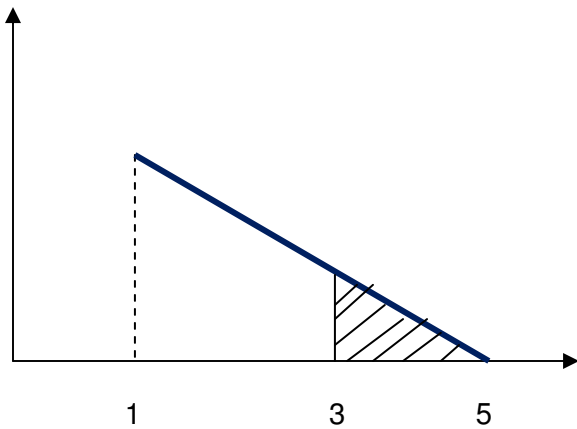


ECO220Y: Homework, Lecture 8 – SOLUTIONS

(1)



$P(X < 3)$ is the un-shaded area of the triangle. To find it we can use Complement rule: $P(X < 3) = 1 - P(X > 3)$. $P(X > 3)$ is the area of the shaded triangle and is $\frac{1}{2} \cdot \text{Base} \cdot \text{Height}$. To solve for it, we need to find height. We cannot read the value of the function directly from the graph (it is intentionally erased). The first step would be to find the equation of the straight line. Start with the height of the function at $x=1$. We know that this is a probability density function, so it should be true that the area of the large triangle is equal to 1.

$$\text{Area} = \frac{1}{2} \cdot \text{Base} \cdot \text{Height}, \text{ so Height} = \frac{(\text{Area} \cdot 2)}{\text{Base}} = \frac{1 \cdot 2}{(5-1)} = \frac{2}{4} = \frac{1}{2}$$

$$\text{Slope is rise/run} = \frac{(0 - 0.5)}{(5 - 1)} = \frac{-0.5}{4} = -\frac{1}{8}$$

$$Y = a + bX, \text{ so } 0 = a - \frac{1}{8} \cdot 5, \text{ and } a = \frac{5}{8}$$

$$Y = \frac{5}{8} - \frac{1}{8} \cdot X$$

$$\text{If } X=3 \text{ then } Y = \frac{5}{8} - \frac{1}{8} \cdot 3 = \frac{2}{8} = \frac{1}{4}$$

$$P(X > 3) = \text{Area (Small triangle)} = \frac{1}{2} \cdot 2 \cdot \frac{1}{4} = \frac{1}{4}$$

$$P(X < 3) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

(2) Parameters of the uniform distribution are a and b .

$$\mu = \frac{(a + b)}{2}; \sigma^2 = \frac{(b - a)^2}{12}$$

$$2\mu = a + b \Rightarrow a = 2\mu - b \Rightarrow \sigma^2 = \frac{(b - 2\mu + b)^2}{12} \Rightarrow 12\sigma^2 = (2b - 2\mu)^2$$

$$\sigma^2 \sqrt{12} = 2b - 2\mu \Rightarrow b = \sigma\sqrt{3} + \mu \Rightarrow a = \mu - \sigma\sqrt{3}$$

Now, plug in values for mean and standard deviation to get:

$$b = 1\sqrt{3} + 0 = 1.73, a = -1.73$$

(3)

$$P(X < 65) = 0.05 \text{ and } P(X < 105) = 0.75$$

Let's standardize both expressions:

$$P\left(\frac{X - \mu}{\sigma} \geq \frac{65 - \mu}{\sigma}\right) = 0.05 \Rightarrow z_A = \frac{65 - \mu}{\sigma}$$

$$P\left(\frac{X - \mu}{\sigma} \leq \frac{105 - \mu}{\sigma}\right) = 0.75 \Rightarrow z_A = \frac{105 - \mu}{\sigma}$$

If we knew values of z-scores at the bottom 5th percentile and 75th percentile, then we could solve the system of two equations with two unknowns. We can find those values from the standard normal table.

$$z_{0.05} = -1.645 \Rightarrow -1.645 = \frac{65 - \mu}{\sigma} \Rightarrow -1.645\sigma = 65 - \mu \Rightarrow \mu = 65 + 1.645\sigma$$

$$z_{0.25} = 0.675 \Rightarrow 0.675 = \frac{105 - \mu}{\sigma} \Rightarrow 0.675\sigma = 105 - \mu \Rightarrow \mu = 105 - 0.675\sigma$$

$$65 + 1.645\sigma = 105 - 0.675\sigma \Rightarrow 40 = 2.32\sigma \rightarrow \sigma = 17.24, \mu = 65 + 1.645 * 17.24 = 93.36$$

(4) This looks like a triangle distribution with parameters [2a, 2b] equal to [10, 20].

Variance of triangle distribution is $(b-a)^2/6=(10-5)^2/6=4.17$

(5) Parameters of the underlying uniform distribution are [a, b]=[10/2, 20/2]=[5, 10]

$$\mu = \frac{b+a}{2} = \frac{10+5}{2} = 7.5$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(10-5)^2}{12} = 2.08$$

(6) IQ ~ N(100, 15²)

$$P(X > 120) = P(Z > (120-100)/15) = P(Z > 1.33) = 0.0918$$

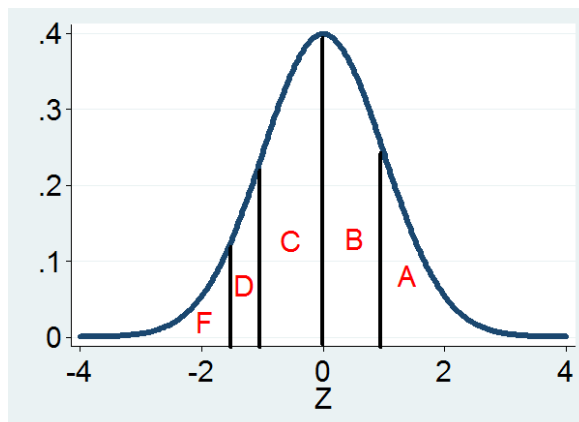
$$P(X > 140) = P(Z > (140-100)/15) = P(Z > 2.67) = 0.0038$$

$$P(X > 150) = P(Z > (150-100)/15) = P(Z > 3.33) = 0.0004$$

$$P(X > \text{Galileo IQ}) = P(Z > (185-100)/15) = P(Z > 5.67) = 0$$

$$P(X > \text{Goethe IQ}) = P(Z > (210-100)/15) = P(Z > 7.33) = 0$$

(7) We are asked to find the z-scores of the standard normal curve such that $P(Z < z_F) = 0.05$, $P(Z < z_D) = 0.15$, $P(Z < z_C) = 0.5$, $P(Z < z_B) = 0.80$, $P(Z > z_A) = 0.20$. Look up the values in the table to find $z_D = -1.645$, $z_C = -1.045$, $z_B = 0$, $z_A = 0.845$



(8) We can think of each item in the batch as a Bernoulli random variable which has probability of 0.98 to be non-defective and probability 0.02 to be defective. When the sample size is 10, we have a sequence of 10 Bernoulli random variables; hence we can use Binomial distribution to model the process.

$$P(\text{batch is rejected} | 10 \text{ items}) = P(X \geq 1) = 1 - P(X=0) = 1 - C_0^{10} (0.02)^0 (0.98)^{10} = 0.183$$

$$P(\text{batch is rejected} | 20 \text{ items}) = P(X \geq 2) = 1 - P(X=0) - P(X=1) = 0.0599$$

With $n=200$, 10% is equal to 20, so we are better off using normal approximation.

Let's check first whether we can use normal approximation: $np=200 \cdot 0.02=4 < 10$! We cannot use normal approximation!

$$P(X \geq 20) = 1 - P(X=0) - P(X=1) - \dots - P(X=19) = 0 \text{ (software calculations)}$$

With normal approximation (let's cheat!):

$$P(X > 20 | np=4, np(1-p)=3.92) = P(Z > (20-4)/1.98) = P(Z > 8.1) = 0$$

The answer is the same although we cannot formally use normal approximation. This is a special case when the probability is extremely low (it is not zero, but it is very close to zero). In other cases, it is better (and safer) check the rule first, and if it doesn't hold, use Binomial distribution.