A Survey on Delayed Linear Agreement Protocol in Robot Rendezvous Problem

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Motivation: Multi-agent Systems

Flocking

Chemical Reaction Network

Flight Formation

Autonomous Underwater Vehicles
Systems of Coupled Dynamics

Robot Rendezvous problem with delays

M.C. Escher: Drawing Hands
Delays in Rendezvous Problems

• Communication delay
  – Uniform delay
  – Non-uniform delay
  – Time-invariant
  – Time-Varying delay

• Delays in computing or response

Without delay:
\[ \dot{x}_1(t) = x_2(t) - x_1(t) \]

With delay:
\[ \dot{x}_1(t) = x_2(t - \tau) - x_1(t - \tau) \]
References:


First Order Linear Time-Delay System

\[ \dot{x}(t) = -\lambda x(t - \tau) \quad \lambda > 0, \tau > 0 \]

(Hale and Lunel, Bliman & F.-Trecate)

The system is exponentially stable if and only if \( \tau < \pi/(2\lambda) \).

Characteristic equation: \( s + \lambda e^{-\tau s} = 0 \)

\[ \tau = \frac{(2k\pi + \pi/2)}{\lambda}, \quad \text{where } k = 0,1,2,\ldots \]

\[ \tau^* = \frac{\pi}{2\lambda} \quad \text{for } \tau > 0 \]

- Due to continuous dependence of roots in \( \tau \), for \( \tau \in (0, \tau^*) \), all the poles are in the OLHP.

- When \( \tau = \tau^* \), the solution will be in the form of

\[ x(t) = c \sin(\lambda t + \varphi) \]
Example

\[ \dot{x}(t) = -x(t - \tau) \quad \tau > 0 \]

\[ \tau^* = \pi / 2 \]
Some Existing Results on Delay Networks

(Olfati & Murray, 2004) Consider a network of integrator agents with uniform communication-delays $\tau > 0$ in all links. Assume:

(1) the information flow $G$ of the network is undirected, and connected.

(2) under the protocol: $u_i(t) = \sum_{i \in N_j} a_{ij} [x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})]$ with $\tau_{ij} = \tau$

It globally asymptotically solves average-consensus problem if and only if

$\tau \in (0, \tau^*)$ with $\tau^* = \pi/(2\lambda_n)$, where $\lambda_n = \lambda_{\text{max}}(L)$. 
Sketch of the Proof

1. Under the assumptions, if the solutions globally asymptotically converge to a limit $x^*$, then

$$x^* = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$$

2. (Stability of the solutions)

$$X(s) = (sI_n + e^{-\tau s} L)^{-1} x(0)$$

$$Z_{\tau}(s) = (sI_n + e^{-\tau s} L)$$

Target: want a condition that zeros of $Z_{\tau}(s)$ are on the OLHP

Difficulties: (1) nonlinearity (2) multi-input and multi-output (MIMO)
**Sketch of the Proof**

**From MIMO to ‘SISO’**

\[ w_k \] be the \( k \)th normalized eigenvector of \( L \), associated with 

\[ \lambda_k \], be the \( k \)th eigenvalue of \( L \) in an increasing order.

For a connected graph \( G \), \( 0 = \lambda_1 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n = \lambda_{\text{max}}(L) \)

\[
Z_\tau(s)w_k = sw_k + e^{-\tau s}Lw_k = (s + e^{-\tau s} \lambda_k)w_k = 0
\]

\[ s + e^{-\tau s} \lambda_k = 0 \]

Solution for \( s \), is dependent on \( \lambda_k \)
Sketch of the Proof

From Nonlinear to ‘Linear’

\[ s + e^{-\tau s} \lambda_k = 0 \]

Use the result from the first-order linear time-delay system,

\[ \tau = \pi / 2\lambda_k \]

\[ \tau^* = \min_{k>1} \tau = \frac{\pi}{2\lambda_{\text{max}}(L)} \]

Note that when \( \lambda=0, s=0, \) no matter what \( \tau \) is, there is a zero on imaginary axis.

Due to the continuous dependence of roots on \( \tau, \)

The solution is asymptotically stable if and only if \( \tau \in (0, \tau^*) \) with \( \tau^* = \pi/(2\lambda_n), \) where \( \lambda_n = \lambda_{\text{max}}(L). \)
Example: Two-robot Rendezvous

\begin{align*}
L &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
\lambda_{\text{max}} &= 2, \tau^* = \frac{\pi}{4}
\end{align*}

Can’t apply to three-robot cyclic pursuit: violates the assumptions!
Some Existing Results on Delay Networks

(Bliman & F.-Trecate) Consider a communication network modeled through an undirected and connected graph with constant delays. Let $\Delta$ be the Laplacian operator, the solution to

$$\dot{v}(x, t) = \Delta v(x, t - \tau)$$

is globally exponentially stable for all possible $\tau \leq \tau^*$, if and only if

<table>
<thead>
<tr>
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<th>Uniform delays</th>
<th>Non-uniform delays</th>
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</thead>
<tbody>
<tr>
<td>Time-invariant</td>
<td>$\tau^* &lt; \frac{\pi}{2|\Delta|}$</td>
<td>$\tau^* &lt; \frac{\pi}{2|\Delta|}$</td>
</tr>
<tr>
<td>Time-varying</td>
<td>$\tau^*(t) &lt; \frac{3\pi}{2|\Delta|}$</td>
<td>$\tau^*(t) &lt; \frac{1}{\sum_{i,i' \in I} |\Delta_i \Delta_{i'}| |\Delta^{-1}|}$</td>
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Robotic Network Approach

Linear Agreement Protocol:
\[ \dot{x}_i(t) = \sum_{j \in N_i} x_j(t) - x_i(t) \]
\[ \dot{x}(t) = -Lx(t) \]

\[ H_i(s) = \frac{G_i(s)K_i(s)}{1 + G_i(s)K_i(s)T_i(s)} \]
Example: 3-Robot Cyclic Pursuit

Cyclic Pursuit Model

\[
\begin{align*}
\dot{x}_1(t) &= [x_2(t - \tau_{21}) - x_1(t - \tau_{21})] \\
\dot{x}_2(t) &= [x_3(t - \tau_{32}) - x_2(t - \tau_{32})] \\
\dot{x}_3(t) &= [x_1(t - \tau_{13}) - x_3(t - \tau_{13})]
\end{align*}
\]

Position Vector

\[X(s) = [X_1(s), X_2(s), X_3(s)]\]

Frequency Domain Transform

\[
\begin{align*}
X_1(s) &= (s + T_{21})^{-1}(T_{21}X_2(s) + x_1(0)) \\
X_2(s) &= (s + T_{32})^{-1}(T_{32}X_3(s) + x_2(0)) \\
X_3(s) &= (s + T_{13})^{-1}(T_{13}X_1(s) + x_3(0))
\end{align*}
\]
Example: 3-Robot Cyclic Pursuit

**Frequency Domain Transform**

\[
X_1(s) = (s + T_{21})^{-1}(T_{21}X_2(s) + x_1(0))
\]

\[
X_2(s) = (s + T_{32})^{-1}(T_{32}X_3(s) + x_2(0))
\]

\[
X_3(s) = (s + T_{13})^{-1}(T_{13}X_1(s) + x_3(0))
\]

**Delay Matrix T**

\[
T(s) = \begin{bmatrix}
T_{11}(s) & 0 & 0 \\
T_{12}(s) & 0 & 0 \\
T_{13}(s) & 0 & 0 \\
0 & T_{21}(s) & 0 \\
0 & T_{22}(s) & 0 \\
0 & T_{23}(s) & 0 \\
0 & 0 & T_{31}(s) \\
0 & 0 & T_{32}(s) \\
0 & 0 & T_{33}(s)
\end{bmatrix}
\]

**System Matrix H**

\[
H(s) = \begin{bmatrix}
(s + T_{21})^{-1} & 0 & 0 \\
0 & (s + T_{32})^{-1} & 0 \\
0 & 0 & (s + T_{13})^{-1}
\end{bmatrix}
\]

**Selection Matrix S**

\[
S = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Example: 3-Robot Cyclic Pursuit

\[ X_1(s) = (s + T_{21})^{-1}(T_{21}X_2(s) + x_1(0)) \]
\[ X_2(s) = (s + T_{32})^{-1}(T_{32}X_3(s) + x_2(0)) \]
\[ X_3(s) = (s + T_{13})^{-1}(T_{13}X_1(s) + x_3(0)) \]
Features of the Established Framework

- A general framework for robotic networks

- **S**: Selection matrix is related to the topology
  - balanced or unbalanced
  - directed or undirected
  - time-varying or time-variant

- **T(s)**: Delay matrix
  - uniform delay or non-uniform delay
  - time-varying or time-invariant

- **H(s)**: System matrix:
  - parametric uncertainties

- Robust Stability Criteria

- Robust Controller design
Results on Delay Networks

• (Modified from Lee & Spong, 2006)
  Suppose that we have $||(\Delta H)_i(j\omega)|| < 1$, $\forall \omega \in (0, +\infty)$ and
  $\lim_{\omega \to 0} ||(\Delta H)_i(j\omega)|| = 1$, and the information graph G is
  strongly connected, then $x_i(t) \to c$, $\forall i \in \{1, 2, 3, \ldots, n\}$, regardless
  of the non-uniform constant delays.

• Used for estimation for the bounds of delay
  – Conservative as a sufficient condition.
Sketch of the Proof

1. Since $\Delta H(jw)$ is diagonal,

the eigenvalues of $\Delta H(jw)$ are located in the union of the following discs in complex plane $\mathbb{C}$:

$$D_i(\omega) := \{ z \in \mathbb{C} : |z| \leq |(\Delta H)_i(jw)| \}$$

with $|(\Delta H)_i(jw)| \leq 1$

Equality only holds for $w=0$.

$$\rho(\Delta H(jw)) < 1 \quad \text{for } w > 0$$

$$\rho(\Delta H(jw)) = 1 \quad \text{for } w = 0$$
Non-zero frequency signal will die out and only the zero-frequency signal remains.

This implies that the feedback-loop is marginally stable with marginal behavior possible only when \( w=0 \).

The DC component of \( X(s) \):

\[
\bar{x} = A(G)\bar{x}
\]

1 is always an eigenvalue of \( A(G) \) and if \( G \) is strongly connected, 1 is a simple eigenvalue and its eigenvector is uniquely given by \( 1^T \). Therefore

\[
\bar{x} = c1^T
\]
Example: Three Robot Cyclic Pursuit

\[ \dot{x}_1(t) = [x_2(t - \tau_{21}) - x_1(t - \tau_{21})] \]
\[ \dot{x}_2(t) = [x_3(t - \tau_{32}) - x_2(t - \tau_{32})] \]
\[ \dot{x}_3(t) = [x_1(t - \tau_{13}) - x_3(t - \tau_{13})] \]

\[ \|\Delta H(jw)\|_\infty < 1 \text{ for } w \in (0, +\infty) \]
\[ \text{and } \lim_{w \to 0} \|\Delta H(jw)\|_\infty = 1 \text{ for } \tau < 0.5268 \]

All \( \tau_{ij} < 0.5258 \) will satisfy the condition;
Thus, rendezvous occurs for any \( \tau_{12}, \tau_{23}, \tau_{13} < 0.5268 \).
Example: Three Robot Cyclic Pursuit

\[ \tau_{co} \approx 0.605 \]

The estimation is conservative.
Example: Multiple Robot Cyclic Pursuit (n=10)

\[ \tau_{co} \approx 0.51 \]
Example: Multiple Robot Cyclic Pursuit (n=32)

\[ \tau_{co} \approx 0.5 \]
Random Updates

- Probabilistic model of failure or delay in communications, computations and updates

Discrete Model Random Updates

\[
\begin{align*}
  x_i(n+1) &= x_i(n) + \varepsilon \left( \sum_{j \in N_i} x_j(n) - x_i(n) \right) \\
  x_i(n+1) &= x_i(n)
\end{align*}
\]

with probability \( \pi_i \)

with probability \( 1-\pi_i \)

\( \pi_i \) describes how robust the mechanism is.
Two Robot Rendezvous

- Uniform random variable for each robot;
- End up in different rendezvous points
- How to increase the robustness of the distributed algorithm?
- How to make sure the robot rendezvous with presence of possible failure in updates?

\[ \pi = [0.95, 0.95, 0.95, 0.95, 0.95] \]
Summary

1st order DODE (Hale and Lunel)

Uniform Time-Invariant Delay (Olfati & Murray)

Recent Advanced Extension (Bliman & F.-Trecate)

Delay

Linear Network Model (Lee & Spong)

Probabilistic Viewpoint
Future Research

- More general and stronger results on delayed communication networks: e.g. directed graphs and time-varying topology
  - Much more involved mathematics: Functional differential equations (Hale & Lunel), Differential-difference equations (Bellman)
- Better heuristic algorithms to compute the bounds for delays
- Controller design with the presence of parametric uncertainties
- Introducing probabilistic approach
- Explain convergences present in
  - the delay in the multi-robot cyclic pursuit
  - Probabilistic algorithm
Questions