A Survey on Delayed Linear Agreement Protocol in Robot Rendezvous Problem

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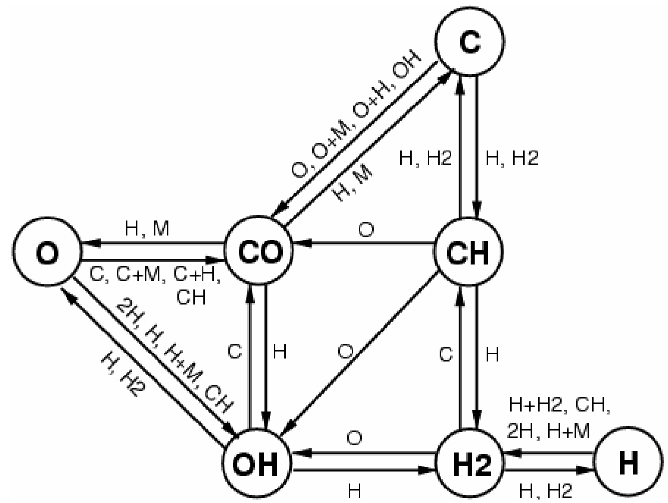
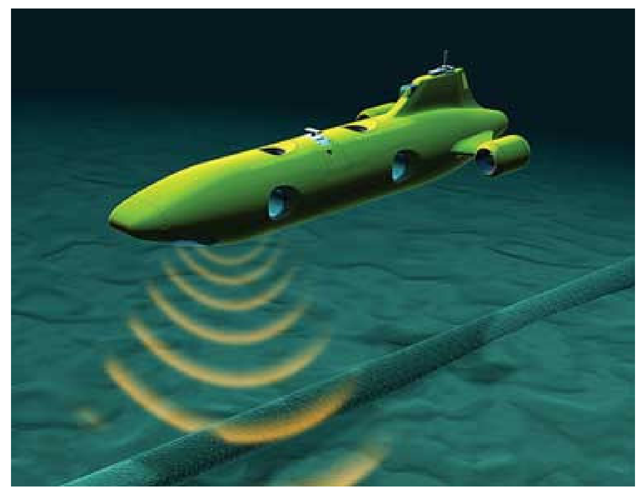
Motivation: Multi-agent Systems

Flocking



Chemical Reaction Network

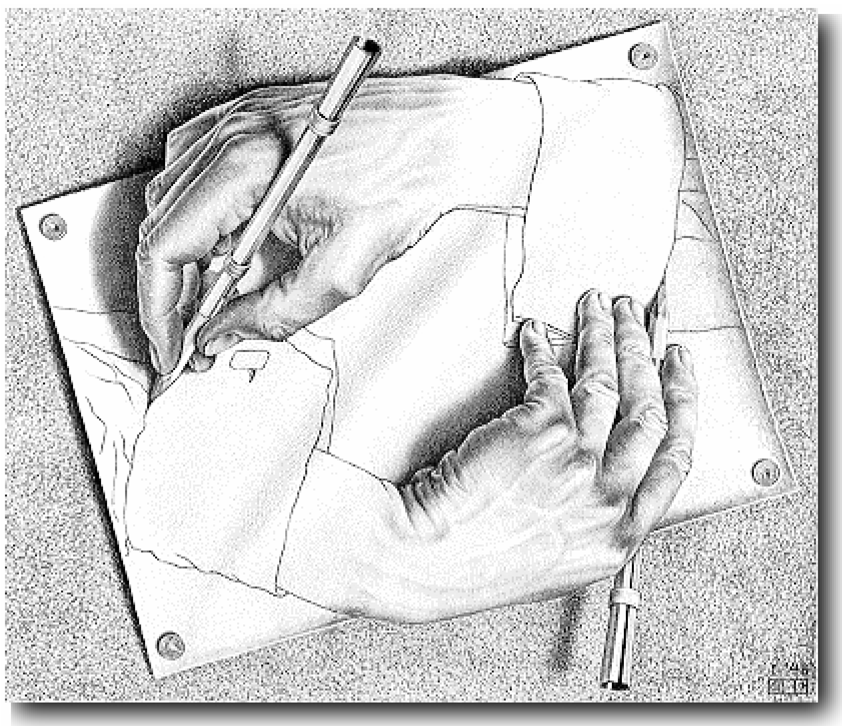
Flight Formation



Autonomous Underwater Vehicles

Systems of Coupled Dynamics

Robot Rendezvous problem with delays

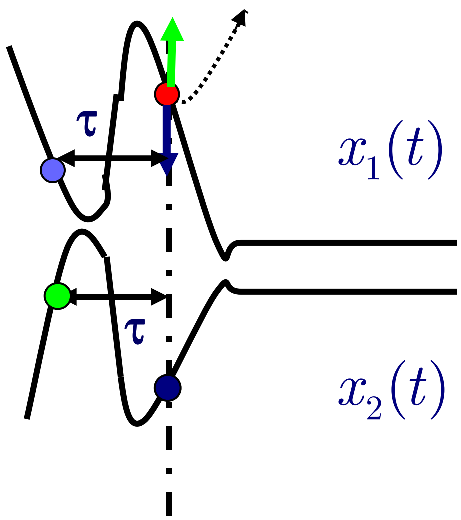


M.C. Escher: Drawing Hands

Delays in Rendezvous Problems

- Communication delay
  - Uniform delay
  - Non-uniform delay
  - Time-invariant
  - Time-Varying delay

- Delays in computing or response



Without delay:

$$\dot{x}_1(t) = x_2(t) - x_1(t)$$

With delay:

$$\dot{x}_1(t) = x_2(t-\tau) - x_1(t-\tau)$$

References:

- Lee, D.J. and Spong, M.W., "Agreement with Non-Uniform Information Delays," *American Control Conference*, pp. 756-761, Minneapolis, MN, June 14-16, 2006

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- P.-A. Bliman, G. F.-Trecate, Average Consensus Problems in Networks of Agents with Delayed Communications, *Automatica*, under review.

- L.Pavel, “Dynamics and stability in optical communication networks: a system theory framework”, *Automatica* 40, pp.1361-1370, 2004.

- K.Zhou, J.Doyle, Essentials of Robust Control, Prentice Hall, 1997.

First Order Linear Time-Delay System

$$\dot{x}(t) = -\lambda x(t-\tau) \qquad \lambda > 0,\ \tau > 0$$

(Hale and Lunel, Bliman & F.-Trecate)
The system is exponentially stable if and only if $\tau < \pi/(2\lambda)$.

Characteristic equation: $s + \lambda e^{-\tau s} = 0$

$$\tau = \frac{(2k\pi + \pi/2)}{\lambda}, \quad \text{where } k = 0, 1, 2, \ldots$$

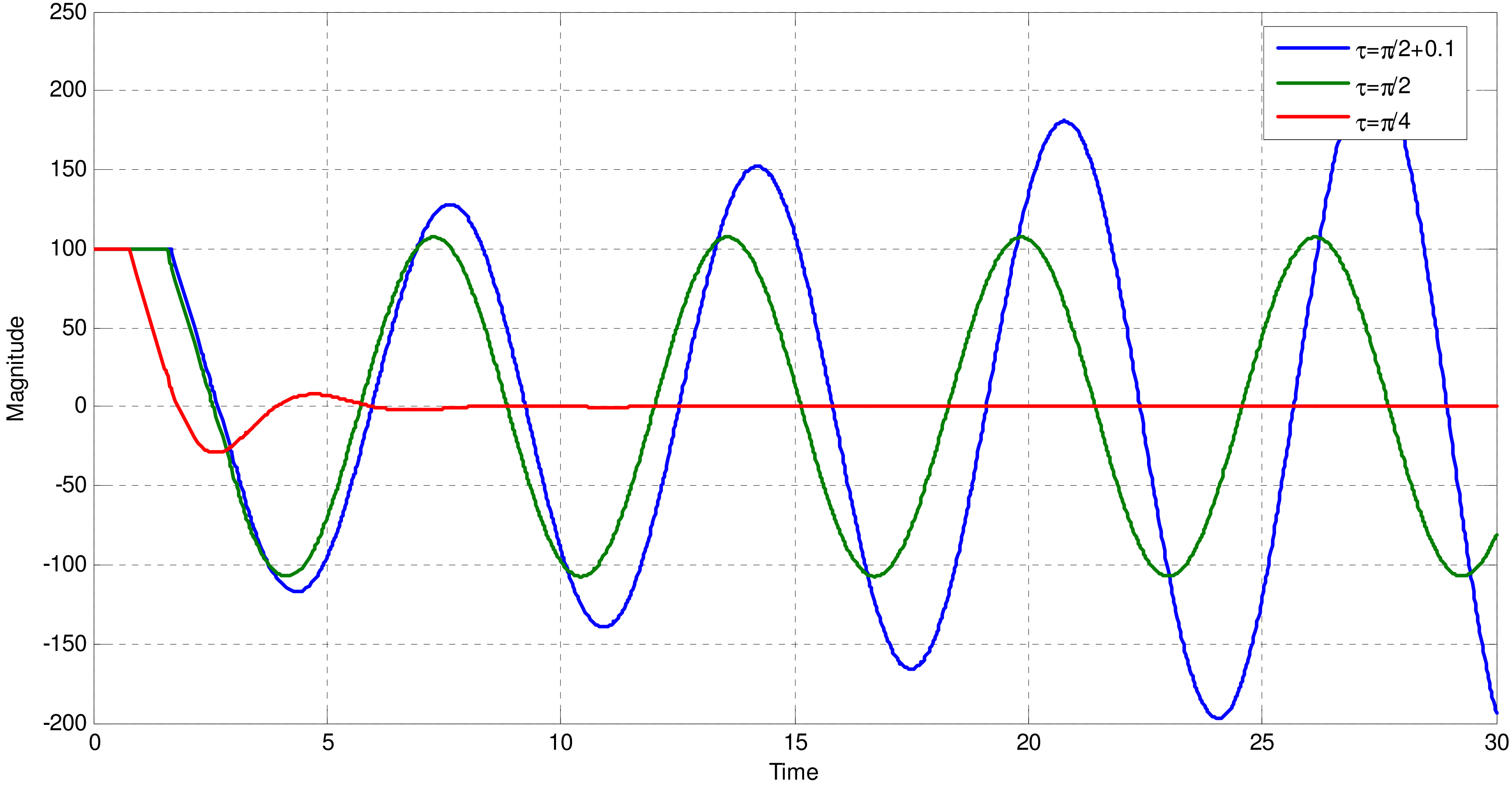
$$\tau^* = \frac{\pi}{2\lambda} \qquad \text{for } \tau > 0$$

- Due to continuous dependence of roots in τ, for $\tau \in (0, \tau^*)$, all the poles are in the OLHP.
- When $\tau = \tau^*$, the solution will be in the form of

$$x(t) = c\sin(\lambda t + \varphi)$$

Example

$$\dot{x}(t) = -x(t-\tau) \qquad \tau > 0$$
$$\tau^* = \pi/2$$



Some Existing Results on Delay Networks

(Olfati & Murray, 2004) Consider a network of integrator agents with uniform communication-delays $\tau > 0$ in all links. Assume:

(1) the information flow G of the network is undirected, and connected.

(2) under the protocol: $u_i(t) = \sum_{i \in N_i} a_{ij} \, [x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})]$ with $\tau_{ij} = \tau$

It globally asymptotically solves average-consensus problem if and only if

$$\boxed{\tau \in (0, \tau^*) \text{ with } \tau^* = \pi/(2\lambda_n), \text{ where } \lambda_n = \lambda_{\max}(L).}$$

Sketch of the Proof

1. Under the assumptions, if the solutions globally asymptotically converge to a limit x^*, then

$$x^* = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$$

2. **(Stability of the solutions)**

$$X(s) = (\underbrace{sI_n + e^{-\tau s}L}_{Z_\tau(s) = (sI_n + e^{-\tau s}L)})^{-1} x(0)$$

Target: want a condition that zeros of $Z_\tau(s)$ are on the OLHP

Difficulties: (1) nonlinearity (2) multi-input and multi-output (MIMO)

Sketch of the Proof

From MIMO to 'SISO'

w_k be the kth normalized eigenvector of $\boldsymbol{L}$, associated with

λ_k, be the kth eigenvalue of $\boldsymbol{L}$ in an increasing order.

For a connected graph G, $0=\lambda_1 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n = \lambda_{\max}(L)$

$$Z_\tau(s)w_k = sw_k + e^{-\tau s}Lw_k = (s + e^{-\tau s}\lambda_k)w_k = 0$$

$$\Rightarrow \qquad s + e^{-\tau s}\lambda_k = 0$$

Solution for s, is dependent on λ_k

Sketch of the Proof

From Nonlinear to 'Linear'

$$s + e^{-\tau s}\lambda_k = 0$$

Use the result from the first-order linear time-delay system,

$$\tau = \pi / 2\lambda_k$$

$$\Rightarrow \qquad \tau^* = \min_{k>1} \tau = \frac{\pi}{2\lambda_{\max}(L)}$$

Note that when $\lambda=0$, $s=0$, no matter what τ is, there is a zero on imaginary axis

Due to the continuous dependence of roots on τ,

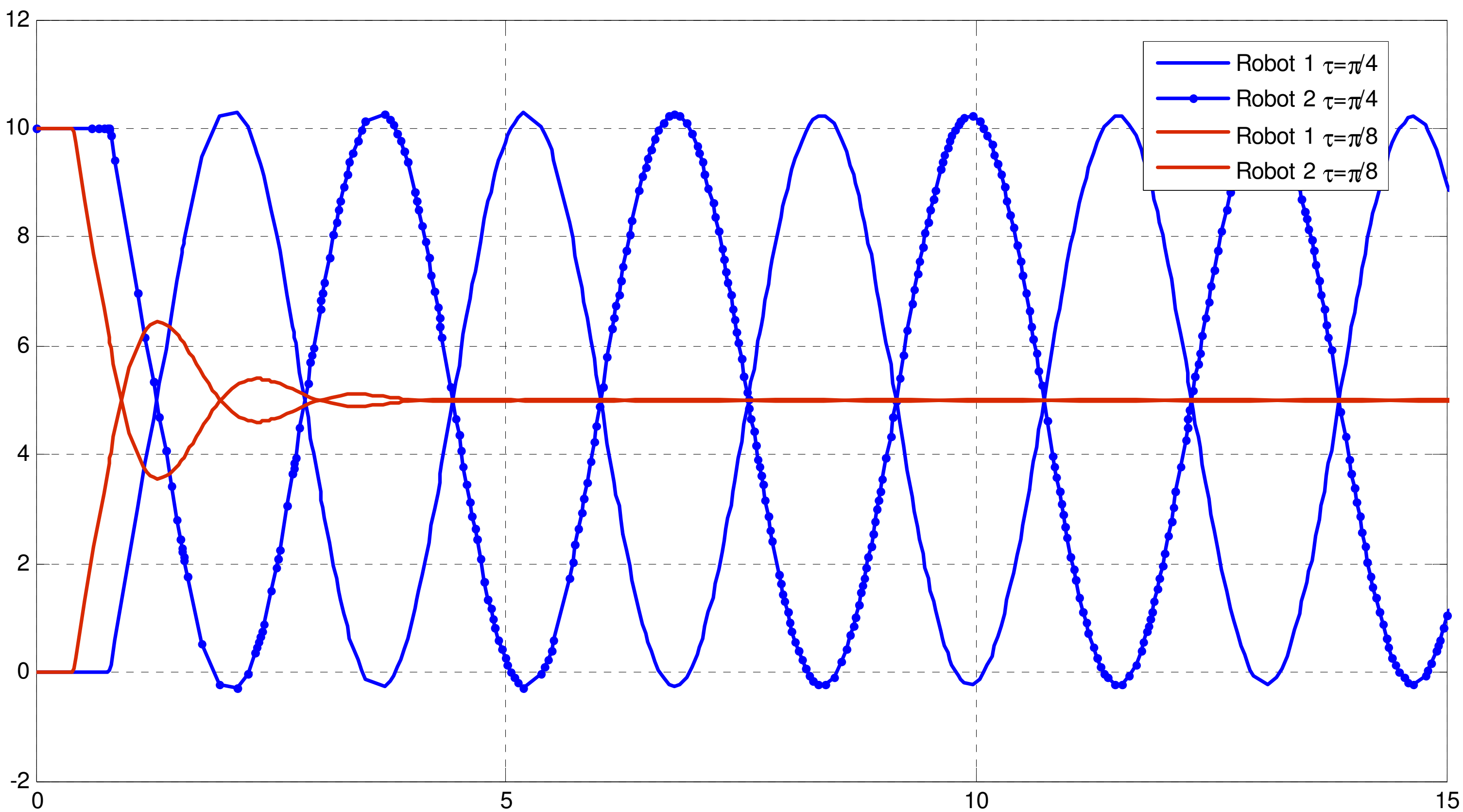
> The solution is asymptotically stable if and only if $\tau \in (0, \tau^*)$ with $\tau^* = \pi/(2\lambda_n)$, where $\lambda_n = \lambda_{\max}(L)$.

Example: Two-robot Rendezvous

undirected, connected

$$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lambda_{\max} = 2, \tau^{*} = \frac{\pi}{4}$$



Can't apply to three-robot cyclic pursuit: violates the assumptions!

Some Existing Results on Delay Networks

(Bliman & F.-Trecate) Consider a communication network modeled through an undirected and connected graph with constant delays. Let Δ be the Laplacian operator, the solution to

$$\dot{v}(x,t) = \Delta v(x, t-\tau)$$

is globally exponentially stable for all possible $\tau \leq \tau^*$, if and only if

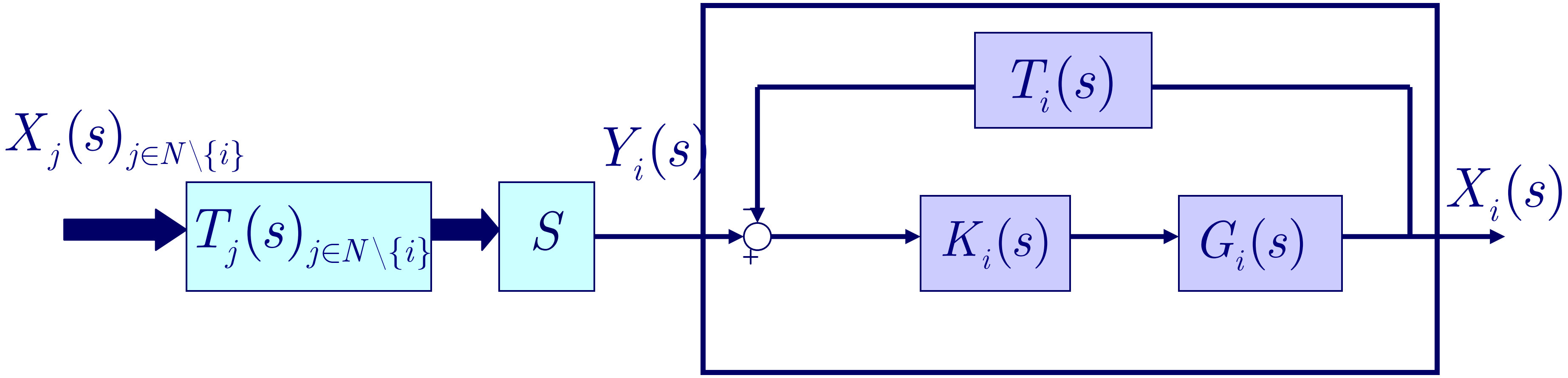
	Uniform delays	Non-uniform delays
Time-invariant	$\tau^* < \dfrac{\pi}{2\|\Delta\|}$	$\tau^* < \dfrac{\pi}{2\|\Delta\|}$
Time-varying	$\tau^*(t) < \dfrac{3\pi}{2\|\Delta\|}$	$\tau^*(t) < \dfrac{1}{\sum_{i,i' \in I} \|\Delta_i \Delta_{i'}\| \|\Delta^{-1}\|}$

Robotic Network Approach

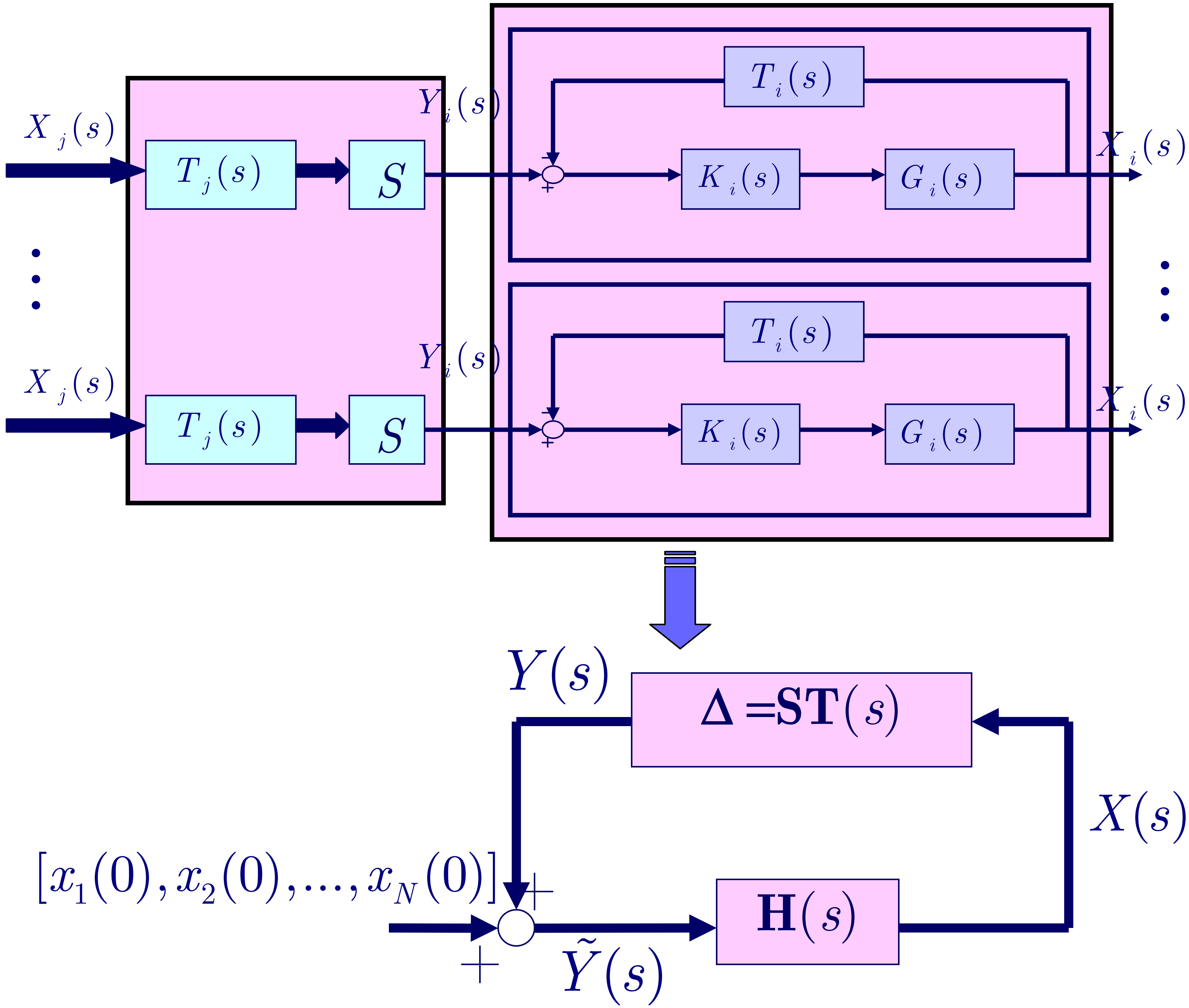
Linear Agreement Protocol:

$$\dot{x}_i(t) = \sum_{j \in N_i} x_j(t) - x_i(t)$$

$$\dot{\boldsymbol{x}}(t) = -L\boldsymbol{x}(t)$$



$$H_i(s) = \frac{G_i(s)K_i(s)}{1 + G_i(s)K_i(s)T_i(s)}$$



$X_j(s)$
$T_j(s)$
S
$Y_i(s)$
$X_j(s)$
$T_j(s)$
S
$Y_i(s)$
$T_i(s)$
$K_i(s)$
$G_i(s)$
$X_i(s)$
$T_i(s)$
$K_i(s)$
$G_i(s)$
$X_i(s)$
$Y(s)$
$\boldsymbol{\Delta} = \mathbf{ST}(s)$
$X(s)$
$[x_1(0), x_2(0), \ldots, x_N(0)]$
$\mathbf{H}(s)$
$\tilde{Y}(s)$

Example: 3-Robot Cyclic Pursuit

Cyclic Pursuit Model

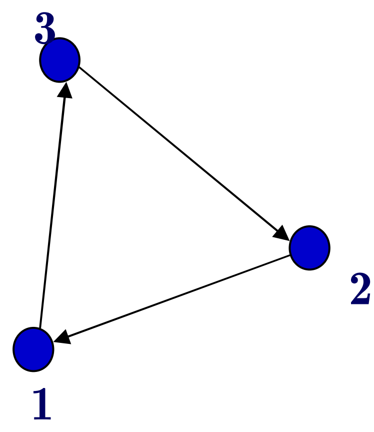
$$\dot{x}_1(t) = [x_2(t-\tau_{21}) - x_1(t-\tau_{21})]$$
$$\dot{x}_2(t) = [x_3(t-\tau_{32}) - x_2(t-\tau_{32})]$$
$$\dot{x}_3(t) = [x_1(t-\tau_{13}) - x_3(t-\tau_{13})]$$

Frequency Domain Transform

$$X_1(s) = (s+T_{21})^{-1}(T_{21}X_2(s) + x_1(0))$$
$$X_2(s) = (s+T_{32})^{-1}(T_{32}X_3(s) + x_2(0))$$
$$X_3(s) = (s+T_{13})^{-1}(T_{13}X_1(s) + x_3(0))$$

Position Vector

$$X(s) = [X_1(s), X_2(s), X_3(s)]$$



Example: 3-Robot Cyclic Pursuit

Frequency Domain Transform

$$X_1(s) = (s + T_{21})^{-1}(T_{21}X_2(s) + x_1(0))$$
$$X_2(s) = (s + T_{32})^{-1}(T_{32}X_3(s) + x_2(0))$$
$$X_3(s) = (s + T_{13})^{-1}(T_{13}X_1(s) + x_3(0))$$

System Matrix H

$$H(s) = \begin{bmatrix} (s+T_{21})^{-1} & 0 & 0 \\ 0 & (s+T_{32})^{-1} & 0 \\ 0 & 0 & (s+T_{13})^{-1} \end{bmatrix};$$

Delay Matrix T

$$\mathrm{T}(s) = \begin{bmatrix} T_{11}(s) & 0 & 0 \\ T_{12}(s) & 0 & 0 \\ T_{13}(s) & 0 & 0 \\ 0 & T_{21}(s) & 0 \\ 0 & T_{22}(s) & 0 \\ 0 & T_{23}(s) & 0 \\ 0 & 0 & T_{31}(s) \\ 0 & 0 & T_{32}(s) \\ 0 & 0 & T_{33}(s) \end{bmatrix}$$

Selection Matrix S

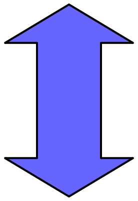
$$S = \left[\begin{array}{ccc:ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

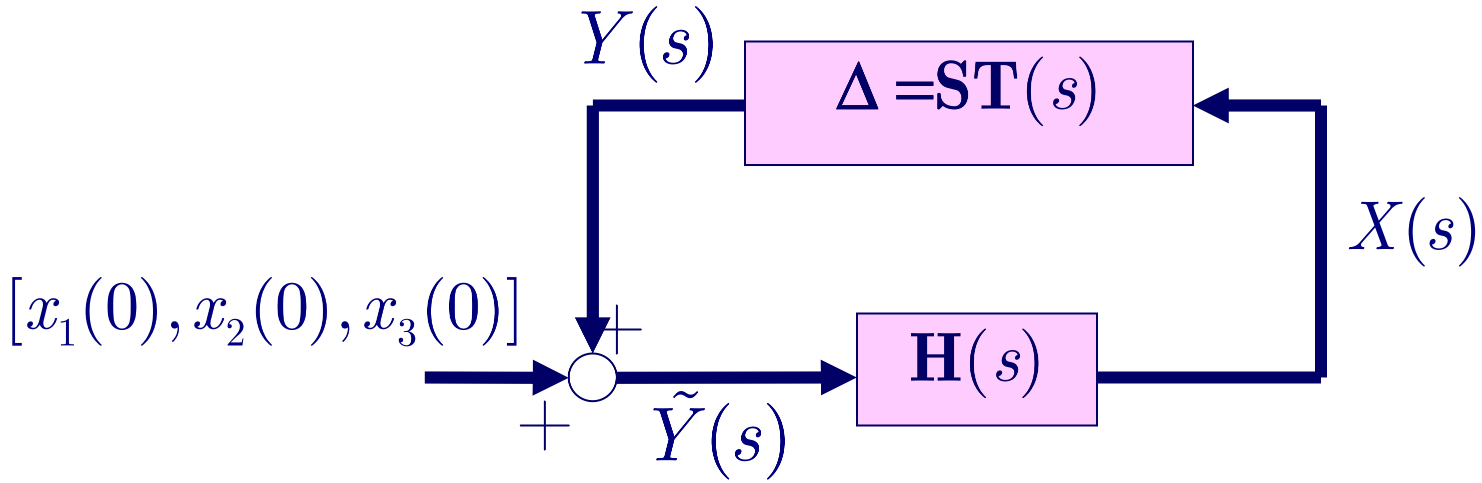
Example: 3-Robot Cyclic Pursuit

$$X_1(s) = (s + T_{21})^{-1}(T_{21}X_2(s) + x_1(0))$$

$$X_2(s) = (s + T_{32})^{-1}(T_{32}X_3(s) + x_2(0))$$

$$X_3(s) = (s + T_{13})^{-1}(T_{13}X_1(s) + x_3(0))$$





Features of the Established Framework

- A general framework for robotic networks
- **S**: Selection matrix is related to the topology
  - balanced or unbalanced
  - directed or undirected
  - time-varying or time-variant
- $\mathbf{T}(s)$: Delay matrix
  - uniform delay or non-uniform delay
  - time-varying or time-invariant
- $\mathbf{H}(s)$**:** System matrix:
  - parametric uncertainties
- Robust Stability Criteria
- Robust Controller design

Results on Delay Networks

- **(Modified from Lee & Spong, 2006)**
  Suppose that we have $||(\Delta H)_i(j\omega)||<1$, $\forall\ \omega \in (0,+\infty]$ and $\lim_{\omega\to 0} ||(\Delta H)_i\ (j\omega)||= 1$, and the information graph G is strongly connected, then $x_i(t) \to$ c, $\forall\ i \in\{1,2,3,\ldots,n\}$, regardless of the non-uniform constant delays.

- Used for estimation for the bounds of delay
  - Conservative as a sufficient condition.

Sketch of the Proof

1. Since $\Delta H(jw)$ is diagonal,

the eigenvalues of $\Delta H(jw)$ are located in the union of the following discs in complex plane C:

$$D_i(\omega) := \left\{ z \in C : | z | \leq \left| (\Delta H)_i (jw) \right| \right\}$$

with $|(\Delta H)_i(jw)| \leq 1$

$$|z| \leq 1$$

Equality only holds for w=0.

$$\rho(\Delta H(jw)) < 1 \quad \text{for } w > 0$$

$$\rho(\Delta H(jw)) = 1 \quad \text{for } w = 0$$

Sketch of the Proof (Cont'd)

2. Non-zero frequency signal will die out and only the zero-frequency signal remains .

   This implies that the feedback-loop is marginally stable with marginal behavior possible only when w=0.

3. The DC component of X(s):

$$\overline{x} = A(G)\overline{x}$$

1 is always an eigenvalue of A(G) and if G is strongly connected, 1 is a simple eigenvalue and its eigenvector is uniquely given by $\mathbf{1}^{\mathrm{T}}$. Therefore

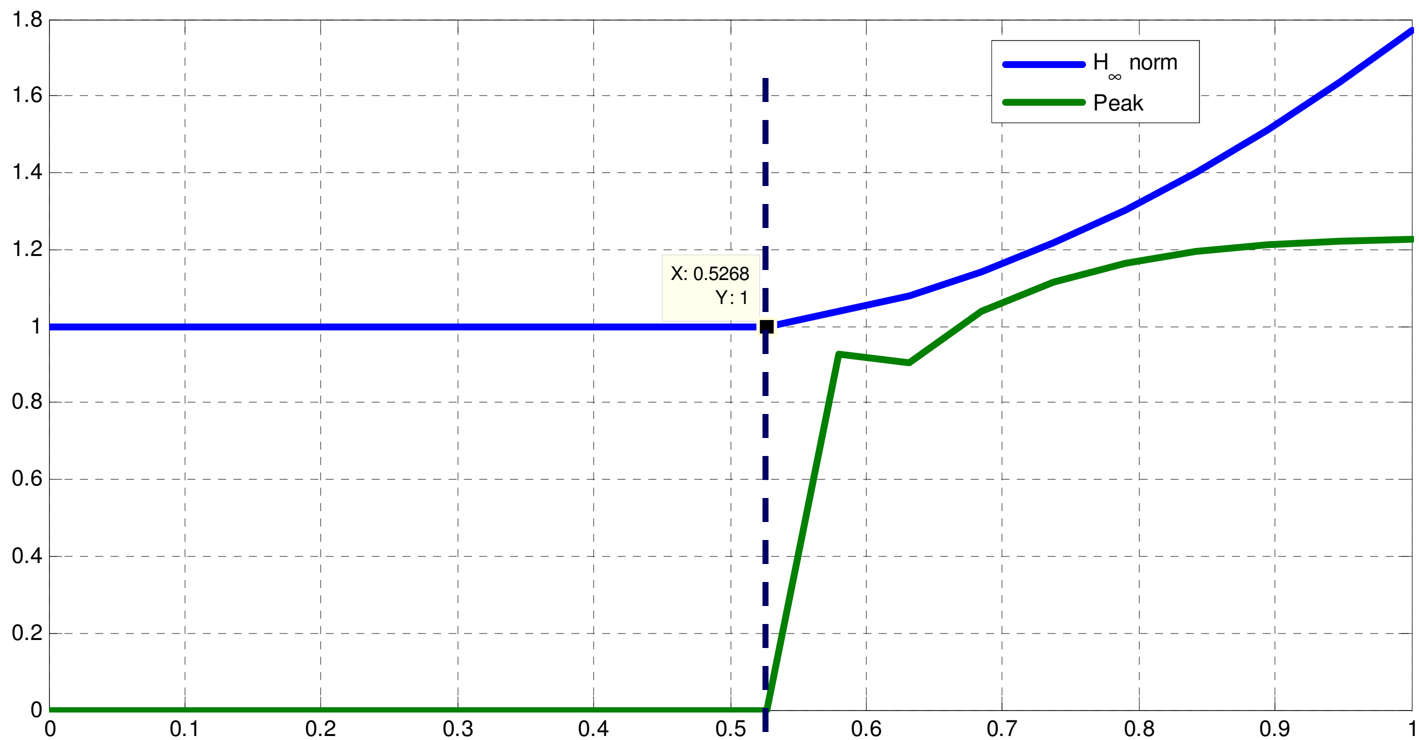
$$\overline{x} = c\mathbf{1}^T$$

Example: Three Robot Cyclic Pursuit

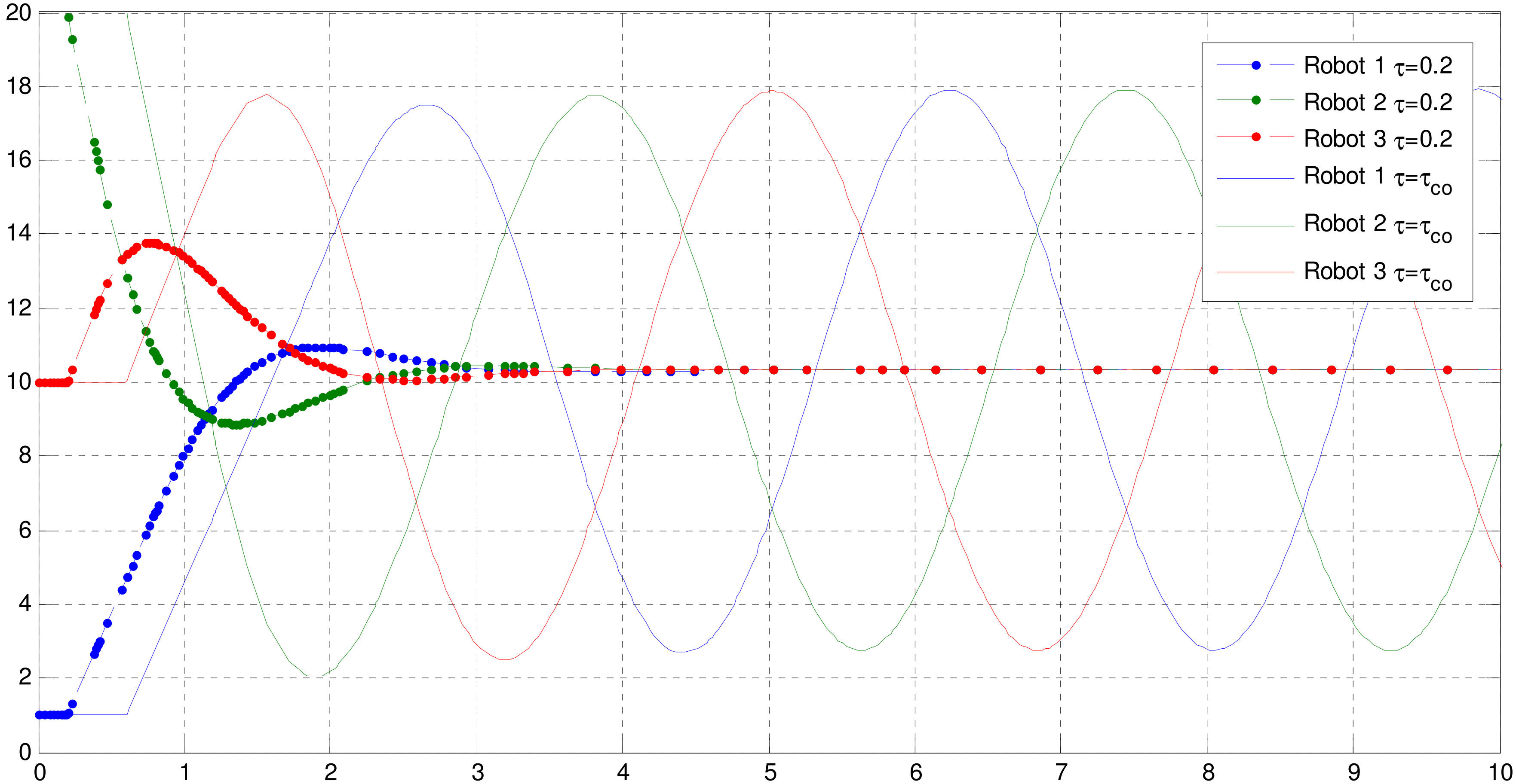
$$\dot{x}_1(t) = [x_2(t-\tau_{21}) - x_1(t-\tau_{21})]$$
$$\dot{x}_2(t) = [x_3(t-\tau_{32}) - x_2(t-\tau_{32})]$$
$$\dot{x}_3(t) = [x_1(t-\tau_{13}) - x_3(t-\tau_{13})]$$

$$\left\|\Delta H(jw)\right\|_\infty < 1 \text{ for } w \in (0, +\infty]$$
$$\text{and} \lim_{w \to 0} \left\|\Delta H(jw)\right\|_\infty = 1 \quad \text{for } \tau < 0.5268$$

All τ_{ij} <0.5258 will satisfy the condition;
Thus, rendezvous occurs for any τ_{12}, τ_{23}, τ_{13}< 0.5268.



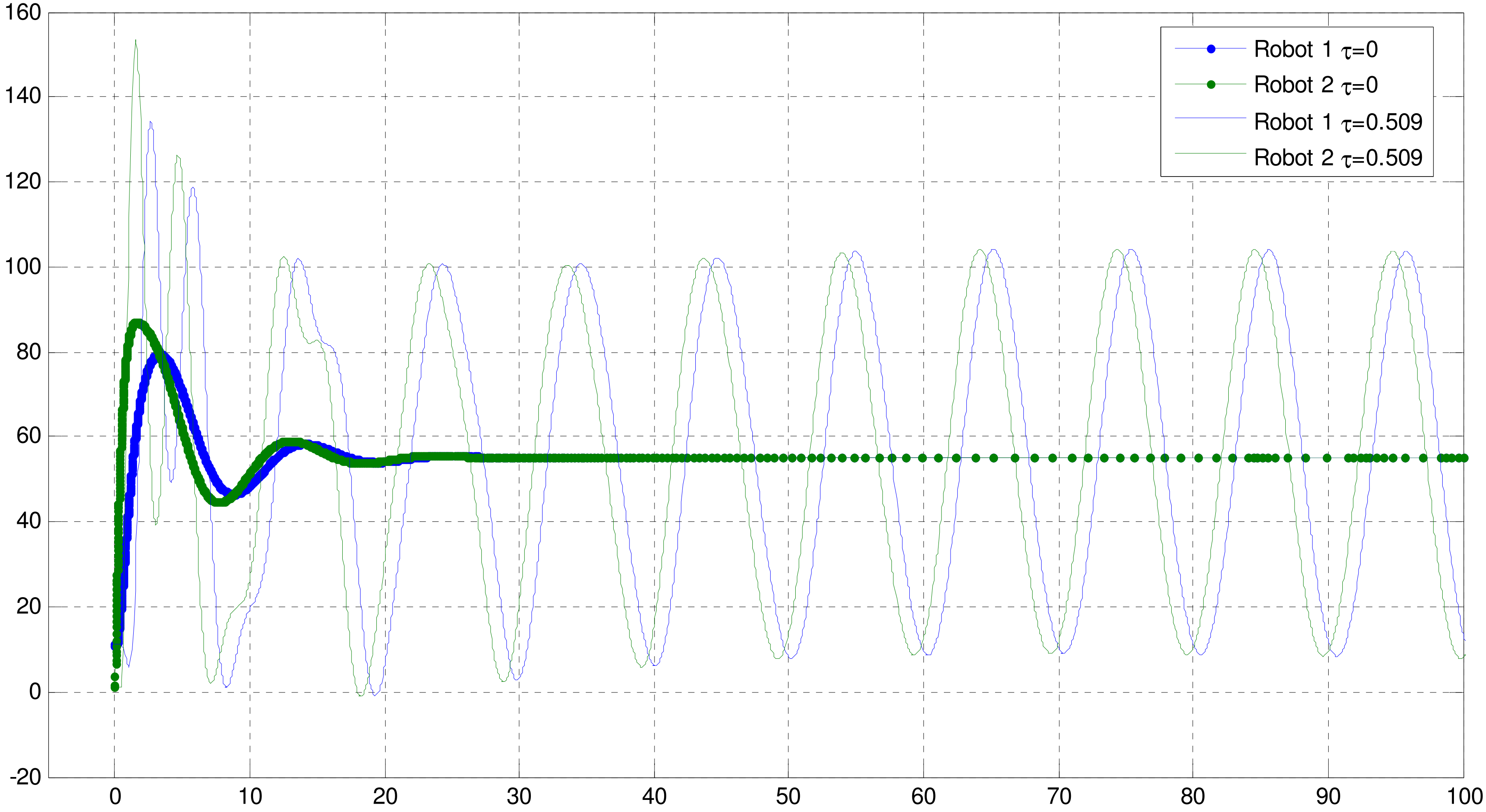
Example: Three Robot Cyclic Pursuit



$$\tau_{co} \cong 0.605$$

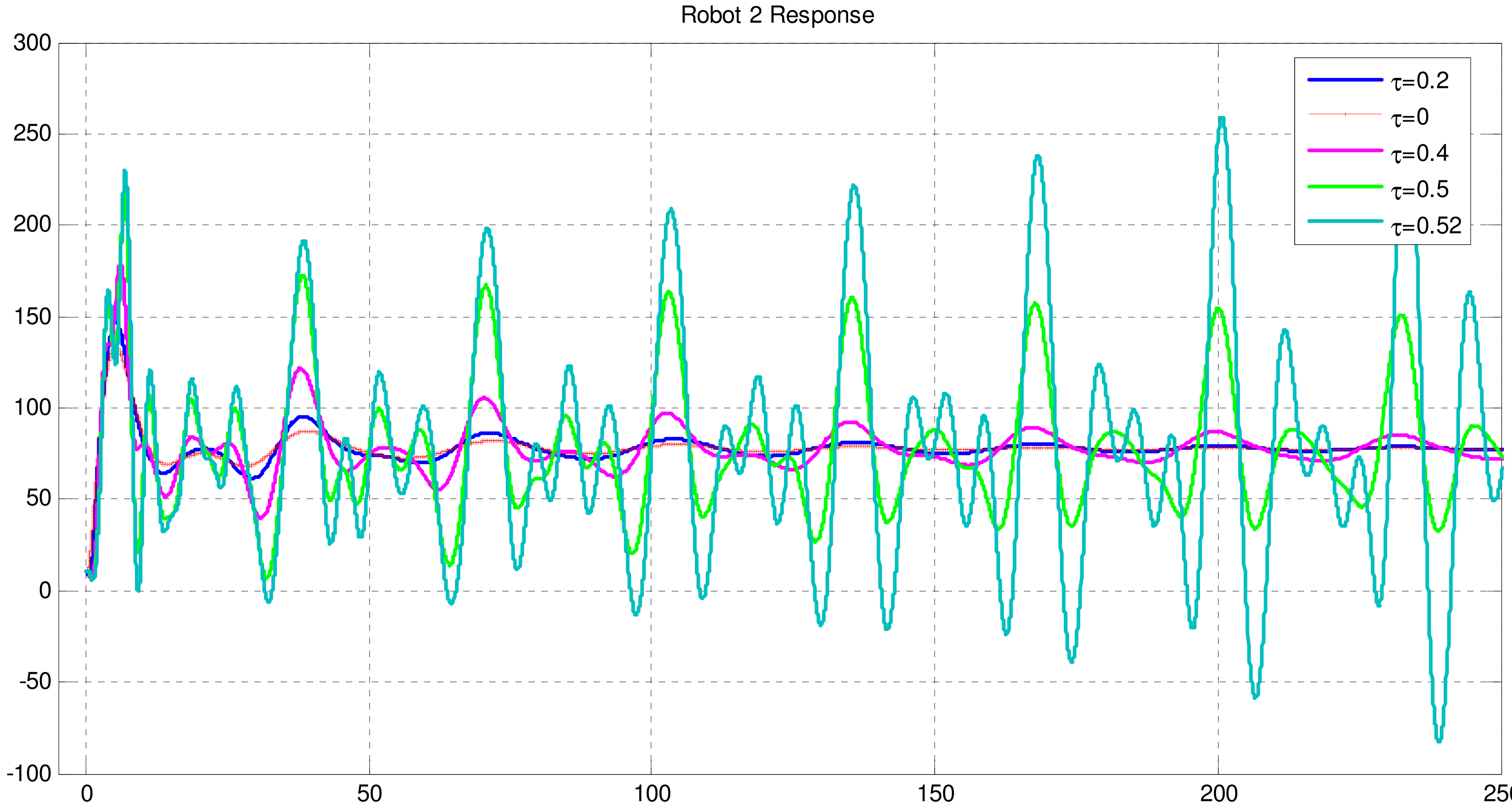
The estimation is conservative.

Example: Multiple Robot Cyclic Pursuit (n=10)



$$\tau_{co} \cong 0.51$$

Example: Multiple Robot Cyclic Pursuit (n=32)



$\tau_{co} \cong 0.5$

Random Updates

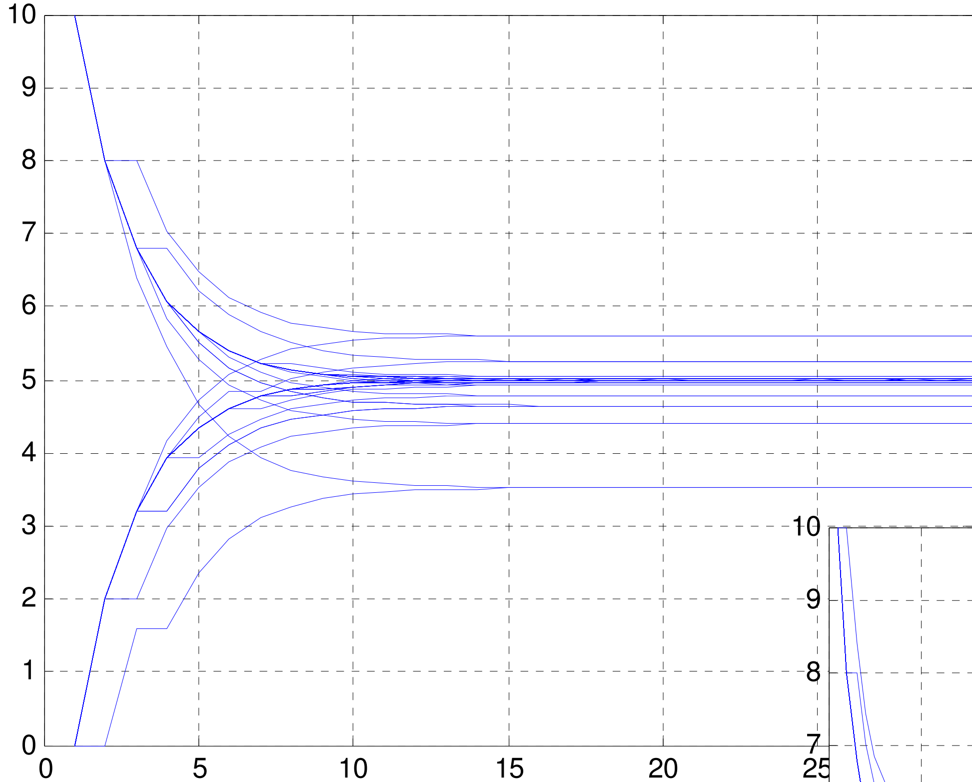
- Probabilistic model of failure or delay in communications, computations and updates

Discrete Model Random Updates

$$\begin{cases} x_i(n+1) = x_i(n) + \varepsilon\left(\sum_{j\in N_i} x_j(n) - x_i(n)\right) & \text{with probability } \pi_i \\ x_i(n+1) = x_i(n) & \text{with probability } 1\text{-}\pi_i \end{cases}$$

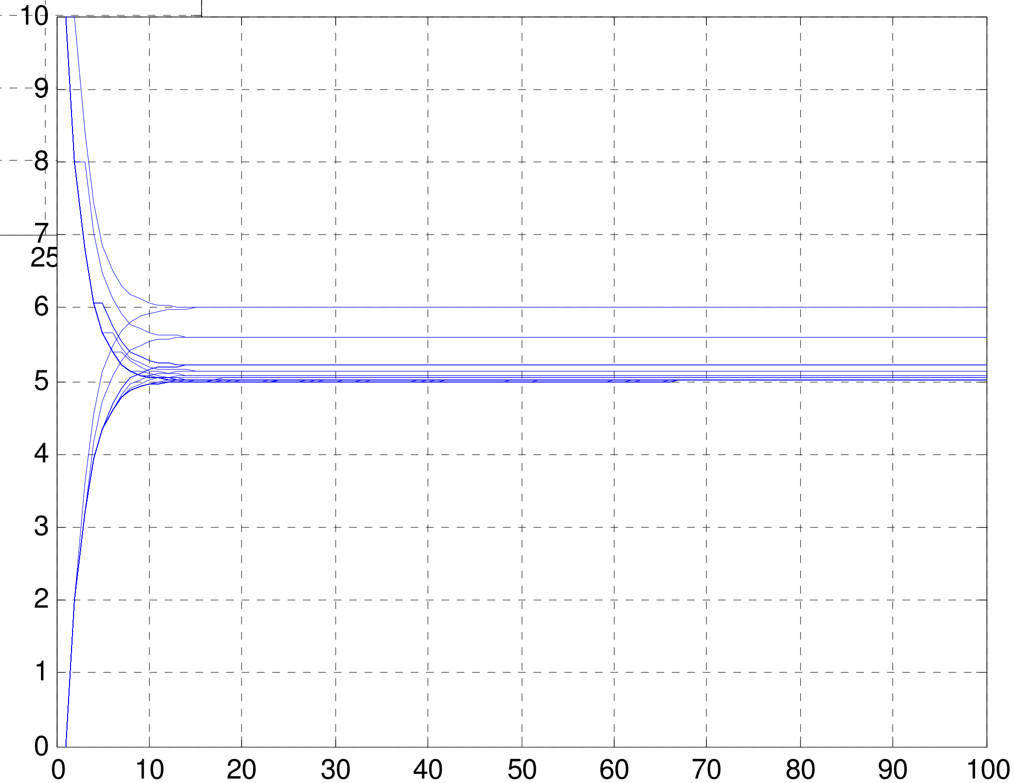
π_i describes how robust the mechanism is.

Two Robot Rendezvous



π=[0.98,0.98]

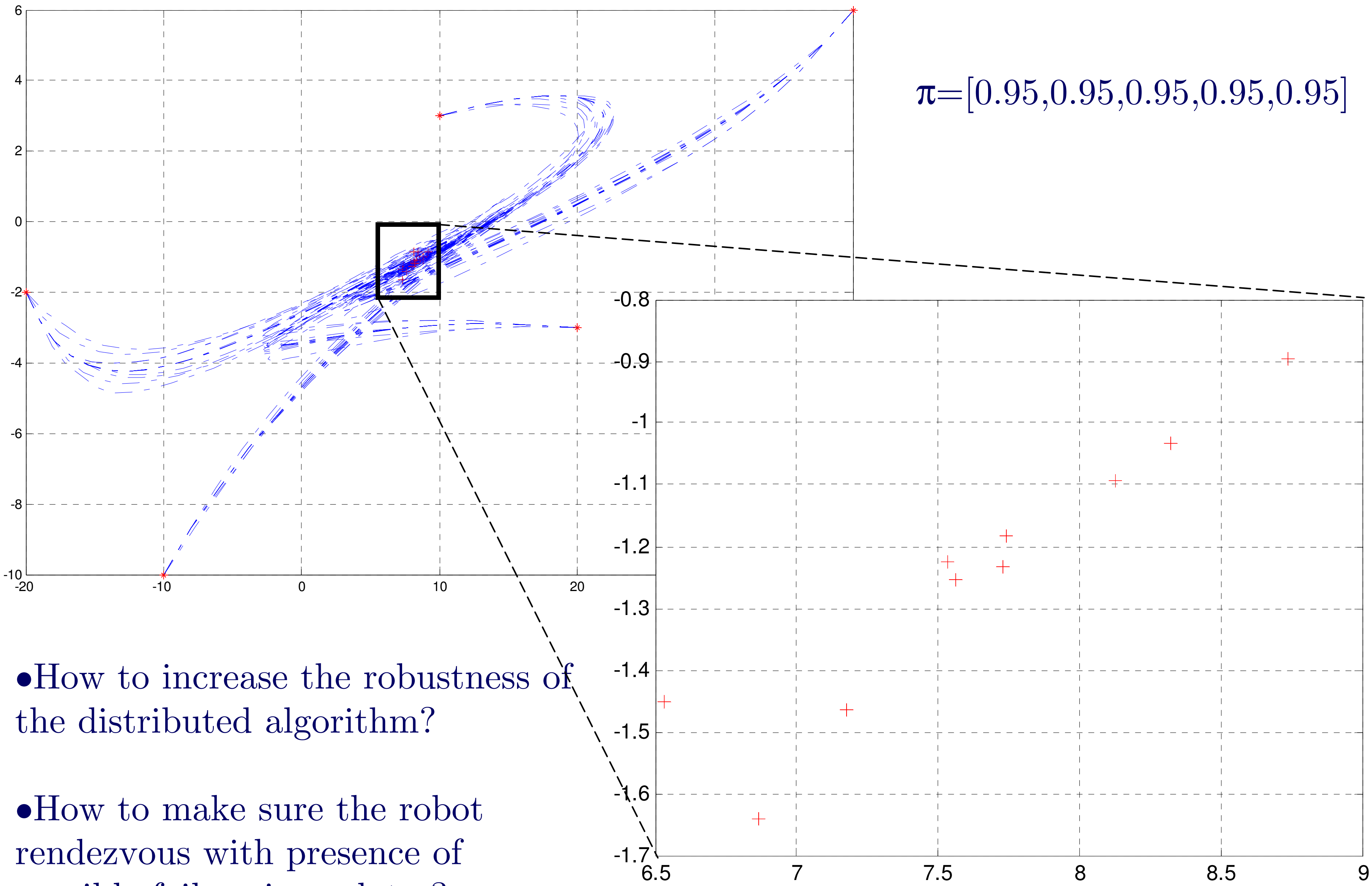
π=[1.0,0.98]



- Uniform random variable for each robot;

- End up in different rendezvous points

Illustration: 5-Robot Cyclic Pursuit

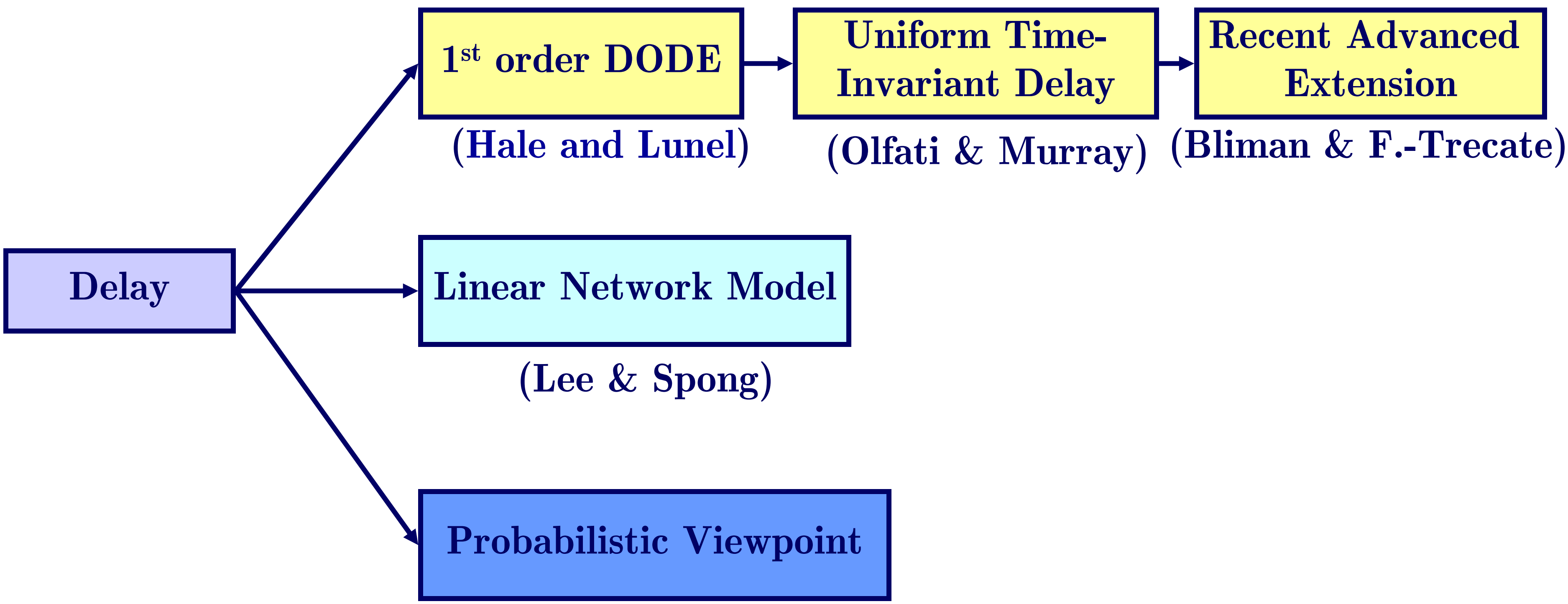


π=[0.95,0.95,0.95,0.95,0.95]

•How to increase the robustness of the distributed algorithm?

•How to make sure the robot rendezvous with presence of possible failure in updates?

Summary


Delay
1st order DODE
(Hale and Lunel)
Uniform Time-Invariant Delay
(Olfati & Murray)
Recent Advanced Extension
(Bliman & F.-Trecate)
Linear Network Model
(Lee & Spong)
Probabilistic Viewpoint

Future Research

- More general and stronger results on delayed communication networks: e.g. directed graphs and time-varying topology
  - Much more involved mathematics : Functional differential equations (Hale & Lunel), Differential-difference equations (Bellman)
- Better heuristic algorithms to compute the bounds for delays
- Controller design with the presence of parametric uncertainties
- Introducing probabilistic approach
- Explain convergences present in
  - the delay in the multi-robot cyclic pursuit
  - Probabilistic algorithm

Questions