Solving Constrained OSNR Nash Game in WDM Optical Networks with a Fictitious Player

Quanyan Zhu
Department of Electrical
and Computer Engineering
University of Toronto
Toronto, Ontario M5S 3L1
Email: qzhu@control.utoronto.ca

Lacra Pavel
Department of Electrical
and Computer Engineering
University of Toronto
Toronto, Ontario M5S 3L1
Email: pavel@control.utoronto.ca

***Abstract*—Non-cooperative game theory is a powerful modeling tool for resource allocation problems in modern communication networks. However, practical concerns of capacity constraints and allocation efficiency have been a challenge for network engineers. In this paper, we base our results in the context of link-level power control of optical networks and propose a special form of games with an additional player to overcome these difficulties. We introduce a novel framework with a fictitious player (GFP) to extend the current OSNR Nash game framework with capacity constraints. We characterize a more analytically tractable solution in comparison to other approaches and propose a first-order iterative algorithm to find the equilibrium.**

I. Introduction

Recent technological advances have enabled a new generation of Optical Wavelength-Division Multiplexed (WDM) communication networks. Devices such as Optical Add/Drop MUXes (OADM), optical cross connects (OXC) and dynamic gain equalizer (DGE) have provided essential building blocks for smart optical networks [1]. With advent of these new technologies, current networks are evolving towards dynamic networks, able to respond to changes in traffic and requirements. A static network management mechanism can no longer service such networks. Therefore, intelligent network management and control systems need to be part of future network design. Complex in their own structure, networks need control on different levels. The first level is an optical device level control, where smart feedback algorithms are used to reduce noise and stabilize the device. Examples have been seen in [2] and [3] where control principles are applied to study EDFA and SOA, respectively. The next level of management is on the link level, where we need to optimize the quality of transmission and reduce the interference and noise in transmission. Optimization-based models have been seen in the case of wireless networks in [4], [5]. However, the unique physical structure of optical networks imposes different challenges on modeling and solution concepts.

The third level is the network level, where problems of interests are optimal routing and congestion control. These problems are on a higher level and they have been well studied in a general network setting such as in [6]. The last but not least is the system level control. This level of research encapsulates optical network as a dynamical system as seen in [7]. Interesting problems are usually on the robustness and stability of large scale networks.

Our focus here is on the link level. Channel optical signal-to-noise ratio (OSNR) is an important performance factor at this level as it directly relates to the bit error rate (BER) in the transmission [8]. In recent years, research work on OSNR-based optimization is making an effort to derive iterative decentralized OSNR optimization algorithms in optical networks. Two dominant methods are commonly seen in literature. One is the centralized optimization as in [9], [10] and the other is non-cooperative game theory as in [11], [12]. The centralized approach embeds OSNR targets in constraints and indirectly minimizes the total power consumption in optical networks. It is relatively easy to find a closed form solution with this approach, however, its indirect minimization of total power consumption doesn't fully make use of the network resource for communication purposes. On the other hand, the non-cooperative game approach naturally deals with OSNR optimization in a decentralized and direct manner. However, it is a well-known fact that the resulting Nash equilibrium may not be Pareto efficient [13], [14]. In addition, under the OSNR game framework, it has been a challenge to find an analytical solution for a game with capacity constraints. Research efforts have been made to solve this problem by integrating constraints into utility functions [15], [16]. And, in particular, work has been done in [17], [18] to deal with such constraints based on classical Lagrangian duality theory. However, complexity of the solution grows in an undesirable way and it is exceedingly difficult to give an analytical solution for OSNR Nash game.

In this paper, we propose a different approach to deal with constraints in OSNR Nash games. We formulate a Nash game with a fictitious player to give a closed form solution to the constrained OSNR Nash game. We may also use the role of fictitious player to achieve an efficient Nash equilibrium under certain conditions. The fictitious player, in reality, can be implemented via a service channel or a transmission channel which only needs a target OSNR.

This paper is organized as the following. In section 2, we review a network OSNR model and give a brief introduction to unconstrained non-cooperative game approach. In section 3,

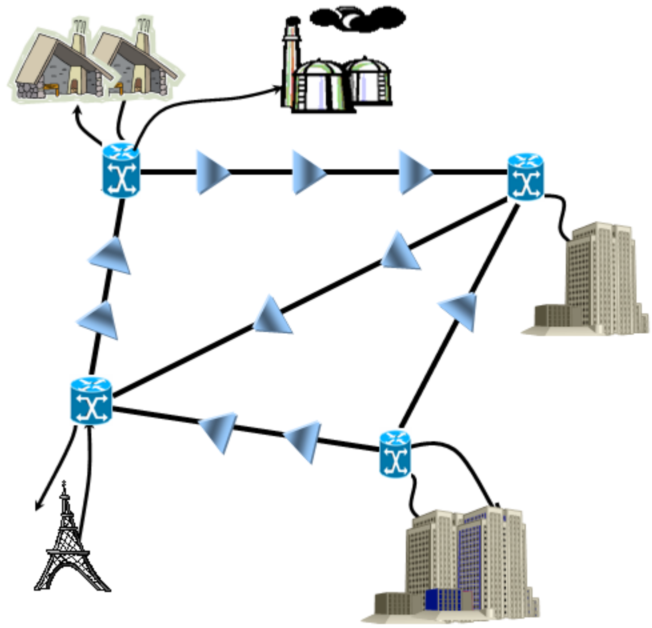

Fig. 1. A Small-scale Optical Networks

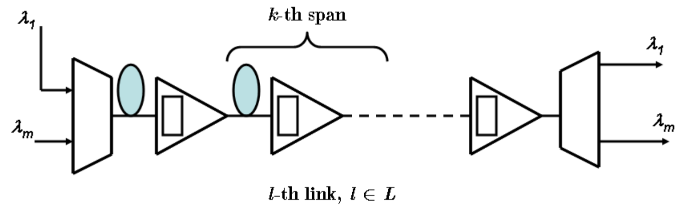

Fig. 2. A Typical Optical Link in DWMW Optical Networks

we establish the framework of game with a fictitious player. We will characterize the Nash equilibrium and discuss the achievable target OSNR of the fictitious player. In section 4, we point out the directions of future research and we will conclude in section 5.

II. Background

A. Review of Optical Network Model

We first review the analytical OSNR model for a WDM optical network and basic unconstrained Nash game as formulated in [11]. Consider a network with a set of optical links $\mathcal{L} = \{1, 2, .., L\}$ connecting the optical nodes, where channel add/drop is realized. A set $\mathcal{N} = \{1, 2, ..., N\}$ of channels are transmitted, corresponding to a set of multiplexed wavelengths. An example of such network is depicted in Figure 1. Illustrated in Figure 2, a link l has K_l cascaded optically amplified spans. Let N_l be the set of channels transmitted over link l. For a channel $i \in \mathcal{N}$, we denote by $\mathcal{R}_i$ its optical path, or collection of links, from source (Tx) to destination (Rx). Let u_i be the ith channel input optical power (at Tx), and $\mathbf{u} = [u_1, ..., u_N]^T$ the vector of all channels' input powers. Let s_i be the ith channel output power (at Rx), and n_i the optical noise power in the ith channel bandwidth at Rx. The ith channel optical OSNR is defined as $OSNR_i = \frac{s_i}{n_i}$. In [9], some assumptions are made to simplify the expression for OSNR, typically for uniformly designed optical links. It is assumed that

1) (A1) ASE noise power does not participate in amplifier gain saturation.
2) (A2) All the amplifiers in a link have the same spectral shape with the same total power target and are operated in automatic power control (APC) mode, with the total power target P_0. P_0 is selected to be below the threshold for nonlinear effects.

Under A1 and A2,the dispersion and nonlinearity effects are considered to be limited, the ASE noise accumulation will be the dominant impairment in the model. The OSNR for the ith channel is given as

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in \mathcal{N}} \Gamma_{i,j} u_j}, \tag{1}$$

where Γ is the full $n \times n$ system matrix which characterizes the coupling between channels. $n_{0,i}$ denotes the ith channel noise power at the transmitter. System matrix Γ encapsulates the basic physics present in optical fiber transmission and implements an abstraction from a network to an input-output system. This approach has been used in [12] for the wireless case to model CDMA uplink communication. Different from the system matrix used in wireless case, the matrix Γ given in (2) is commonly asymmetric and is more complicatedly dependent on parameters such as spontaneous emission noise, wavelength-dependent gain, and the path channels take.

$$\Gamma_{i,j} = \sum_{i \in \mathcal{R}_i} \sum_{k=1}^{K_l} \frac{G_{l,j}^k}{G_{l,i}^k} \left(\prod_{q=1}^{l-1} \frac{\mathbf{T}_{q,j}}{\mathbf{T}_{q,i}} \right) \frac{ASE_{l,k,i}}{P_{0,l}}, \forall j \in \mathcal{N}_l. \tag{2}$$

where $G_{l,k,i}$ is the wavelength dependent gain at kth span in lth link for channel i; $\mathbf{T}_{l,i} = \prod_{q=1}^{K_l} G_{l,k,i} L_{l,k}$ with $L_{l,k}$ being the wavelength independent loss at kth span in lth link; $ASE_{l,k,i}$ is the wavelength dependent spontaneous emission noise accumulated across cascaded amplifiers; $P_{0,l}$ is the output power at each span.

B. Non-cooperative Game Approach

Let's review the basic game-theoretical model for power control in optical networks without constraints. Consider a game defined by a triplet $\langle \mathcal{N}, (A_i), (J_i) \rangle$. $\mathcal{N}$ is the index set of players or channels; A_i is the strategy set $\{u_i \mid u_i \in [u_{i,\min}, u_{i,\max}]\}$; and, J_i is the cost function, chosen such that minimizing the cost is related to maximizing OSNR level. In [11], J_i is defined as

$$J_i(u_i, u_{-i}) = \alpha_i u_i - \beta_i \ln\left(1 + a_i \frac{u_i}{X_{-i}}\right), i \in \mathcal{N} \tag{3}$$

where α_i, β_i are channel specific parameters, that quantify the willingness to pay the price and the desire to maximize its OSNR, respectively, a_i is a channel specific parameter, X_{-i} is defined as $X_{-i} = \sum_{j \neq i} \Gamma_{i,j} u_j + n_{0,i}$. This specific choice of utility function is non-separable, nonlinear and coupled. However, J_i is strictly convex in u_i and takes a specially designed form such that its first-order derivative takes a linear form with respect to $\mathbf{u}$, i.e. is in the class of linear games defined in section 2.

The solution from the game approach is usually characterized by Nash equilibrium (NE). Provided that $\sum_{j\neq i}\Gamma_{i,j}\leq a_i$,the resulting NE solution is given in a closed form by

$$\widetilde{\mathbf{\Gamma}}\mathbf{u}^* = \widetilde{\mathbf{b}}, \tag{4}$$

where $\widetilde{\Gamma}_{i,j} = a_i$, for $j = i$; $\widetilde{\Gamma}_{i,j} = \Gamma_{i,j}$, for $j \neq i$ and $\widetilde{b}_i = \frac{a_i b_i}{\alpha_i} - n_{0,i}$.

Similar to the wireless case [12], we are able to construct iterative algorithms to achieve the Nash equilibrium. A simple deterministic first order parallel update algorithm can be found by $u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{X_{-i}(n)}{a_i}$, or equivalently in terms of $OSNR_i$,

$$u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(n)} - \Gamma_{i,i}\right)u_i(n). \tag{5}$$

As proved in [11], the algorithm (5) converges to Nash equilibrium $\mathbf{u}^*$ provided that $\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j}\leq 1, \forall i$.

III. Game with a Fictitious Player (GFP)

In optical networks, a saturation power level exists in each link of channel paths [15]. A launched power has to be below or equal to this threshold so that the nonlinear effects in the span following each amplifier are kept minimum [19]. We can easily interpret this effect as a capacity constraint on an optical link in the network. In this section, we tackle the game described in section 2 with such constraint by considering a non-cooperative game with an additional fictitious player, labeled F. The fictitious player can be regarded as an additional player implemented via a channel that doesn't participate in the game for its need for quality of transmission. An example is the service channel in optical networks. It only requires certain amount of power to transmit network information and doesn't aim for OSNR optimization. It rather behaves as a player to regulate the performance of the network. We will use this interpretation to solve an $(N+1)$-person non-cooperative game with constraint of

$$\sum_{i\in\mathcal{N}\cup\{F\}} u_i \leq C. \tag{6}$$

Let the payoff function of user $i\in\mathcal{N}$ given by Equation (3) and we choose the payoff function of user F to be

$$J_F(u_F, u_{-F}) = \alpha_F u_F - \beta_F\left(C - \sum_{j\neq F} u_j\right)\ln a_F u_F. \tag{7}$$

Function J_F is convex when $\sum_{j\neq F} u_j \leq C$. Since the fictitious player may not ask for an optimal quality of transmission, we do not design function (7) directly related to OSNR, but in terms of power and capacity constraint instead. It is composed of two parts with the first term describing the cost on power usage u_F and the second term the capacity-dependent utility. With the assumption of convexity, the best response function for J_F is given by an implicit expression in (8).

$$\omega_F u_F + \sum_{j\neq F} u_j = C. \tag{8}$$

where $\omega_F = \frac{\alpha_F}{\beta_F}$. We let $u_i \in [u_{i,\min}, u_{i,\max}]$,where $u_{i,\min}\in\mathcal{R}^+$ and $u_{i,\max}\in\mathcal{R}^+$ can be chosen to be sufficiently small and large so that they will not be the solution to the minimization of the cost function $J_i, i\in\mathcal{N}\cup\{F\}$.

Proposition 3.1: If $\omega_F\geq 1$, then any solution $\mathbf{u}$ that satisfies (8) is within the feasible set described by the constraint (6).

Proof: Observe from (8), we can conclude that for any $\mathbf{u}\in\{\mathbf{u}\mid\sum_{i\in\mathcal{N}} u_i + \omega_F u_F\}$, the following holds.

$$C = \sum_{i\in\mathcal{N}} u_i + \omega_F u_F \geq \sum_{i\in\mathcal{N}} u_i + u_F, \forall\omega_F\geq 1. \tag{9}$$

Therefore, $\mathbf{u}\in\{\mathbf{u}\mid\sum_{i\in\mathcal{N}\cup\{F\}} u_i\leq C.\}$ ■

Following the proof, we also can observe that when $\omega_F = 1$, the best response function of user F will impose an equality capacity constraint of $\sum_{i\in\mathcal{N}\cup\{F\}} u_i = C$ and the solution will be efficiently achieved on the boundary of the feasible set. However, increasing ω_F to be strictly greater than 1 will result in less efficient solution.

The construction of the best response function (8) can be seen as a slacked constraint from (10)

$$\omega'_F u_F + u_F + \sum_{j\neq F} u_j = C. \tag{10}$$

where $\omega'_F = \omega_F - 1$, and $\omega'_F u_F > 0$ as a slack variable. Similar to the interpretation of Lagrangian multiplier in classical Lagrangian theory, $\omega'_F u_F$ can be seen as an analog of Lagrange multiplier and tells how efficient the system is with respect to the constraint.

A. Characterization of Nash Equilibrium

We use the same approach in [11] to characterize the equilibrium of the game. By the definition of Nash equilibrium in [20], a Nash equilibrium $\mathbf{u}^F$ with a fictitious player is a point which jointly satisfies the best response functions as follows.

$$a_i u_i^F + X_{-i}^F = \frac{a_i\beta_i}{\alpha_i} \quad , \quad \text{for } i\in\mathcal{N}. \tag{11}$$

$$\omega_F u_F^F + \sum_{j\neq F} u_j^F = C \quad , \quad \text{for } i = F. \tag{12}$$

Expressed in matrix form, they become

$$\overline{\mathbf{\Gamma}}\mathbf{u}^F = \overline{\mathbf{b}}. \tag{13}$$

where $\mathbf{u}^F = [u_1^F, \cdots, u_N^F, u_F^F]^T$,

$$\overline{\mathbf{\Gamma}} = \begin{pmatrix} a_1 & \Gamma_{12} & \cdots & \Gamma_{1N} & \Gamma_{1F} \\ \Gamma_{21} & a_2 & \cdots & \Gamma_{2N} & \Gamma_{2F} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \Gamma_{N1} & \Gamma N_2 & \cdots & a_N & \Gamma_{NF} \\ 1 & 1 & \cdots & 1 & \frac{\alpha_F a_F}{\beta_F} \end{pmatrix}, \overline{\mathbf{b}} = \begin{pmatrix} \frac{a_1\beta_1}{\alpha_1} - n_{0,1} \\ \vdots \\ \vdots \\ \frac{a_N\beta_N}{\alpha_N} - n_{0,N} \\ C \end{pmatrix}.$$

A necessary and sufficient condition for Nash equilibrium to exist is to require

$$\overline{\mathbf{b}} \in \mathbf{R}(\overline{\mathbf{\Gamma}}). \tag{14}$$

where $\mathbf{R}(\cdot)$ denotes the range space. However, to ensure the existence and uniqueness of Nash equilibrium, we may need to assume some special features of the game, for example, diagonal dominance of the matrix $\overline{\mathbf{\Gamma}}$ and the convexity of the utility functions J_i. Theorem 3.2 summarizes these conditions and gives a sufficient condition on the uniqueness and existence of the Nash equilibrium to GFP.

Theorem 3.2: Let $\rho(\cdot)$ denote the spectral radius of a matrix. If $\frac{\max_i \overline{\mathbf{b}}_i \sqrt{N+1}}{\sqrt{\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}})}} \leq C$ and $a_i > \sum_{j \neq i} \Gamma_{ij}$, $\omega_F > N$, then the game with a fictitious player (GFP) will have a unique Nash equilibrium.

Proof: First of all, we need to show that the utility functions are convex and there exists a minimizing $\mathbf{u}^F$. It has been proved in [11] that functions (3) is convex in u_i. We just need to show the convexity of J_F in u_F. Knowing that J_F is formed by an addition of two functions and that sum of convex functions results in a convex function, we only need to guarantee the pricing and utility functions are convex. The linear pricing function is already convex. With the condition that $\sum_{j\neq F} u_j \leq C$, the convexity of J_F in u_F will follow. Due to the fact that $u_i \in [0, u_{\max}]$ gives a closed compact set, there exists a minimizing $\mathbf{u}^F$, for any given $\mathbf{u}_{-i}$, such that

$$J(u_i^F, \mathbf{u}_{-i}) < J(u_i, \mathbf{u}_{-i}), \forall u_i \neq u_i^F, i \in \mathcal{N} \cup \{F\}.$$

Secondly, we derive a sufficient condition for convexity of J_F in u_F. Starting with the condition $\sum_{j\neq F} u_j \leq C$, we use matrix norm inequality $\|\overline{\mathbf{\Gamma}}_{m\times n}\|_2 \leq \sqrt{m}\|\overline{\mathbf{\Gamma}}_{m\times n}\|_\infty$ [21] to obtain an upper bound on $\|\overline{\mathbf{1}}^T\overline{\mathbf{\Gamma}}^{-1}\overline{\mathbf{b}}\|_\infty$, where $\overline{\mathbf{1}}^T = [1, ..., 1, 0]$.

$$\|\overline{\mathbf{1}}^T\overline{\mathbf{\Gamma}}^{-1}\overline{\mathbf{b}}\|_\infty \leq \|\overline{\mathbf{1}}^T\overline{\mathbf{\Gamma}}^{-1}\|_\infty \|\overline{\mathbf{b}}\|_\infty \leq \frac{1}{\|\overline{\mathbf{\Gamma}}\|_\infty}\|\overline{\mathbf{b}}\|_\infty \leq$$

$$\frac{\max_i \overline{\mathbf{b}}_i}{\|\overline{\mathbf{\Gamma}}\|_\infty} \leq \frac{\max_i \overline{\mathbf{b}}_i\sqrt{N+1}}{\|\overline{\mathbf{\Gamma}}\|_2} = \frac{\max_i \overline{\mathbf{b}}_i\sqrt{N+1}}{\sqrt{\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}})}}.$$

Therefore, if inequality $\frac{\max_i \overline{\mathbf{b}}_i \sqrt{N+1}}{\sqrt{\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}})}} \leq C$ holds, then the condition of convexity of the fictitious player will hold.

Lastly, we prove that there exists a unique solution under the assumption of diagonal dominance of matrix $\overline{\mathbf{\Gamma}}$. With $a_i > \sum_{j\neq i} \Gamma_{ij}$, $\omega_F > N$, matrix $\overline{\mathbf{\Gamma}}$ becomes diagonal dominant. From Gershgorins Theorem [22], it follows that $\overline{\mathbf{\Gamma}}$ is nonsingular and there exists a unique solution to linear system (13). ■

Remark 3.1: If we further assume that $C \geq \frac{a_i b_i}{\alpha_i} - n_{0,i}, \forall i$, then it will reduce the condition to $\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}}) \geq N+1$. This result alludes to the maximum number of channels to be admitted in the network for a fixed capacity.

Though we notice that some portion of the power is allocated to the service channel or the fictitious player, we need to accept that this amount of power is a necessary allocation for the network to operate. Furthermore, this power consumption can be adjusted through parameter ω_F. On the other hand, we should also note that the strong assumption of diagonal dominance, in particular, $\omega_F > N > 1$ may not lead to an efficient solution as has been indicated by Inequality (9). However, letting $\omega_F = 1$ may still give rise to a unique and efficient solution, since Theorem 3.2 only describes a sufficient condition.

B. Iterative Algorithm

Following (5), the algorithm for the game with a fictitious player is given by a synchronous algorithm given in (15). A step of update includes two sub-steps: an initial update on $u_i(n+1), i \in \mathbf{N}$ and a update sub-step on u_F.

$$\begin{cases} u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(n)} - \Gamma_{i,i}\right)u_i(n), & \forall i \in \mathcal{N}; \\ u_F(n+1) = \frac{1}{\omega_F}\left(C - \sum_{j\neq F} u_j(n)\right), & \text{for } F. \end{cases} \tag{15}$$

Proposition 3.3: The algorithm described by (15) converges to $\mathbf{u}^F$ provided that $a_i > \sum_{j\in\mathcal{N}} \Gamma_{ij}$ and $\omega_F > N$.

Proof: Define $e_i(n) = u_i(n) - u_i^*$ and $e_i(n+1) = -\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j}e_j(n)$ will follow. Letting $\mathbf{e}(n) = [e_1(n), \cdots, e_N(n), e_F(n)]^T$ and taking the infinity norm on $\mathbf{e}(n+1)$, we can arrive at

$$\begin{aligned} \|\mathbf{e}(n+1)\|_\infty &= \max_{i\in\mathcal{N}} |e_i(n+1)| \\ &\leq \max_{i\in\mathcal{N}} \left(\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j}|e_j(n)|\right) \\ &\leq \max_{i\in\mathcal{N}} \left(\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j}\right)\|\mathbf{e}(n)\|_\infty. \end{aligned} \tag{16}$$

Under the assumption of strictly diagonal dominance, i.e., $\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j} < 1, \forall i$, the contraption mapping theorem will show $e(n) \to 0$ and hence, $u_i(n) \to u_i^F$.

For user F's algorithm, in a similar way, we define $e_F(n) = u_F(n) - u_F^F$ and $e_F(n+1) = -\frac{1}{\omega_F}\sum_{i\neq F}(u_i - u_i^F)$.

$$\begin{aligned} |e_F(n+1)| &= \frac{1}{\omega_F}\left|\sum_{i\neq F}\left(u_i(n) - u_i^F(n)\right)\right| \\ &\leq \frac{N}{\omega_F}\max_{i\neq F}|u_i(n) - u_i^F(n)| \\ &= \frac{N}{\omega_F}\|\mathbf{e}(n)\|_\infty. \end{aligned} \tag{17}$$

Using inequality (16),

$$|e_F(n+1)| \leq \frac{N}{\omega_F}\max_{i\in\mathcal{N}}\left(\frac{1}{a_i}\sum_{j\neq i}\Gamma_{i,j}\right)\|\mathbf{e}(n-1)\|_\infty. \tag{18}$$

Since $\|\mathbf{e}(n)\|_\infty \to 0$, then $|e_F(n+1)| \to 0$; and thus, $u_F(n)$ will converge to u_F^F. ■

Parameters $a_i, i \in \mathcal{N}$, and ω_F, as shown in the proof, determines the rate of convergence. On average, increasing

$a_i, i \in \mathcal{N}$ results in a faster convergence for $u_i, i \in \mathcal{N} \cup \{F\}$. And increasing ω_F will lead to a boost in convergence speed of user F's algorithm.

We also can observe a similarity with the algorithm derived based on duality theory in [17], where u_F is more closely related to the dual variable μ. The difference between the two is that we used a fictitious player in the game in the position of the dual variable and the player has it own rule of interactions with other players.

We need to point out that this similarity is not surprising to us because we can see the way a constraint is associated with an additional player in a constrained Nash game in analogy to the way constraints are associated with lagrangian multipliers in classical optimization theory. The user F's algorithm turns can be seen analogously as the algorithm for the Lagrangian multiplier.

C. GFP with OSNR constraint

In the above game with fictitious player, we have considered a utility function for user F without OSNR requirement. However, being an internode communications channel for management and user data, the optical service channel may require a certain OSNR level to support intelligent optical network communication [1], [23]. In this regard, we may impose a target OSNR as a constraint for the user F to guarantee its minimum requirement of quality of transmission. Let γ_F be the target OSNR for F and require $OSNR_F \geq \gamma_F$, that is, by (1),

$$OSNR_F \geq \gamma_F. \tag{19}$$

that is,

$$\begin{aligned} \frac{u_F}{\sum_{j\in\mathcal{N}} \Gamma_{Fj} u_j + n_{0,F}} &\geq \gamma_F. &(20)\\ u_F - \gamma_F \sum_{j\in\mathcal{N}} \Gamma_{Fj} u_j &\geq \gamma_F n_{0,F}. &(21)\\ \mathbf{q}_F \mathbf{u}_F &\geq \gamma_F n_{0,F}. &(22)\end{aligned}$$

where $\mathbf{q_F} = [-\gamma_F\Gamma_{F1}, -\gamma_F\Gamma_{F2}, \cdots, 1]^T$, and $\mathbf{u}_F = [u_1, u_2, \cdots, u_N, u_F]$.

We can show the OSNR constraint of user F, together with the capacity constraint, will give a nonempty convex feasible set when $C \geq \gamma_F n_{0,F}$ or $\gamma_F \leq C/n_{0,F}$. It is illustrated in Figure 19 and stated in Proposition 3.4.

Proposition 3.4: The feasible set $X_F = F_1 \cap F_2$ is nonempty if and only if $C \geq \gamma_F n_{0,F}$, i.e., $\gamma_F \leq C/n_{0,F}$, where $F_1 = \{\mathbf{u} \mid \sum_{i\in\mathcal{N}\cup\{F\}} u_i \leq C\}$ and $F_2 = \{\mathbf{u}_F \mid \mathbf{q}_F\mathbf{u}_F \geq \gamma_F n_{0,F}\}$.

Proof: Let's prove the necessity first, i.e., if the feasible set X_F is nonempty, then $C \geq \gamma_F n_{0,F}$. The proof starts with $\mathbf{q}_F\mathbf{u}_F \geq \gamma_F n_{0,F}$ and substitute the inequality $u_F \leq C - u_1 - u_2 - \cdots - u_N$ into u_F. We obtain the following inequality if both constraints are satisfied.

$$-(\gamma_F\Gamma_{F1}+1)u_1 - \cdots - (\gamma_F\Gamma_{FN}+1)u_N + C \geq \gamma_F n_{0,F}.$$

Since $\gamma_F, \Gamma_{Fi}, u_i$ are nonnegative,

$$C \geq -(\gamma_F\Gamma_{F1}+1)u_1 - \cdots - (\gamma_F\Gamma_{FN}+1)u_N + C \geq \gamma_F n_{0,F}$$

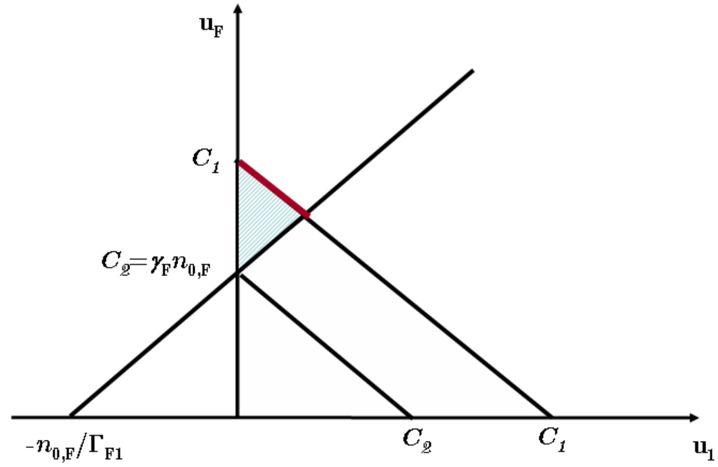

Fig. 3. The feasible set of two player game is nonempty if $C \geq \gamma_F n_{0,F}$.

Therefore, we have $C \geq \gamma_F n_{0,F}$.

In the following, we show the sufficiency, i.e., if $C \geq \gamma_F n_{0,F}$, then there always exist a point in X_F that satisfies both constraints. Suppose there exists a point $\mathbf{u}' = [0, \cdots, u'_F]$ that satisfies the constraint $\mathbf{q}_F\mathbf{u}_F \geq \gamma_F n_{0,F}$. This yields $u'_F \geq \gamma_F n_{0,F}$. With the condition $C \geq \gamma_F n_{0,F}$, we can find at least one u'_F such that $C \geq u'_F \geq \gamma_F n_{0,F}$, for example, $u'_F = \gamma_F n_{0,F}$. This shows that we can find points $\mathbf{u}'$ that satisfy the inequality $\sum_i u_i \leq C$. Therefore, there always exists feasible points in the feasible set and the feasible set is not empty. ■

Proposition 3.5: Suppose the non-cooperative game with a fictitious player has a unique solution $\mathbf{u}^F$, then if target OSNR γ_F is met, then γ_F should satisfy the inequality:

$$\ln\gamma_F \leq \min\{\ln(C/n_{0,F}), \ln(1/\overline{\gamma}) - \hat{\gamma}^T(\overline{\mathbf{\Gamma}}^{-1}\overline{\mathbf{b}})\}.$$

Proof: From Proposition 3.4, we require $\gamma_F < C/n_{0,F}$. Use the OSNR expression in (1) and we obtain

$$\begin{aligned} OSNR_F(\mathbf{u}^F) &= \frac{u_F}{\sum_{j\in\mathcal{N}} \Gamma_{Fj} u_j + n_{0,F}} \\ &= \frac{u_F}{\overline{\gamma}(\Gamma'_{F1}u_1 \cdots + \Gamma'_{FN}u_N) + n_{0,F}} \\ &\leq \frac{u_F}{\overline{\gamma}(u_1^{\Gamma'_{F1}} \cdots u_N^{\Gamma'_{FN}})} \end{aligned} \tag{23}$$

where $\overline{\gamma} = \sum_{i\in\mathcal{N}} \Gamma_{Fi}$. Inequality (23) comes from the arithmetic-geometric mean inequality [24]. With $\sum_i \Gamma'_{Fi} = 1, u_i, \Gamma'_{Fi} > 0$, we have

$$u_1^{\Gamma'_{F1}} u_2^{\Gamma'_{F2}} \cdots u_N^{\Gamma'_{FN}} \leq \Gamma'_{F1}u_1 + \Gamma'_{F2}u_2 + \cdots + \Gamma'_{FN}u_N. \tag{24}$$

Equality holds only when $u_1 = u_2 = \cdots = u_N$. Requiring $\gamma_F \leq OSNR_F(\mathbf{u}^F)$, we can further determine an upper bound on the target γ_F.

$$\begin{aligned}\ln\gamma_F &\leq \ln(1/\overline{\gamma})+\ln(u_1^{-\Gamma'_{F1}})+\cdots+\ln(u_N^{-\Gamma'_{FN}})+\ln u_F\\ &= \ln(1/\overline{\gamma})-\Gamma'_{F1}\ln(u_1)+\cdots-\Gamma'_{FN}\ln(u_N)+\ln u_F\\ &= \ln(1/\overline{\gamma})-\hat{\gamma}^T\ln(\mathbf{u}) \qquad (25)\\ &= \ln(1/\overline{\gamma})-\hat{\gamma}^T\ln(\overline{\mathbf{\Gamma}}^{-1}\overline{\mathbf{b}}) \qquad (26)\end{aligned}$$

where $\hat{\gamma}^T=[\Gamma'_{F1},\cdots,\Gamma'_{FN},-1]$ and $\Gamma'_{Fi}=\frac{\Gamma_{Fi}}{\overline{\gamma}}$. ■

From the above result, by assuming the power allocated to user F is negligible, we can further simplify the inequality into an estimate of γ_F by

$$\gamma_F\leq\min\{C_0/n_{0,F},1/\overline{\gamma}\},$$

because $\hat{\gamma}^T\ln(\mathbf{u})>0$ if u_F is sufficiently small. Therefore, we have a rough estimate on the upper bound on γ_F, i.e.,$\gamma_F<\frac{1}{\overline{\gamma}}$. It also means that we can't not make the transmission for user u_F better than $1/\overline{\gamma}$ or $C_0/n_{0,F}$ in terms of OSNR.

IV. Direction of Future Work

This paper outlines a way to deal with coupled constraints by including a fictitious player. In the context of optical networks, we have only considered the coupled capacity constraints. It can be further extended to include multiple linearly coupled constraints by including more fictitious players. As a result, analogous to classical Lagrangian method in which each constraint is associated with a Lagrangian multiplier, in noncooperative game, we deal with each constraint by associating it with a fictitious player. It will be possible to crystalize this analogy into a theory for dealing with a general class of Nash games with coupled constraints.

V. Conclusion

In this paper, we study the constrained OSNR game in the context of optical networks. We addressed the issue of constrained optimization and its efficiency in non-cooperative games. We characterize the Nash equilibrium with a closed form solution by including a fictitious player with her target OSNR due to the linearity of the coupled constraint and the best response function. This unique approach allows us to derive an iterative algorithm similar to the duality approach in a much simpler way.

References

[1] R. Ramaswami and K. Sivarajan, *Optical Networks: A Practical Perspective*. Academic Press, 2002.

[2] N. Stefanovic and L. Pavel, "Application of robust L2 control to erbium-doped fiber amplifier: input and state uncertainty," *Proc. IEEE Conference on Control Applications*, pp. 686–692.

[3] S. Kuntze, L. Pavel, and J. Aitchison, "Controlling a semiconductor optical amplifier using a state-space model," *IEEE Journal of Quantum Electronics*, vol. 43, no. 2, pp. 123–129, Feb 2007.

[4] C. Saraydar, N. Mandayam, and D. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Transactions on Communications*, vol. 50, no. 2.

[5] C. Saraydar and D. Goodman, "Pricing and power control in a multicell wireless data network," *IEEE Journal of Selected Areas of Communications*, vol. 19, no. 10.

[6] A. Yassine, O. Kabranov, and D. Makrakis, "Competitive game theoretic optimal routing in optical networks," *Proc. SPIE on Optical Networking II*, vol. 4910, no. 11, pp. 27–36, September 2002.

[7] L. Pavel, "Dynamics and stability in optical communication networks: A system theoretic framework," *Automatica*, vol. 40, no. 8, pp. 1361–1370, August 2004.

[8] G. Agrawal, *Lightwave Technology*. Wiley-Interscience, 2005.

[9] L. Pavel, "OSNR optimization in optical networks: Modeling and distributed algorithms via a central cost approach," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 4, pp. 54–65, April 2006.

[10] T. Alpcan, X. Fan, T. Basar, M. Arcak, and J.T.Wen, "Power control for multicell CDMA wireless networks: A team optimization approach," *Third International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks*, pp. 379–388, 2005.

[11] L. Pavel, "A noncooperative game approach to OSNR optimization in optical networks," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 848–852, May 2006.

[12] R. Srikant, E. Altman, T. Alpcan, and T. Basar, "CDMA uplink power control as noncooperative game," *Wireless Networks*, vol. 8, p. 659 690, 2002.

[13] P. Dubey, "Inefficiency of Nash equilibria," *Mathematics of Operations Research*, vol. 11, no. 1, February 1986.

[14] T. Roughgarden and E. Tardos, "Bounding the inefficiency of equilibria in nonatomic congestion games," *Games and Economic Behavior*, vol. 47, pp. 389–403, 2004.

[15] Y. Pan and L. Pavel, "OSNR optimization in optical networks: Extension for capacity constraints," *Proceedings of 2005 American Control Conference*, pp. 2379–2385, June 2005.

[16] ——, "Global convergence of an iterative gradient algorithm for the Nash equilibrium in an extended OSNR game," *Proc. IEEE INFOCOM*, May 2007.

[17] L. Pavel, "An extension of duality to a game-theoretic framework," *Automatica*, vol. 43, no. 2, pp. 226–237, 2007.

[18] ——, "Hierarchical iterative algorithm for a coupled constrained OSNR nash game," *Proceedings of IEEE GLOBECOM*, November 2006.

[19] R. T. F. Forghieri and D. Favin, "Simple model of optical amplifier chains to evaluate penalties in wdm systems," *Journal of Lightwave Technology*, vol. 16, no. 9, p. 15701576, 1998.

[20] M. Osborne and A. Rubinstein, *A Course in Game Theory*. MIT Press, 1994.

[21] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1990.

[22] A. Berman and R. Plemmons, *Nonnegative Matrices in Mathematical Sciences*. SIAM, 1994.

[23] J. Walrand and P. Varaiya, *High-Performance Communication Networks*, 2nd ed. Morgan Kaufmann, 2000.

[24] D. Bertsekas, A. Nedic, and A. E. Ozdaglar, *Convex Analysis and Optimization*. Athena Scientific, 2003.