that g(*B*) = **f** and g(*A*) = **n** if $A \neq B$. Note that $\sigma 1_M$(g)(*B*) = **f**. So g ≤ $\sigma 1_M$(g). By the monotony of σ1, there is a unique σ1-sequence ***S*** such that $\boldsymbol{S}_0$ = g. Furthermore, ***S*** is increasing (not strictly) and culminates in a fixed point, say h_1. Note that h_1(*B*) = **f**. But h_0 is also a fixed point of σ1, and h_0(*B*) = **t**. So gifp($\sigma 1_M$) is not classical. ⊣

Example 5.17. This example will show that $\mathbf{T}^{\text{gifp}, \sigma 1} \not\leq_2 \mathbf{T}^{\text{gifp}, \sigma 2}$. Consider a language *L* with exactly four nonquote names, *b*, *c*, *d* and *e*, no function symbols and no nonlogical predicates. Let M = ⟨S, I⟩ be that ground model such that I(*b*) = *B* = ***T****b* ∨ (¬***T****d* & ¬***T****e*), I(*c*) = *C* = ***T****b* ∨ ¬***T****c*, I(*d*) = *D* = ***T****c* and I(*e*) = *E* = ¬***T****c*. The facts in the following table can be established by calculating. The asterisks are wildcards, and the question marks can vary with the wildcards:

If ⟨h(*B*), h(*C*), h(*D*), h(*E*)⟩	=	tt**	ft**	*f**
then ⟨τ_M(h)(*B*), τ_M(h)(*C*), τ_M(h)(*C*), τ_M(h)(*E*)⟩	=	tttf	?ftf	?tft

From Gupta and Belnap's Transfer Theorem, we can conclude that τ_M has a unique fixed point, say h_0, where h_0(*B*) = h_0(*C*) = h_0(*D*) = **t** and h_0(*E*) = **f**. Since h_0 is a fixed point of τ_M, it is also a fixed point of $\sigma 1_M$ and of $\sigma 2_M$.

Furthermore, by an argument similar to that given for the Transfer Theorem, we can conclude that the fixed points of σ1 are completely determined by the values ⟨h(*B*), h(*C*), h(*D*), h(*E*)⟩. Thus, we can conclude that h_0 is the only classical fixed point of $\sigma 1_M$, and the only fixed point h of $\sigma 1_M$ for which h(*B*) = h(*C*) = h(*D*) = **t** and h(*E*) = **f**. In fact, h_0 is the only fixed point h of σ1 such that h(*B*) = h(*C*) = **t**, since any fixed point satisfying this also satisfies h(*D*) = **t** and h(*E*) = **f**.

Now we will show that σ1 has no fixed points h such that h(*B*) = **f**. For a reductio, suppose that h is a fixed point of σ1 with h(*B*) = **f**. h(*C*) cannot be **t**, otherwise we would have h(*C*) = $\sigma 1_M$(h)(*C*) = **f**. Similarly h(*C*) cannot be **f**, otherwise we would have h(*C*) = $\sigma 1_M$(h)(*C*) = **t**. So h(*C*) = **n**. Thus h(***T***'*C*') = h(***T****c*) = **n** = h(¬***T****c*), since h is a fixed point. Consider the classical hypothesis h′ such that h′(*A*) = **t** iff h(*A*) = **t** for every *A* ∈ S. h′ is weakly consistent, since the set {A: h(A) = **t**} is consistent, h being a fixed point. Also note that h′(***T**c*) = h′(¬***T**c*) = **f**, and h′(B) = **f**. Thus $\mathrm{Val}_{M + h', \tau}$(***T**d*) = $\mathrm{Val}_{M + h', \tau}$(***T***‘***T**c*’) = **f** = $\mathrm{Val}_{M + h', \tau}$(***T***’¬***T**c*’) = $\mathrm{Val}_{M + h', \tau}$(***T**e*). So $\mathrm{Val}_{M + h', \tau}$($B$) = $\mathrm{Val}_{M + h', \tau}$(***T**b* ∨ (¬***T**d* & ¬***T**e*)) = **t**. So by the definition of the jump operator $\sigma 1_M$, $\sigma 1_M$(h)(B) ≠ **f** = h(B), which contradicts h's being a fixed point of σ1.

Furthermore, σ1 has no fixed points h such that h(C) = **f**. For a reductio, suppose that h is a fixed point of σ1 with h(C) = **f**. So h(***T**c*) = **f**, since h is a fixed point. So h(C) = h(***T**b* ∨ ¬***T**c*) = **t**, a contradiction.

So for every fixed point h of σ1, the possible values for the quartuple ⟨h(B), h(C), h(D), h(E)⟩ are tttf, tnnn, nttf, and nnnn. As already pointed out, each fixed point h of σ1 is uniquely determined by ⟨h(B), h(C), h(D), h(E)⟩, and the ordering on them is isomorphic to the ordering induced on the four quartuples tttf, tnnn, nttf, and nnnn:

```
          tttf
         /    \
     tnnn      nttf
         \    /
          nnnn
```

Thus h_0 is the greatest fixed point of σ1. Thus h_0 = gifp($\sigma 1_M$), which is classical.

As for σ2, let g be the (strongly consistent) hypothesis such that g(B) = **f** and g(A) = **n** if A ≠ B. Note that $\sigma 2_M$(g)(B) = **f**. So g ≤ $\sigma 2_M$(g). By the monotony of σ2, there is a unique σ2-sequence ***S*** such that $\boldsymbol{S}_0$ = g. Furthermore, ***S*** is increasing (not strictly) and culminates in a fixed point, say h_1. Note that h_1(B) = **f**. But h_0 is also a fixed point of σ2, and h_0(B) = **t**. So gifp($\sigma 2_M$) is not classical. ⊣

Example 5.18. This example will show that $\mathbf{T}^{\mathrm{gifp},\, \sigma 2} \nleq_2 \mathbf{T}^{c}$. Consider a language L with exactly two nonquote names, b and c, no function symbols and no nonlogical predicates. Let M = ⟨S, I⟩ be that ground model such that I(b) = B = ***T**c*, and I(c) = C = ***T**b* & ¬***T**c*. The facts in the following table can easily be established by calculating:

If ⟨h(B), h(C)⟩	=	tt	tf	tn	ft	ff	fn	nt	nf	nn
then ⟨$\sigma 2_M$(h)(B), $\sigma 2_M$(h)(C)⟩	=	tf	ft		tf	ff				

Note that we have not filled in all the spaces in the table. These are not trivial: in order to calculate these values, we must know which classical $h' \geq h$ are strongly consistent. Right away we know that there are no strongly consistent hypotheses h such that $\langle h(B), h(C)\rangle = \langle \mathbf{t}, \mathbf{t}\rangle$, so that we can fill in the third column of the table with "ft". For our purposes, we do not really need all the other columns. All we need is the following:

If $\langle h(B), h(C)\rangle$	=	tt	tf	tn	ft	ff	fn	nt	nf	nn
then $\langle \sigma 2_M(h)(B), \sigma 2_M(h)(C)\rangle$	=	tf	ft	ft	tf	ff	?f	tf	??	??

Given this, by an argument similar to Gupta and Belnap's argument for the Transfer Theorem, we can conclude that each fixed point h of $\sigma 2_M$ is uniquely determined by the values $\langle h(B), h(C)\rangle$, and that the fixed point h_0 determined by the values $\langle \mathbf{f}, \mathbf{f}\rangle$ is classical. We can furthermore conclude that the only other potential fixed points are determined by the values $\langle \mathbf{n}, \mathbf{f}\rangle$ and $\langle \mathbf{n}, \mathbf{n}\rangle$. If such fixed points exist, they are both $\leq h_0$. So, whatever other fixed points there might be, $h_0 = \text{gifp}(\sigma 2_M)$. So $\text{gifp}(\sigma 2_M)$ is classical.

Now we will show that $\mathbf{T}^c$ does not dictate that truth behaves like a classical concept in M. Choose any strongly consistent hypothesis h such that $h(B) = \mathbf{t}$ and $h(C) = \mathbf{f}$. This can be done since $(B \mathbin{\&} \neg C)$ is consistent. Note that if n is even then $\tau_M^n(h)(B) = \mathbf{t}$ and $\tau_M^n(h)(C) = \mathbf{f}$, and if n is odd then $\tau_M^n(h)(B) = \mathbf{f}$ and $\tau_M^n(h)(C) = \mathbf{t}$. So there is some maximally consistent τ_M-sequence $\boldsymbol{S}$ such that neither B nor C is stable in $\boldsymbol{S}$. This suffices. ⊣

Example 5.19. This example will show that (1) $\mathbf{T}^{lfp, \kappa} \not\leq_1 \mathbf{T}^{lfp, \mu}$, and (2) $\mathbf{T}^{lfp, \rho'} \not\leq_1 \mathbf{T}^{gifp, \rho}$, where ρ and ρ' are chosen from the list μ, κ, σ, $\sigma 1$, $\sigma 2$, with ρ strictly to the left of ρ' on this list. Consider a language L with exactly two nonquote names, b and c, no function symbols and no nonlogical predicates. Let $M = \langle S, I\rangle$ be that ground model such that $I(b) = B = \neg \boldsymbol{T}b$. Let $C = \exists x(x = x)$. Note that for any fixed point h of μ, κ, σ, $\sigma 1$, or $\sigma 2$, $h(B) = \mathbf{f}$. Thus $\text{lfp}(\kappa_M)(B \vee C) = \text{lfp}(\sigma_M)(B \vee \neg B) = \text{lfp}(\sigma 1_M)(\neg \boldsymbol{T}\text{‘}B\text{’} \vee \neg \boldsymbol{T}\text{‘}\neg B\text{’}) = \text{lfp}(\sigma 2_M)(\boldsymbol{T}\text{‘}B\text{’} \vee \boldsymbol{T}\text{‘}\neg B\text{’}) = \mathbf{t}$. Meanwhile, $\text{lfp}(\mu_M)(B \vee C) = \text{gifp}(\mu_M)(B \vee C) = \text{gifp}(\kappa_M)(B \vee \neg B) = \text{gifp}(\sigma_M)(\neg \boldsymbol{T}\text{‘}B\text{’} \vee \neg \boldsymbol{T}\text{‘}\neg B\text{’}) = \text{gifp}(\sigma 1_M)(\boldsymbol{T}\text{‘}B\text{’} \vee \boldsymbol{T}\text{‘}\neg B\text{’}) = \mathbf{n}$. ⊣

Example 5.20. This example will show that $\mathbf{T}^* \not\leq_1 \mathbf{T}^{\text{gifp}, \rho}$, for ρ = σ, σ1 or σ2. Consider a language *L* with countably infinitely many nonquote names, $b_0, b_1, ..., b_n, ...$, no function symbols and exactly one non-logical predicate, the unary predicate *G*. Let M = ⟨S, I⟩ be that classical ground model such that $I(b_0) = B_0 = \boldsymbol{T}b_0 \vee \exists x \exists y(Gx \,\&\, Gy \,\&\, \boldsymbol{T}x \,\&\, \boldsymbol{T}y \,\&\, x \neq y) \vee \forall x(Gx \supset \neg \boldsymbol{T}x)$, and $I(b_i) = B_i = \forall x(Gx \supset (\boldsymbol{T}x \equiv x = b_i))$ for i ≥ 1, and $I(G)(A) = \mathbf{t}$ iff $A \in Y = \{B_0, B_1, ..., B_n, ...\}$. For each n ≥ 0, let H_n = {h: h is a classical hypothesis, and $h(B_n) = \mathbf{t}$ and $h(B_m) = \mathbf{f}$ for m ≠ n}. Let $H_{\mathbf{f}}$ = {h: h is a classical hypothesis, and $h(B_m) = \mathbf{f}$ for every m} and let $H_{\mathbf{t}}$ = {h: h is a classical hypothesis, and $h(B_m) = \mathbf{t}$ for every m}. Note the following:

if $h \in H_n$ then $\tau_M(h) \in H_n$;

if $h \in H_{\mathbf{f}}$ then $\tau_M(h) \in H_0$;

if $h \in H_{\mathbf{t}}$ then $\tau_M(h) \in H_0$; and

if $h \notin \cup_n H_n \cup H_{\mathbf{f}} \cup H_{\mathbf{t}}$ then $\tau_M(h) \in H_0$.

Thus, for any τ_M-sequence $\boldsymbol{S}$, we have $\boldsymbol{S}_1 \in \cup_n H_n$. We also have, for every m ≥ 1, $\boldsymbol{S}_m =_Y \boldsymbol{S}_{m+1}$. Thus by the Major Corollary (Corollary 5.6), $\boldsymbol{S}_\omega \in \cup_n H_n$ is a fixed point. Thus, not only does every τ_M-sequence culminate in some fixed point in $\cup_n H_n$, but τ_M has infinitely many fixed points, exactly one in each H_n. Let h_n be the unique fixed point of τ_M in H_n. Note that $\mathbf{V}^*_M = \{A: h_n(A) = \mathbf{t}$ for each n}. So $\exists x(Gx \,\&\, \boldsymbol{T}x) \in \mathbf{V}^*_M$. Furthermore, suppose we define the hypothesis h^* as follows: $h^*(A) = \mathbf{t}$ if $A \in \mathbf{V}^*_M$; $h^*(A) = \mathbf{f}$ if $\neg A \in \mathbf{V}^*_M$; $h^*(A) = \mathbf{n}$ otherwise. Then h^* is the greatest lower bound of the h_n. Also note that h^* is strongly consistent.

We will now argue that $\text{gifp}(\sigma2_M)(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \text{gifp}(\sigma1_M)(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \text{gifp}(\sigma_M)(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \mathbf{n}$. We will only give the argument for $\text{gifp}(\sigma2_M)(\exists x(Gx \,\&\, \boldsymbol{T}x))$; the other arguments are similar.

Any intrinsic point of $\sigma2_M$ must be ≤ any classical fixed point of τ_M. Thus $\text{gifp}(\sigma2_M) \leq h_n$, for each n. Thus $\text{gifp}(\sigma2_M) \leq h^*$. Now we claim that $\mathbf{V}^*_M \cup \{\neg B_0, \neg B_1, ..., \neg B_n, ...\}$ is a consistent set. To see this, note that $\mathbf{V}^*_M \cup \{\neg B_0, \neg B_1, ..., \neg B_n\}$ is a consistent set, since $\mathbf{V}^*_M \cup \{\neg B_0, \neg B_1, ..., \neg B_n\} \subseteq \{A: h_{n+1}(A) = \mathbf{t}\}$. Given that $\mathbf{V}^*_M \cup \{\neg B_0, \neg B_1, ..., \neg B_n, ...\}$ is consistent, the following hypothesis $h' \geq h^*$ is strongly consistent: $h'(A) = \mathbf{t}$ if $A \in \mathbf{V}^*_M$; $h'(A) = \mathbf{f}$ if $\neg A \in \mathbf{V}^*_M$ or $A \in Y$; $h'(A) = \mathbf{n}$ otherwise. Since h' is strongly consistent, it can be extended to a *classical* strongly consistent hypothesis $h'' \geq h' \geq h^* \geq \text{gifp}(\sigma 2_M)$. Note that $h''(A) = \mathbf{f}$ for each $A \in Y$. So $\tau_M(h'')(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \mathbf{f}$. Thus $\text{gifp}(\sigma 2_M)(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \sigma 2_M(\text{gifp}(\sigma 2_M))(\exists x(Gx \,\&\, \boldsymbol{T}x)) \neq \mathbf{t}$, by the definition of $\sigma 2_M$. Also, $\text{gifp}(\sigma 2_M)(\exists x(Gx \,\&\, \boldsymbol{T}x)) \neq \mathbf{f}$, since $\text{gifp}(\sigma 2_M) \leq h_0$ and $h_0(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \mathbf{t}$. Thus $\text{gifp}(\sigma 2_M)(\exists x(Gx \,\&\, \boldsymbol{T}x)) = \mathbf{n}$, as desired. ⊣

So far, we have the following results.

Positive results proved in §4. $\mathbf{T}^{\text{lfp}, \mu} \leq_1 \mathbf{T}^{\text{lfp}, \kappa} \leq_1 \mathbf{T}^{\text{lfp}, \sigma} \leq_1 \mathbf{T}^{\text{lfp}, \sigma 1} \leq_1 \mathbf{T}^{\text{lfp}, \sigma 2}$. $\mathbf{T}^{\text{lfp}, \rho} \leq_1 \mathbf{T}^{\text{gifp}, \rho}$ for $\rho = \mu, \kappa, \sigma, \sigma 1$, or $\sigma 2$. $\mathbf{T}^* \leq_1 \mathbf{T}^\#$. $\mathbf{T}^* \leq_1 \mathbf{T}^c$. $\mathbf{T}^{\text{lfp}, \sigma} \leq_1 \mathbf{T}^*$. $\mathbf{T}^{\text{lfp}, \sigma 2} \leq_1 \mathbf{T}^c$. $\mathbf{T}^{\text{lfp}, \mu} \equiv_2 \mathbf{T}^{\text{lfp}, \kappa}$. $\mathbf{T}^{\text{lfp}, \kappa} \leq_2 \mathbf{T}^{\text{lfp}, \sigma} \leq_2 \mathbf{T}^{\text{lfp}, \sigma 1} \leq_2 \mathbf{T}^{\text{lfp}, \sigma 2} \leq_2 \mathbf{T}^c$. $\mathbf{T}^{\text{lfp}, \sigma} \leq_2 \mathbf{T}^* \leq_2 \mathbf{T}^\#$. $\mathbf{T}^* \leq_2 \mathbf{T}^c$. $\mathbf{T}^c \leq_2 \mathbf{T}^{\text{gifp}, \sigma 2}$. $\mathbf{T}^{\text{gifp}, \sigma 2} \leq_2 \mathbf{T}^{\text{gifp}, \sigma 1} \leq_2 \mathbf{T}^{\text{gifp}, \sigma} \leq_2 \mathbf{T}^{\text{gifp}, \kappa} \leq_2 \mathbf{T}^{\text{gifp}, \mu}$. $\mathbf{T}^* \leq_3 \mathbf{T}^c \leq_3 \mathbf{T}^{\text{gifp}, \sigma 2} \leq_3 \mathbf{T}^{\text{gifp}, \sigma 1} \leq_3 \mathbf{T}^{\text{gifp}, \sigma} \leq_3 \mathbf{T}^{\text{gifp}, \kappa} \leq_3 \mathbf{T}^{\text{gifp}, \mu}$. $\mathbf{T}^{\text{lfp}, \sigma 2} \leq_3 \mathbf{T}^c$. $\mathbf{T}^{\text{lfp}, \sigma 1} \leq_3 \mathbf{T}^{\text{lfp}, \sigma 2}$. $\mathbf{T}^{\text{lfp}, \mu} \leq_3 \mathbf{T}^{\text{lfp}, \sigma} \leq_3 \mathbf{T}^{\text{lfp}, \sigma 1}$. $\mathbf{T}^{\text{lfp}, \kappa} \leq_3 \mathbf{T}^{\text{lfp}, \sigma}$.

Positive results proved in §5. $\mathbf{T}^* \leq_3 \mathbf{T}^*$. $\mathbf{T}^c \leq_3 \mathbf{T}^c$. $\mathbf{T}^{\text{lfp}, \sigma 2} \leq_3 \mathbf{T}^{\text{lfp}, \sigma 2}$. $\mathbf{T}^{\text{gifp}, \rho} \leq_3 \mathbf{T}^{\text{gifp}, \rho}$ for $\rho = \mu, \kappa, \sigma, \sigma 1$ or $\sigma 2$.

Negative results from the examples in §5. $\mathbf{T}^\# \not\leq_3 \mathbf{T}^\#$. $\mathbf{T}^\# \not\leq_2 \mathbf{T}^*$. $\mathbf{T}^\# \not\leq_2 \mathbf{T}^{\text{gifp}, \mu}$. $\mathbf{T}^{\text{lfp}, \sigma 1} \not\leq_2 \mathbf{T}^*$. $\mathbf{T}^{\text{lfp}, \sigma 1} \not\leq_2 \mathbf{T}^\#$. $\mathbf{T}^{\text{lfp}, \sigma 2} \not\leq_2 \mathbf{T}^*$. $\mathbf{T}^{\text{lfp}, \sigma 2} \not\leq_2 \mathbf{T}^\#$. $\mathbf{T}^c \not\leq_3 \mathbf{T}^*$. $\mathbf{T}^c \not\leq_3 \mathbf{T}^\#$. $\mathbf{T}^c \not\leq_2 \mathbf{T}^*$. $\mathbf{T}^c \not\leq_2 \mathbf{T}^\#$. $\mathbf{T}^{\text{lfp}, \sigma} \not\leq_3 \mathbf{T}^{\text{lfp}, \sigma}$. $\mathbf{T}^{\text{lfp}, \sigma 1} \not\leq_3 \mathbf{T}^{\text{lfp}, \sigma 1}$. $\mathbf{T}^{\text{lfp}, \sigma 2} \not\leq_2 \mathbf{T}^{\text{lfp}, \sigma 1}$. $\mathbf{T}^{\text{gifp}, \mu} \not\leq_2 \mathbf{T}^{\text{gifp}, \kappa}$. $\mathbf{T}^{\text{gifp}, \kappa} \not\leq_2 \mathbf{T}^{\text{gifp}, \sigma}$. $\mathbf{T}^{\text{gifp}, \sigma} \not\leq_2 \mathbf{T}^{\text{gifp}, \sigma 1}$. $\mathbf{T}^{\text{gifp}, \sigma 1} \not\leq_2 \mathbf{T}^{\text{gifp}, \sigma 2}$. $\mathbf{T}^{\text{gifp}, \sigma 2} \not\leq_2 \mathbf{T}^c$. $\mathbf{T}^{\text{lfp}, \kappa} \not\leq_1 \mathbf{T}^{\text{lfp}, \mu}$. $\mathbf{T}^{\text{lfp}, \rho'} \not\leq_1 \mathbf{T}^{\text{gifp}, \rho}$, where ρ and ρ' are chosen from the list $\mu, \kappa, \sigma, \sigma 1, \sigma 2$, so that ρ is strictly to the left of ρ' on this list. $\mathbf{T}^* \not\leq_1 \mathbf{T}^{\text{gifp}, \rho}$, for $\rho = \sigma, \sigma 1$ or $\sigma 2$.

We add the following three negative results. (i) $\mathbf{T}^* \not\leq_2 \mathbf{T}^{\text{lfp}, \sigma}$. See [3], Example 6B.7. (ii) $\mathbf{T}^* \not\leq_2 \mathbf{T}^{\text{lfp}, \sigma 2}$. See [3], Example 6B.13. (iii) $\mathbf{T}^{\text{lfp}, \sigma} \not\leq_2 \mathbf{T}^{\text{lfp}, \kappa}$. Choose any S-neutral ground model. By Corollary 4.24, lfp(σ) is classical. But, by the proof of Theorem 4.5, lfp(κ) is not classical.

The negative parts of Theorems 4.2 and 4.5 follow from these results, together with (1) Lemma 4.20; (2) the fact that if $\mathbf{T} \leq_1 \mathbf{T}'$ then $\mathbf{T} \leq_2 \mathbf{T}'$; (3) the fact that $\leq_1$ and $\leq_2$ are reflexive and transitive; (4) the fact that $\leq_1$, $\leq_2$ and $\leq_3$ are transitive; and (5) the positive parts of Theorems 4.2 and 4.5.

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