ECO220Y

Discrete Probability Distributions: Bernoulli and Binomial

Readings: Chapter 9, Sections 9.4 and 9.6 only

Fall 2011

Lecture 7

Bernoulli Experiment

Definition of **Bernoulli trial** (after James Bernoulli (1654-1705):

1. There are two possible outcomes, not necessarily equally likely, that are labeled generically "success" and "failure".
2. If the uncertain situation is repeated, the probabilities of success and failure are unchanged. The probability of success is not affected by the successes or failures that already have been experienced.

Examples:

- A coin toss
- It's a boy! (or a girl)
- Rolling a 6 on one die.

Bernoulli Random Variable

Bernoulli random variable records an outcome of Bernoulli experiment.

x	P(X=x)
0 (failure)	1-p
1 (success)	p

$$E[X] = 1 * p + 0 * (1 - p) = p$$

$$V[X] = (1 - p)^2 * p + (0 - p)^2 * (1 - p) = p * (1 - p)$$

$$E[X] = \mu = p \text{ and } V[X] = \sigma^2 = p * (1 - p)$$

Bernoulli - discrete or continuous RV?

Binomial Experiment

1. There are two possible outcomes, not necessarily equally likely, that are labeled generically "success" and "failure".
2. If the uncertain situation is repeated, the probabilities of success and failure are unchanged. The probability of success is not affected by the successes or failures that already have been experienced.
3. We have a sequence of Bernoulli trials.

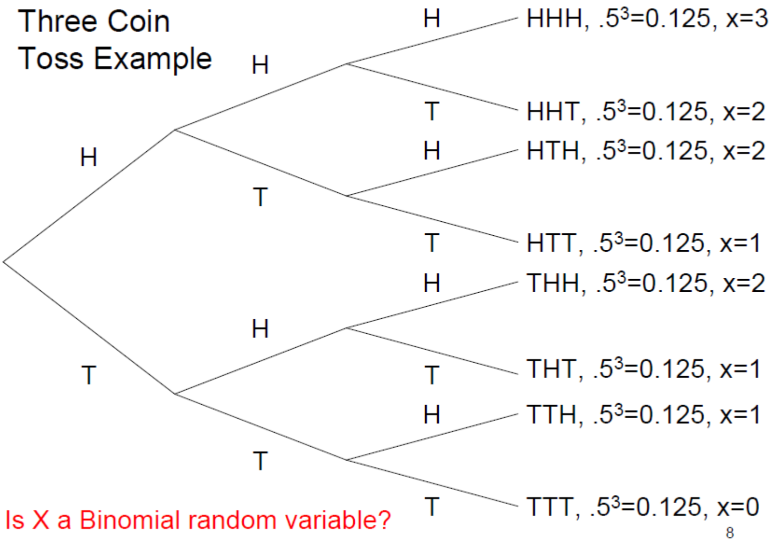
Binomial Random Variable

- Binomial random variable counts the number of successes in a Binomial experiment.
- Binomial distribution returns probability of each possible value of binomial random variable X.

Examples:

- A number of heads in a series of 5 coin tosses.
- A number of boys in families with three children.

Probability Tree

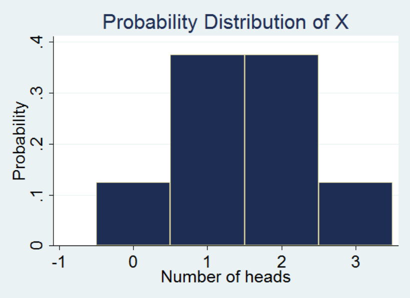


Is X a Binomial random variable?

Probability distribution

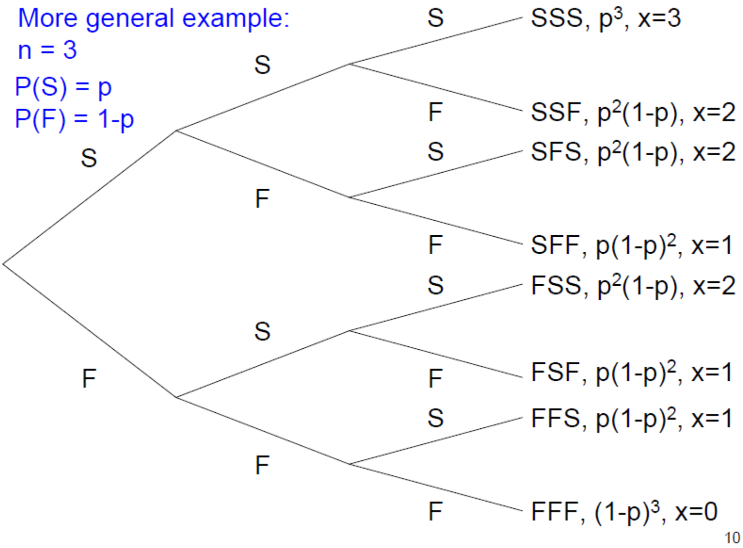
x	P(x)
0	0.125
1	0.375
2	0.375
3	0.125
Total	1

$\Longrightarrow$



Which probability rules have we used to derive this probability distribution?

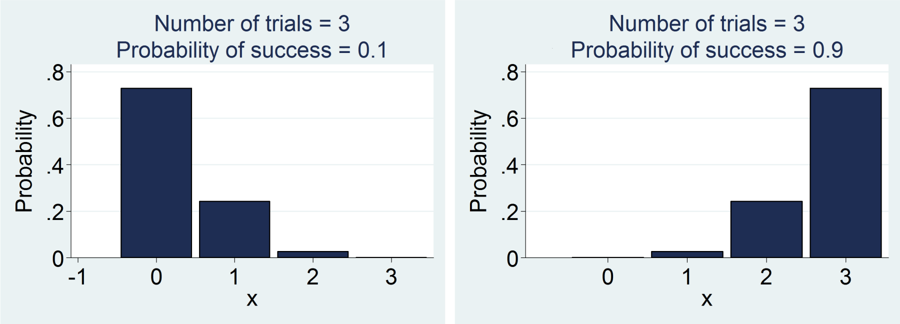
General Probability Tree



Probability Distribution

x	p(x)
0	$(1-p)^3$
1	$3 * p(1-p)^2$
2	$3 * p^2(1-p)$
3	p^3
Total	1

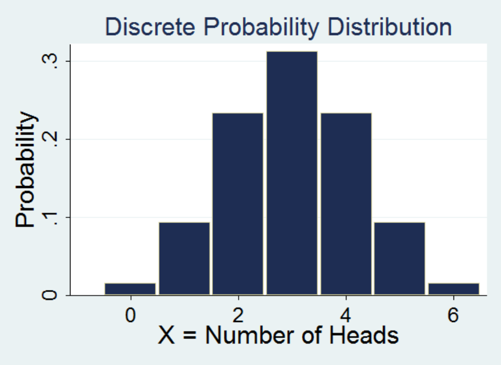
Given that there are three trials (n=3), what determines the shape of the probability distribution? (What it would look like if we graphed it?)



Number of trials = 3
Probability of success = 0.1
Probability
.8
.6
.4
.2
0
-1
0
1
2
3
x
Number of trials = 3
Probability of success = 0.9
Probability
.8
.6
.4
.2
0
0
1
2
3
x

When p=.5, the binomial probability distribution is perfectly symmetrical. For 6 coin flips, for instance, it is:

x	0	1	2	3	4	5	6
P(x)	$\frac{1}{2}^6$	$\frac{1}{2}^6 * 6$	$\frac{1}{2}^6 * 15$	$\frac{1}{2}^6 * 20$	$\frac{1}{2}^6 * 15$	$\frac{1}{2}^6 * 6$	$\frac{1}{2}^6$



What if n is large?

If n (number of trials) in binomial experiment is large, we can compute the probabilities for all of the possible values of X as:

$$P(\text{x successes}) = P(x) = C_x^n p^x (1-p)^{n-x}$$

where C_x^n is number of all possible sequences with x successes:

$$C_x^n = \frac{n!}{x!(n-x)!}$$

$$n! = n * (n-1) * (n-2) * (n-3) * ...3 * 2 * 1$$

$$C_n^n = C_0^n = 1, \quad C_x^n = C_{n-x}^n, \quad 0! = 1, \quad 1! = 1$$

Note: C_x^n is the *binomial coefficient*, read "n choose x"

Practice Using the Combinatorial Formula

- We have constructed and graphed probability distribution of a random variable X=number of heads in three coin tosses.
- We know that there are 3 ways to get 1 head in three tosses: HTT, THT, or TTH.

$$C_1^3 = \frac{3!}{1!(3-1)!} = \frac{3*2*1}{1*2*1} = 3$$

- We also know that there is only 1 way to get 3 heads: HHH.

$$C_3^3 = \frac{3!}{3!(3-3)!} = \frac{3*2*1}{3*2*1*1} = 1$$

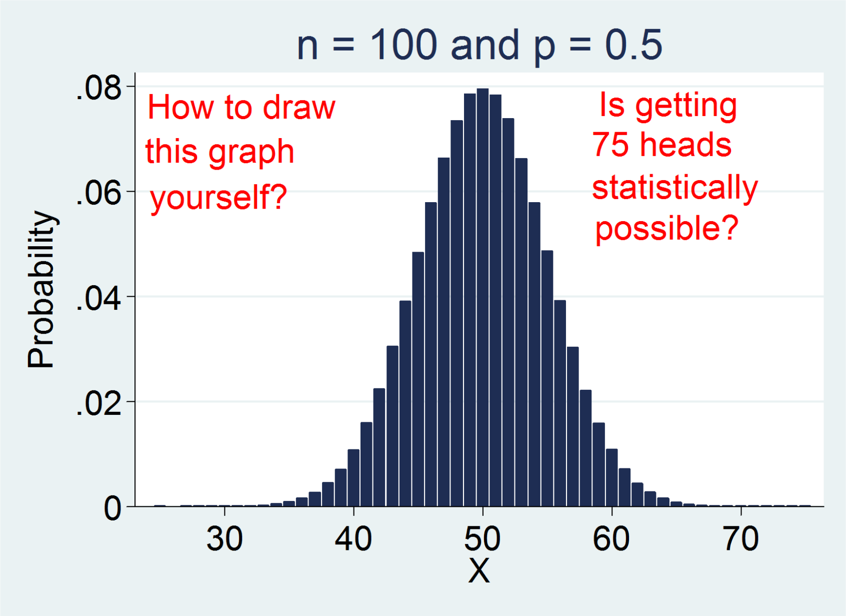
Combinatorial Formula The Easy Way

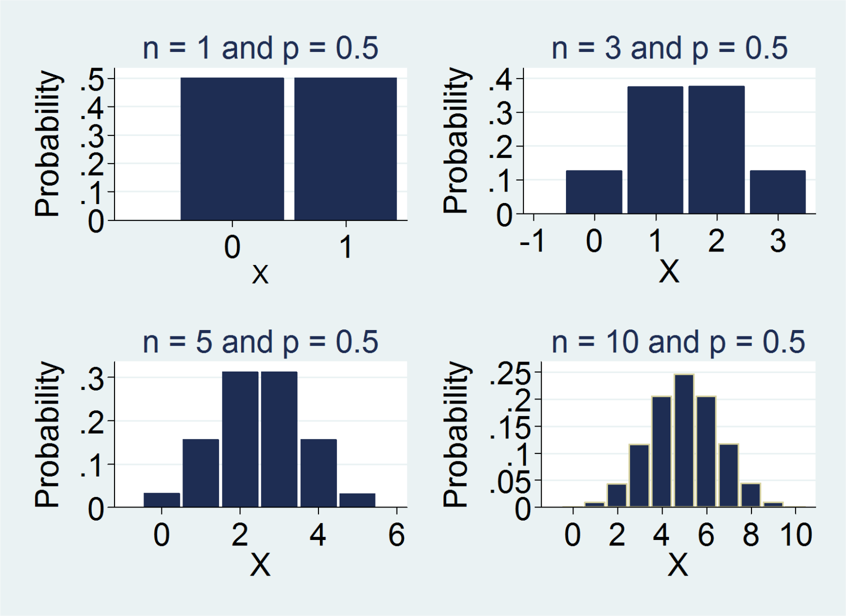
$$C_{11}^{16} = \frac{16!}{11!5!} = \frac{16{\cdot}15{\cdot}14{\cdot}13{\cdot}12{\cdot}\overbrace{11\cdot\ldots\cdot 1}^{11!}}{\underbrace{(11\cdot\ldots\cdot 1)}_{11!}*(5{\cdot}4{\cdot}3{\cdot}2{\cdot}1)} = \frac{16{\cdot}15{\cdot}14{\cdot}13{\cdot}12}{5{\cdot}4{\cdot}3{\cdot}2{\cdot}1}$$

Binomial Probability Distribution

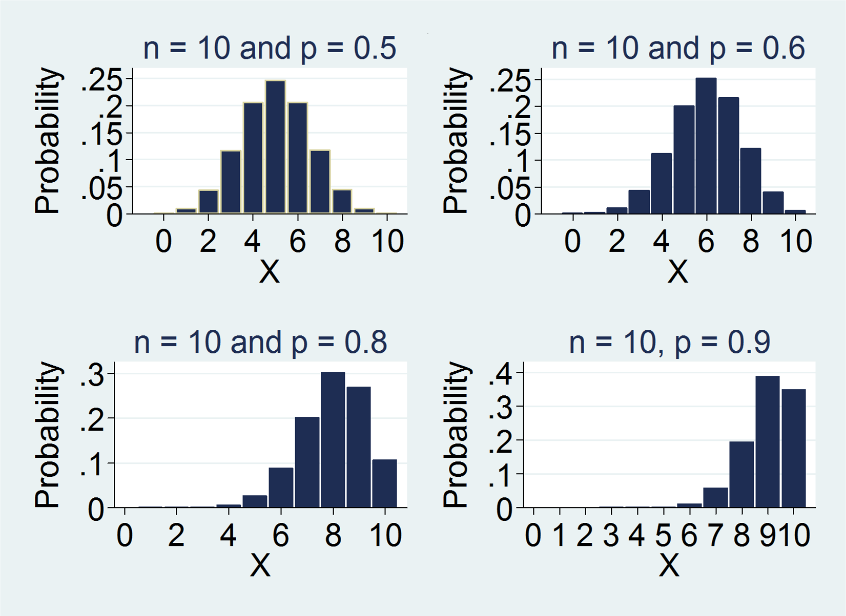
To describe binomial distribution, we need two parameters: n and p. The binomial distribution states that, in n Bernoulli trials, with probability p of success on each trial, the probability of exactly x successes is:

$$P(X = x) = C_x^n * p^x(1-p)^{n-x} = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

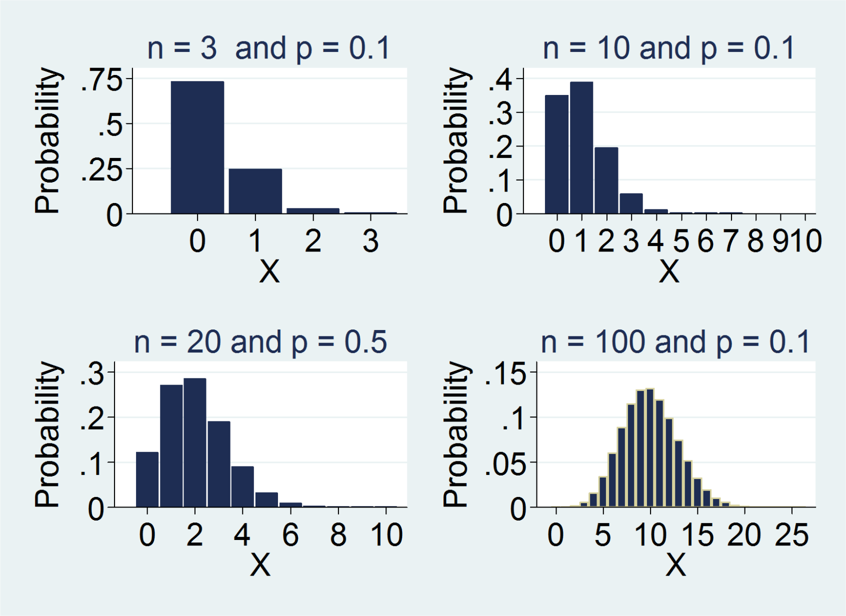

n = 100 and p = 0.5
How to draw this graph yourself?
Is getting 75 heads statistically possible?
Probability
.08
.06
.04
.02
0
30
40
50
60
70
X



How is shape affected by the *n* parameter?



How is shape affected by the p parameter?



How is shape affected by the *n* parameter?

Expected Value and Variance of Binomial RV

Binomial RV is a sum of Bernoulli RVs

Which laws we are going to use to find μ and σ^2?

$$Y = \sum_{i=1}^{n} X_i$$

where n is a number of Bernoulli trials and $E[X] = p$ and $V[X] = p(1-p)$

$$E[Y] = E[\sum_{i=1}^{n} X_i] = E[nX_i] = n * E[X_i] = np$$

$$V[Y] = V[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} V[X_i] = nV[X_i] = np(1-p)$$

Summary

- Binomial distribution is described by two parameters: n and p
- Shape of the probability distribution for binomial RV depends on parameters
- For any p and n we can find the probability of obtaining x successes in n trials as $P(X = x) = C_x^n p^x (1-p)^{n-x}$
- $E[X] = \mu = np$
- $V[X] = \sigma^2 = np(1-p)$

Example

The Adams are planning their family and both want an equal number of boys and girls. Mrs Adams says their chances are best if they plan on having two children. Mr Adams says that they have a better chance of having an equal number of boys and girls if they plan on having four children. Who is right? (Let's assume that boy and girl babies are equally likely)

Statistically Plausible?

Statistics Professor claims that 90% of the students are extremely satisfied with the course he is teaching. We randomly sample 5 students to find that only 1 student is actually "extremely satisfied". Is this result statistically plausible given that the claim is true?

Cumulative Probability

- Cumulative probability is probability that a random variable is less than or equal to a particular value: $P(X \leq x)$
- Example: Toss 6 coins and X counts heads:
  $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

How to Use Cumulative Probabilities

1. Cumulative Probability
   - Probability "less or equal" - $P(X \leq x)$
2. Probability "greater or equal"
   - $P(X \geq x) = 1 - P(X \leq x - 1)$
3. Probability "equal to"
   - $P(X = x) = P(X \leq x) - P(X \leq x - 1) = p(x)$
4. Range probability
   - $P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1 - 1)$
   - where $x_1 \geq 0$, $x_2 \leq n$, and $x_1 \leq x_2$

Example

Consider binomial distribution with p=.5 and n=100

- $P(X \leq 50) = 0.5397$
- $P(X \geq 65) = 1 - P(X \leq 65 - 1) = 1 - P(X \leq 64) = 1 - 0.9982 = 0.0018$
- $P(X = 45) = P(X \leq 45) - P(X \leq 44) = 0.1841 - 0.1356 = 0.0485$
- $P(45 \leq X \leq 55) = P(X \leq 55) - P(X \leq 44) = 0.8643 - 0.1356 = 0.7287$

Applications of Binomial Distribution

To pretest a survey for U of T students, 5 students are randomly selected for a focus group. Among U of T students, 25% live on campus, 35% live off-campus with their parents and 40% live off-campus not with their parents.

What is the chance that the focus group does not contain any student who lives on-campus?

Solution: Let's define X to be a Binomial random variable which counts the number of students who live on-campus with probability of success equal to 0.25.

P(0 students live on-campus out of 5 students)

$= C_0^5 \cdot (0.25)^0 (1 - 0.25)^{5-0} = 0.75^5 = 0.2373$

What is the chance that more than two-thirds of the focus group is students who live off-campus not with their parents?

Solution: Let's define X to be a Binomial random variable which counts the number of students who live off-campus not with their parents with probability 0.4. Two-thirds of 5 is 3, not 3.(3). "More than two-thirds" means 4 or 5 students. We need to find P(X>3)=P(X=4) + P(X=5).

$$P(X=4)+P(X=5)=C_4^5 \cdot (0.4)^4(1-0.4)^1 + C_5^5 \cdot 0.4^5(1-0.4)^0$$

$$=0.078+0.0102=0.087$$