Pricing Design of Power Control Game in WDM Optical Networks via State-space Approach

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***Abstract*—The static nature of the noncooperative power control game model in optical networks makes it difficult to study and design an appropriate pricing scheme. In this paper, we derive a first-order best response dynamics from the game-theoretical model and formulate a general multi-input and multi-output (MIMO) state-space model. We use classical linear system theory to explain the controllability of the pricing and the observability of the power states. We use the output regulator theory to design a pricing policy for the network for a given optical signal-to-noise ratio (OSNR) target. At the end of the paper, we will illustrate the pricing design on a typical end-to-end optical link in DWDM optical networks.**

I. Introduction

Recent investigations of the dynamic and performance aspect of optical wavelength-division multiplexed (WDM) communication networks are inspired by the interest in an intelligent network management system that can maintain network stability and optical channel performance in an on-line fashion [1], [2], [3]. Channel performance is closely dependent on the optical signal-to-noise ratio (OSNR), dispersion and nonlinear effects, [4]. In [5], [6], some static approaches have been developed for a single link optimization. However, for a modern reconfigurable optical networks, where different channels can travel via different optical paths, it is desirable to implement a decentralized and iterative algorithm to intelligently control the network.

As an alternative to traditional system-wide optimization, non-cooperative game theory has been used to control and optimize network performances. In a large-scale networks, decisions are made independently with local network information, as it is difficult to gather real-time complete information for decision-making. Game theory's inherent property of distributedness and noncooperativeness makes itself an appropriate framework in the OSNR performance optimization.

Such non-cooperative model is considered in [7], where an OSNR network model has been developed for decentralized optimization. Each user has a payoff function that is composed of utility and the cost calculated from network price. Under such framework, a closed-form solution of the Nash equilibrium (NE) is found and an iterative algorithm is designed to achieve the solution. However, the NE's static nature makes it difficult to further study and design the network pricing policy that affects each channel's utility function. Pricing of networks is one of the crucial control mechanisms. Proposed in [8], pricing is introduced to provide incentives or a control signal to motivate users to adopt a social behavior, i.e., reach some social optimal solution. A pricing policy is needed to enforce a Nash equilibrium to attain a certain target solution.

In [7], [9], a limited investigation has been on some special type of pricing schemes, such as uniform pricing and proportional pricing. In [8], pricing algorithms are developed in a heuristic way without a rigorous demonstration of convergence and its uniform pricing policy doesn't fully motivate the service of differentiation. Therefore, it still remains a challenge to find an appropriate framework to study the pricing issue analytically.

In this regard, we develop a state-space model from each channel's best response dynamics and offer a different perspective towards the non-cooperative game in optical networks. In our model, we view pricing as a controller determined by the network manager and channel power as a network state. Using the classical control theory, we are able to study the pricing controllability of our system and design a pricing scheme to drive the network to a desirable OSNR level. The systematic approach adopted in this paper allows us to investigate other interesting problems in the networks, such as robustness and sensitivity.

The main contribution of this paper is to connect the state-space control theory to the non-cooperative power control in networks and build a novel framework to address the issue of pricing in optical networks. We give a closed-form non-uniform pricing policy to achieve given desired OSNR levels. This paper is organized as follows. In section II, we review the OSNR network model and basic concepts from the non-cooperative game theory in optical networks. In section III, we formulate the state-space model and use it to design pricing mechanism in section IV. In section V, we will give examples to illustrate the pricing design. We will point out future directions of research within this framework and conclude in section VI and VII, respectively.

II. OSNR Game in Optical Networks

A. Review of Optical Network Model

In this section, we will review the optical network model and the basic game-theoretical framework. Consider a network with a set of optical links $\mathcal{L} = \{1, 2, .., L\}$ connecting the optical nodes, where channel add/drop is realized. A set $\mathcal{N} = \{1, 2, ..., N\}$ of channels are transmitted, corresponding to a set of multiplexed wavelengths. Illustrated in Figure 1, a link l has K_l cascaded optically amplified spans. Let N_l be the set of channels transmitted over link l. For a channel $i \in \mathcal{N}$, we denote by $\mathcal{R}_i$ its optical path, or collection of links, from source (Tx) to destination (Rx). Let u_i be the ith channel input optical power (at Tx), and $\mathbf{u} = [u_1, ..., u_N]^T$ the vector of all channels' input powers. Let s_i be the ith channel output power (at Rx), and n_i the optical noise power in the ith channel bandwidth at Rx. The ith channel optical OSNR is defined as $OSNR_i = \frac{s_i}{n_i}$.

In optical networks, the dominant impairment affecting OSNR is the noise accumulation in chains of optical amplifiers and its spectral dependence. In [10], some assumptions are made to simplify the expression for OSNR, typically for uniformly designed optical links. It is assumed that

(i) (A1) ASE noise power does not participate in amplifier gain saturation.
(ii) (A2) All the amplifiers in a link have the same same spectral shape with the same total power target and are operated in automatic power control (APC) mode.

Under A1 and A2, the dispersion and nonlinearity effects are considered to be limited, the ASE noise accumulation will be the dominant impairment in the model. The OSNR for the ith channel is given as

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in \mathcal{N}} \Gamma_{i,j} u_j}, \forall i \in \mathcal{N} \tag{1}$$

where $\mathbf{\Gamma}$ is the full $n \times n$ system matrix which characterizes the coupling between channels. $n_{0,i}$ denotes the ith channel noise power at the transmitter. System matrix $\mathbf{\Gamma}$ encapsulates the basic physics present in optical fiber transmission and implements an abstraction from a network to an input-output system. This approach has been used in [9] for the wireless case to model CDMA uplink communication. Different from the system matrix used in wireless case, the matrix $\mathbf{\Gamma}$ given in (2) is commonly asymmetric and is more complicatedly dependent on parameters such as spontaneous emission noise, wavelength-dependent gain, and the path channels take.

$$\Gamma_{i,j} = \sum_{i \in \mathcal{R}_i} \sum_{k=1}^{K_l} \frac{G^k_{l,j}}{G^k_{l,i}} \left(\prod_{q=1}^{l-1} \frac{\mathbf{T}_{q,j}}{\mathbf{T}_{q,i}} \right) \frac{ASE_{l,k,i}}{P_{0,l}}, \forall j \in \mathcal{N}_l. \tag{2}$$

where $G_{l,k,i}$ is the wavelength dependent gain at kth span in lth link for channel i; $\mathbf{T}_{l,i} = \prod_{q=1}^{K_l} G_{l,k,i} L_{l,k}$ with $L_{l,k}$ being the wavelength independent loss at kth span in lth link; $ASE_{l,k,i}$ is the wavelength dependent spontaneous emission noise accumulated across cascaded amplifiers; $P_{0,l}$ is the output power at each span.

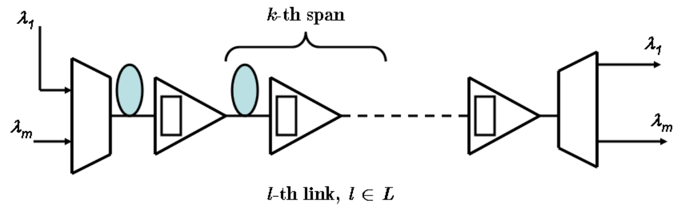

Fig. 1. A Typical Optical Link in DWMW Optical Networks

B. Non-cooperative Game Approach

Let's review the basic game-theoretical model for power control in optical networks. Consider a game defined by a triplet $\langle \mathcal{N}, (A_i), (J_i) \rangle$. $\mathcal{N}$ is the index set of players or channels; A_i is the strategy set $\{u_i \mid u_i \in [u_{i,\min}, u_{i,\max}]\}$; and, J_i is the payoff function. In [7], J_i is defined as

$$J_i(u_i, u_{-i}) = \alpha_i u_i - \beta_i \ln \left(1 + a_i \frac{u_i}{X_{-i}} \right), \tag{3}$$

where X_{-i} is defined as $X_{-i} = \sum_{j \neq i} \Gamma_{i,j} u_j + n_{0,i}$. This specific choice of utility function is non-separable, nonlinear and coupled. However, J_i is strictly convex in u_i and takes a specially designed form such that its first-order derivative takes a linear form with respect to $\mathbf{u}$.

The solution from the game approach is usually characterized by Nash equilibrium (NE) [11], which is defined in the context of optical networks as $\mathbf{u}^*$ such that

$$J_i(\mathbf{u}^*) = \inf_{u_i^* \in [0, u_{max}]} J_i(u_i, \mathbf{u}^*_{-i}), \forall i \tag{4}$$

Provided that $\sum_{j \neq i} \Gamma_{i,j} \leq a_i$,the resulting NE solution is given in a closed form by

$$\tilde{\mathbf{\Gamma}} \mathbf{u}^* = \tilde{\mathbf{b}}, \tag{5}$$

where $\widetilde{\Gamma}_{i,j} = a_i$, for $j = i$; $\widetilde{\Gamma}_{i,j} = \Gamma_{i,j}$, for $j \neq i$ and $\widetilde{b} = \frac{a_i b_i}{\alpha_i} - n_{0,i}$.

III. State-Space Model of OSNR Game

State-space method is a powerful tool to study dynamical systems. It provides a different viewpoint from the input-output frequency domain method and allows a way of systematic study of coupled systems. In this section, we will use the feature of state-space methods based on the derivation of best response dynamics of the OSNR game; and we will design pricing schemes that can regulate the output to track a given reference signal.

The static best response function for payoff functions in (3) can be derived by taking its first derivative with respect to u_i as follows.

$$u_i = BR(\mathbf{u}_{-i}) = \arg\min_{u_i} J_i(u_i, \mathbf{u}_{-i}) = \frac{\beta_i}{\alpha_i} - \frac{X_{-i}}{a_i}. \tag{6}$$

From (6), we can derive the first-order best response dynamics as in (7).

$$\dot{x}_i = \frac{a_i \beta_i}{\alpha_i} - X_{-i} - a_i x_i. \tag{7}$$

or equivalently,

$$\dot{x}_i = -a_i x_i - \sum_{j\neq i} \Gamma_{i,j} x_j + \frac{a_i\beta_i}{\alpha_i} - n_{0,i}, \forall i \qquad (8)$$

where $x_i = u_i$ is the state variable of the channel power i.

To keep the state-space in a linear form, we can define optical signal-noise difference (OSND), based on (1) as

$$OSND_i = u_i - m_i = (1-\Gamma_{i,j})u_i - \sum_{i\neq j} \Gamma_{i,j} u_j, \forall i \qquad (9)$$

where m_i is defined from (1) as $m_i = \Gamma_{i,j}u_i + \sum_{i\neq j} \Gamma_{i,j} u_j$. OSND measures transmission quality just as OSNR does. Since OSND measures the difference of the optical power, we will use unit of dBm for it. It is obvious that the higher the value of OSND, the better the transmission quality will be.

For the simplicity of notation, let $\gamma_i^d = OSND_i$ and $\gamma_i^r = OSNR_i$. Since $\gamma_i^d = u_i - m_i$ and $\gamma_i^r = u_i/m_i$, we can relate γ_i^d and γ_i^o by equation (10).

$$\gamma_i^r = \frac{u_i}{u_i - \gamma_i^d}. \qquad (10)$$

Definition 3.1: A γ^r-feasible power vector $\mathbf{u} \in \mathcal{R}^N$ is such that $\gamma_i^r = u_i - n_i = OSNR_i(\mathbf{u}) = \frac{u_i}{\Gamma_{i,i}u_i + X_{-i}}, \forall i$.
Since not all given γ^r can be realized by a power vector $\mathbf{u} \in \mathcal{R}^N$, Definition (3.1) gives a condition on the feasibility of γ^r that can be chosen. With a given γ_i^d, we can calculate γ_i^r from (10) by a γ^r-feasible choice of signal $\mathbf{u}$.

With (8) and (9), the state-space form of the best response dynamics is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} - \mathbf{n} \qquad (11)$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} - \mathbf{n} \qquad (12)$$

where $\mathbf{x} \in \mathcal{R}^N$ is the state-vector physically modelling the evolution of the power vector $\mathbf{u}$ in optical networks; $\mathbf{v} \in \mathcal{R}^N$ is a vector of control variables relating to the pricing parameters component-wise by $v_i = 1/\alpha_i, \forall i$; $\mathbf{y}$ is the output vector that observes the OSND. The vector $\mathbf{n}$ and matrices $\mathbf{A}$,$\mathbf{B}$ and $\mathbf{C}$ are given as follows.

$$\mathbf{A} \in \mathbf{R}^{N\times N} = \begin{pmatrix} -a_1 & -\Gamma_{12} & \cdots & -\Gamma_{1N} \\ -\Gamma_{21} & -a_2 & \cdots & -\Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_{N1} & \cdots & \cdots & -a_N \end{pmatrix},$$

$$\mathbf{B} \in \mathbf{R}^{N\times N} = \begin{pmatrix} a_1\beta_1 & 0 & \cdots & 0 \\ 0 & a_2\beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_N\beta_N \end{pmatrix},$$

$$\mathbf{C} \in \mathbf{R}^{N\times N} = \begin{pmatrix} 1-\Gamma_{11} & -\Gamma_{12} & \cdots & -\Gamma_{1N} \\ -\Gamma_{21} & 1-\Gamma_{22} & \cdots & -\Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_{N1} & \cdots & \cdots & 1-\Gamma_{NN} \end{pmatrix},$$

$$\mathbf{n} \in \mathbf{R}^N = \begin{pmatrix} n_{0,1} \\ n_{0,2} \\ \vdots \\ n_{0,N} \end{pmatrix}$$

State-space model (11) is a multi-input and multi-output (MIMO) system; however, the model is single-input and multi-output (SIMO) system, if $\mathbf{B}\mathbf{v}$ in (11) is replaced by $\overline{\mathbf{B}}v$, where $\overline{\mathbf{B}} \in \mathcal{R}^N$ is a vector given by $[a_1\beta_1, \cdots, a_N\beta_N]^T$ and $v_i \in \mathcal{R}, v_i = 1/\alpha_i$ is a scalar pricing parameter. SIMO represents a uniformly priced Nash game, in which the network assigns a single network price to every user.

Due to the nonlinearity of OSNR expression, a direct OSNR output formulation will result in solving for a difficult nonlinear set of equations. Without losing generaltiy, we next study a design of pricing to achieve desirable OSNR in the form of OSND as OSNR can be determined from OSND by (10) if there exist a γ^r-feasible power vector $\mathbf{x}$. In this way, we are able to take the advantage of linearity of OSND and derive a closed form for the pricing scheme.

The state-space model (11) naturally allows us to examine the pricing design problems based on classical control theory by viewing it a controller. In the following development, we will ignore the term $\mathbf{n}$, since it is important to first develop some insightful results and then complicate the model by viewing the noise as a disturbance to the system. Furthermore, the term $\mathbf{n}$ in a typical network is usually on the magnitude of 1.0×10^{-4}mW, that is, less than $1\% - 5\%$ of the common signal power.

Lemma 3.1: The steady-state of the first order dynamics described in (8) is given by the Nash Equilibrium if the system is stable. If (8) has only one equilibrium, it will corresponds to a unique Nash equilibrium in the OSNR game.

Proof: The Nash equilibrium $\mathbf{u}^*$ to the OSNR game is determined by (5). The steady-state of (8) is at its equilibrium point(s) $\mathbf{u}^0$ determined by letting

$$\frac{a_i\beta_i}{\alpha_i} - X_{-i} - a_i u_i = 0.\forall i \qquad (13)$$

It it obvious that for a $\mathbf{u}^0$ that satisfies (13) will also satisfy (5). Similarly, if $\mathbf{u}^*$ satisfies (5), it will also satisfy (13). Therefore, if (8) results in a unique equilibrium, it will corresponds to the unique Nash equilibrium to OSNR game. ■

IV. Pricing Design

A common problem in OSNR Nash game is to design a pricing mechanism so that players can reach their OSNR targets at their steady-state, i.e., the best response dynamics $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}$ will give a solution $\mathbf{x}(t) = \mathbf{x}'$ at a sufficiently large t and for some given $\mathbf{x}'$. In this section, we investigate the problem using OSND targets, as its linearity allows us to give some fundamental results in controllability and observability.

Theorem 4.1: The entire OSNR game is *pricing controllable* if we can reach every given state $\mathbf{x}'$ at a given time t, i.e. the solution to (11) will yield $\mathbf{x}(t) = \mathbf{x}'$ for some initial condition $\mathbf{x}(0)$. Necessary and sufficient conditions of *pricing controllability* are given by the following equivalent statements.

(i) The controllability Grammian $\mathbf{W}_c(t)$ in (14) is positive definite for any given $t > 0$.

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{A^T\tau} d\tau \tag{14}$$

(ii) The controllability matrix C has full rank, i.e. rank$(C) = N$.

$$C = \left[\mathbf{B}, \mathbf{AB}, \cdots, \mathbf{A}^{n-1}\mathbf{B}\right]. \tag{15}$$

(iii) The matrix $[\mathbf{A} - \mathbf{I}\lambda, \mathbf{B}]$ has full row rank for all λ's, the eigenvalues, of $\mathbf{A}$.

Theorem 4.1 is a direct application of Theorem 3.1 from [12] (pp. 47-49). It provides a very strong condition because it ensures that every state $\mathbf{x}'$ to be reachable at any given time t. If we know $(\mathbf{A}, \mathbf{B})$ gives pricing controllability, we can thus design a pricing policy to drive the OSNR game to any γ^r-feasible set of power vector $\mathbf{x}$ that can attain given OSNR levels.

Theorem 4.2: The state-space model described in (11) is *power observable* if, for any given $t_1 > 0$, the initial state $\mathbf{x}(0)$ can be determined from the time history of the input $\mathbf{u}(t)$ and the output $\mathbf{y}(t)$ in the interval $[0, t_1]$. The necessary and sufficient conditions for observability is given in the following equivalent statements.

(i) The observability Grammian $\mathbf{W}_o(t)$ in (16) is positive definite for any given $t > 0$.

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{C} e^{A\tau} d\tau \tag{16}$$

(ii) The controllability matrix O in (17) has full rank, i.e. rank$(O) = N$.

$$O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{N-1} \end{bmatrix}. \tag{17}$$

(iii) The matrix $\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{C} \end{bmatrix}$ has full row rank for all λ's, the eigenvalues, of $\mathbf{A}$.

Theorem (4.2) is adapted from [12](pp. 50-51). Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. If an OSNR game is observable, then, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs. In other words, this means all the current values of its states can be determined through output sensors. In our state-space model in terms of OSND, we wish to know the systems state, the power vectors, from our OSND output and, therefore, the corresponding OSNR is calculable via (10).

A. Constant Reference Signal Tracking

In this subsection, we study a regulator problem in which the output is desired to track given feasible OSNR levels. We will use classical regulator theory [13] to develop insights into this pricing problem. Let's construct a dynamical system whose output is the given OSND $\overline{\gamma^d}$. Such a reference system is given by (18).

$$\dot{\mathbf{w}} = \mathbf{S}\mathbf{w} \tag{18}$$
$$\mathbf{y} = \mathbf{y}_d \tag{19}$$

Since the given performance target OSND is a constant signal, we let $\mathbf{S} \in \mathcal{R}^{N\times N} = \mathbf{0}$, $\mathbf{y}_d = \overline{\gamma^d}$ and $\mathbf{w} \in \mathcal{R}^N, \mathbf{w}(0) = \mathbf{1}$. It is obvious that (18) will give $\mathbf{w}(t) = \mathbf{1}, \forall t$. We need to find a $\mathbf{v}$ such that $\mathbf{e}(t) = \mathbf{y} - \mathbf{y}_d$ will converge to $\mathbf{0}$.

Theorem 4.3: Suppose there exists $\mathbf{v} = \overline{\mathbf{F}}_2$, where $\overline{\mathbf{F}}_2 \in \mathcal{R}^N$ and a map $\mathbf{\Pi} : \mathcal{R}^N \to \mathcal{R}^N$ such that

$$0 = \mathbf{A\Pi} + \mathbf{B}\overline{\mathbf{F}}_2 \tag{20}$$
$$\mathbf{C\Pi} = \overline{\gamma^d} \tag{21}$$

If $\mathbf{x}(0) = \mathbf{\Pi}$, then $\mathbf{y}(t) \to \overline{\gamma^d}$, as $t \to 0$.

Proof: Suppose there exists $\mathbf{u} = \overline{\mathbf{F}}_2\mathbf{w}$ that satisfies (20). Let's define $\mathbf{z} = \mathbf{x} - \mathbf{\Pi w}$.

$$\begin{aligned} \dot{\mathbf{z}} &= \dot{\mathbf{x}} - \mathbf{\Pi}\dot{\mathbf{w}} && (22)\\ &= \mathbf{Ax} + \mathbf{B}\overline{\mathbf{F}}_2\mathbf{w} - \mathbf{\Pi}\dot{\mathbf{w}} && (23)\\ &= \mathbf{Ax} - \mathbf{A\Pi w} + \mathbf{A\Pi w} + \mathbf{B}\overline{\mathbf{F}}_2\mathbf{w} - \mathbf{\Pi}\dot{\mathbf{w}} && (24)\\ &= \mathbf{Az} && (25) \end{aligned}$$

The last step comes from the regulator equations (20). From the assumption $\mathbf{z}(0) = \mathbf{x}(0) - \mathbf{\Pi w}(0) = 0$, we can conclude that $\mathbf{z}(t) = 0, \forall t > 0$.

Let $\mathbf{e}(t) = \mathbf{y}(t) - \overline{\gamma^d} = \mathbf{Cx}(t) - \overline{\gamma^d}$. Using (20) and the fact that $\mathbf{w}(t) = \mathbf{w}(0) = \mathbf{1}$, we obtain

$$\begin{aligned} \mathbf{e}(t) &= \mathbf{Cz}(t) + \mathbf{C\Pi w}(t) - \overline{\gamma^d} && (26)\\ &= \mathbf{Cz}(t) + \mathbf{C\Pi w}(t) - \mathbf{C\Pi} && (27)\\ &= \mathbf{Cz}(t) && (28) \end{aligned}$$

Therefore, $\mathbf{e}(t) \to 0$ as $\mathbf{z}(t) \to 0$. As a result, $\mathbf{y}(t) \to \overline{\gamma^d}$. ∎

Equations (20) are called classic regulator equations or FBI equations named after Francis-Byrnes-Isidori [13]. This set of equations can be structured into a matrix form of linear system of equations (29) and solved under some regularity conditions.

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{\Pi} \\ \overline{\mathbf{F}}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\gamma^d} \end{pmatrix} \tag{29}$$

Equation (29) will yield

$$\overline{\mathbf{F}}_2 = -\mathbf{B}^{-1}\mathbf{A}\mathbf{C}^{-1}\overline{\gamma^d} \tag{30}$$
$$\mathbf{\Pi} = \mathbf{C}^{-1}\overline{\gamma^d} \tag{31}$$

under the assumption that

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \text{ is non-singular.} \tag{32}$$

and $\mathbf{B}, \mathbf{C} \in \mathcal{R}^{N\times N}$. Condition (32) is actually equivalent to $\mathbf{C}$ and $\mathbf{B}$ being non-singular. This result is summarized in the following proposition.

Proposition 4.4: Consider the MIMO system described in (11). Suppose $\mathbf{B} \in \mathcal{R}^{N\times N}$. There exists a unique solution (30) to (20) if and only if $\mathbf{B}$ and $\mathbf{C}$ are nonsingular.

Proof: Firstly, we show that ,under the necessary and sufficient condition of (32), (29) gives a unique solution $(\mathbf{\Pi}, \overline{\mathbf{F}}_2)$ in (30).

Under the assumption that (32), we can solve (29) by inverting the incident matrix $\mathbf{\Phi}$ and the solution given in (33).

$$\begin{pmatrix} \overline{\mathbf{F}}_2 \\ \mathbf{\Pi} \end{pmatrix} = \begin{pmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ -\mathbf{B}^{-1}\mathbf{A}\mathbf{C}^{-1} & \mathbf{B}^{-1} \end{pmatrix} \begin{pmatrix} \overline{\gamma^d} \\ \mathbf{0} \end{pmatrix} \tag{33}$$

Equation (33) will eventually lead to the solution in (30).

We next show that condition (32) holds if and only if $\mathbf{B}$ and $\mathbf{C}$ are nonsingular. $\det(\mathbf{\Phi}) = \det(\mathbf{B})\det(\mathbf{C})$. Therefore $\det(\mathbf{\Phi}) \neq 0$ if and only if both $\det(\mathbf{B})$ and $\det(\mathbf{C})$ are non-zeros, i.e., $\mathbf{B}, \mathbf{C}$ are non-singular. Therefore, it completes the proof. ■

Using the regulator equation, we can find that a control $\mathbf{v}$ that can maintain an output $\mathbf{y}$ at given $\overline{\gamma^d}$. It is analytically found to be $\mathbf{v} = \overline{\mathbf{F}}_2\mathbf{w} = -\mathbf{B}^{-1}\mathbf{A}\mathbf{C}^{-1}\overline{\gamma^d}$ for an initial condition $\mathbf{x} = \mathbf{\Pi}$; and thus the pricing parameter in OSNR game is found as $\alpha_i = 1/v_i$.

B. Asymptotic Tracking

With an appropriate initial condition, the above result shows a perfect tracking of the constant reference signal. However, in most cases, we wish to find an asymptotic tracking for any given initial conditions. We will consider modifying the control $\mathbf{v} = \overline{\mathbf{F}}_2\mathbf{w} + \mathbf{F}_1(\mathbf{x} - \mathbf{\Pi}\mathbf{w}) = \mathbf{F}_1\mathbf{x} + \mathbf{F}_2\mathbf{w}$.

Theorem 4.5: Suppose $\overline{\mathbf{A}} = \mathbf{A} + \mathbf{B}\mathbf{F}_1$ is Hurwitz. A regulator $\mathbf{u} = \overline{\mathbf{F}}_2\mathbf{w} + \mathbf{F}_1(\mathbf{x} - \mathbf{\Pi}\mathbf{w})$ exists if and only if there exists maps $\mathbf{\Pi}$ and $\overline{\mathbf{F}}_2$ satisfying equations (20).

Proof: Firstly, let's prove the sufficiency. Suppose there exists $(\Pi, \overline{\mathbf{F}}_2)$ satisfying (20). Define $\mathbf{z} = \mathbf{x} - \mathbf{\Pi}\mathbf{w}$ and take its derivative with respect to time as follows.

$$\begin{aligned} \dot{\mathbf{z}} &= \dot{\mathbf{x}} - \mathbf{\Pi}\dot{\mathbf{w}} \\ &= (\mathbf{A} + \mathbf{B}\mathbf{F}_1)\mathbf{x} + (\mathbf{B}\overline{\mathbf{F}}_2 - \mathbf{B}\mathbf{F}_1\mathbf{\Pi})\mathbf{w} - \mathbf{\Pi}\dot{\mathbf{w}} \\ &= \overline{\mathbf{A}}\mathbf{x} - \overline{\mathbf{A}}\mathbf{\Pi}\mathbf{w} + \overline{\mathbf{A}}\mathbf{\Pi}\mathbf{w} + (\mathbf{B}\overline{\mathbf{F}}_2 - \mathbf{B}\mathbf{F}_1\mathbf{\Pi})\mathbf{w} - \mathbf{\Pi}\dot{\mathbf{w}} \\ &= \overline{\mathbf{A}}\mathbf{z} \end{aligned}$$

Since $\overline{\mathbf{A}}$ is Hurwitz, $\mathbf{z}(t) \to 0$ for any given initial condition $\mathbf{z}(0)$. $\mathbf{y} = \mathbf{C}\mathbf{x} - \mathbf{C}\mathbf{\Pi}\mathbf{w} + \mathbf{C}\mathbf{\Pi}\mathbf{w} = \mathbf{C}\mathbf{z} + \overline{\gamma^d}$. As $t \to \infty$, $\mathbf{y}(t) - \overline{\gamma^d} \to 0$; and therefore, $\mathbf{y}(t)$ asymptotically tracks $\overline{\gamma^d}$.

Secondly, we will show the necessity. Since eigenvalue of $\mathbf{S} = \mathbf{0}$ is in the closed right-half complex plane, by Sylvester's Theorem [14], there exists a unique solution $\mathbf{\Pi}$ satisfying

$$\mathbf{\Pi} - \overline{\mathbf{A}}\mathbf{\Pi} = \mathbf{B}\mathbf{F}_2.$$

Letting $\overline{\mathbf{F}}_2 = \mathbf{F}_2 + \mathbf{F}_1\mathbf{\Pi}$, we obtain (20). As shown above, $\mathbf{z} = \overline{\mathbf{A}}\mathbf{z}$. Hence $\mathbf{z}(t) \to 0$ as $t \to \infty$. Also as above, $\mathbf{e}(t) = \mathbf{C}\mathbf{z} + \mathbf{C}\mathbf{\Pi}\mathbf{w} - \overline{\gamma^d}\mathbf{w}$. By assumption $\mathbf{e} \to 0$ and since $\mathbf{z} \to 0$, it must be that $\mathbf{C}\mathbf{\Pi}\mathbf{w} - \overline{\gamma^d}\mathbf{w} \to 0$ for all initial conditions $\mathbf{w}(0)$. Since eigenvalue of $\mathbf{S} = 0$ is on the closed right half plane, $\mathbf{w}(t)$ doesn't converge to zero. Hence, $\mathbf{C}\mathbf{\Pi} = \overline{\gamma^d}$. ■

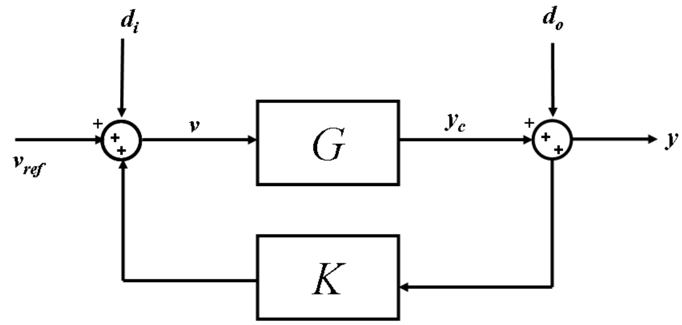

Fig. 2. System Diagram of Best Response Dynamics of OSNR Nash Game

The above gives a necessary and sufficient condition to design pricing values for an asymptotic tracking of given transmission performance OSND targets in OSNR Nash games. If we choose a_i such that $-\mathbf{A}$ is diagonally dominant, i.e., $a_i > \sum_i \Gamma_{i,j}, \forall i$. $\overline{\mathbf{A}} = \mathbf{A}$ itself is already Hurwitz. Thus, we can let $\mathbf{F}_1 = 0$ and the pricing design can be simply found by $\mathbf{v} = \overline{\mathbf{F}}_2\mathbf{w}$. Different from an open-loop style pricing described in [9], [7], the general controller described in Theorem (4.5) provides a feedback mechanism on the OSNR game dynamics and gives another degree of freedom in $\mathbf{F}_1$ to adjust the dynamical response in addition to the choice of a_i. It is desirable because the network is able to adjust the dynamics via $\mathbf{F}_1$ once a_i are set and submitted to the network.

We can summarize the state-space with the controller in Theorem 4.5 in Figure 2, where $\mathbf{v}, \mathbf{y}, \mathbf{d}_i, \mathbf{d}_0 \in \mathcal{R}^N$, $\mathbf{v}_{ref} \in \mathcal{R}^N = \overline{\mathbf{F}}_2 - \mathbf{F}_1\mathbf{\Pi}$, $\mathbf{K} \in \mathcal{R}^{N\times N} = \mathbf{F}_1\mathbf{C}^{-1}$, It is obvious that when $a_i > \sum_i \Gamma_{i,j}, \forall i$ and $\mathbf{F}_1 = 0$, it will be reduced to an open loop system

V. Numerical Examples

We consider an end-to-end optical link described in Figure 1. A common $\mathbf{\Gamma}$ has its entries varying from 1.2×10^{-4} to 1.3×10^{-4}. First, we will consider a two-channel case, where we set $a_1 = 3.419, a_2 = 3.032$; $\beta_1 = 3, \beta_2 = 4$, and

$$\mathbf{\Gamma} = \begin{pmatrix} -1.210 \times 10^{-4} & -1.271 \times 10^{-4} \\ -1.242 \times 10^{-4} & -1.277 \times 10^{-4} \end{pmatrix},$$

In addition, we wish to the power level to be on the magnitude of 0.05mW and the OSNR to be around 35dB. Therefore, we choose $\gamma_1^d = 0.0337$mW and $\gamma_2^d = 0.06$mW. Fig. 3 shows the converging OSNR for channel 1 to 37.06dB and channel 2 34.56dB. By Fig. 4, we demonstrate the similar concept for 20 channels.

VI. Direction of Future Research

This paper outlines a state-space approach to design pricing for the optical networks. It builds a framework for a generalized study of pricing policy in the networks. In this paper,

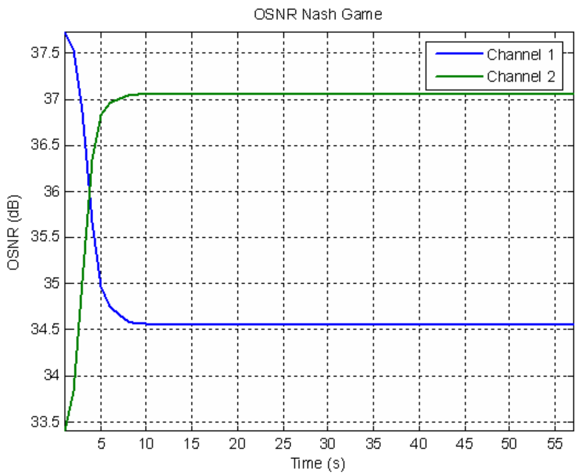

Fig. 3. Pricing Controlled Optical Link: 2 channels

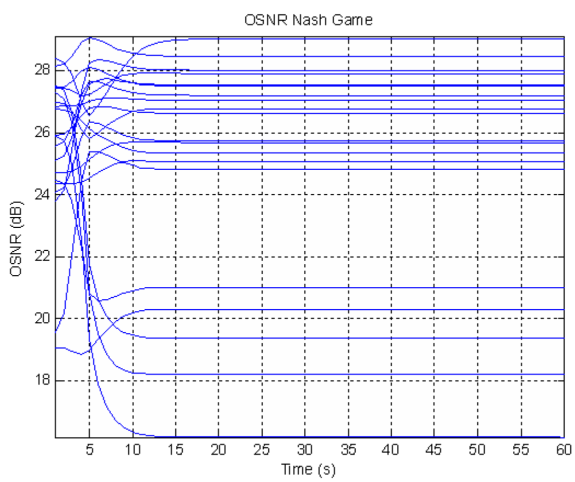

Fig. 4. Pricing Controlled Optical Link: 20 channels

we only considered a non-uniform design which leads to a closed form solution described in Theorems (4.3) and (4.5). Using the same framework, we can easily extend those results to a dynamical pricing policy that depends on the states. In addition, with more advanced tools from robust control theory, we are also able to analyze the robustness and sensitivity of the OSNR game.

Throughout the paper, we used OSND to replace OSNR so that we have a linear system in (11), which is relatively easy to tackle and to give constructive analytical results. Constructing an output directly from OSNR can give rise to a nonlinear state-space model. Its analysis will be more difficult as the paper describes that it involves solving a non-linear set of equations and nonlinear description of the observability criteria. However, as an extension, it is still possible to use nonlinear control theory to give some results for nonlinear state-space models.

Another possible extension of the paper is on a constrained state-space which corresponds to a constrained OSNR game. A typical constraint in optical networks is capacity constraints: the sum of optical power within the network needs to be within an operational range. In the language of state-space model, it means that our states needs to be in a simplex. The pricing design may need to consider a penalty on the states. Furthermore, the result for reachability or controllability will be much more involved. In addition, it is also desirable to design an efficient pricing scheme such that the constrained states will results in a Pareto efficiency.

The state-space model gives a rich insight into the OSNR games. It opens a few doors of possibility of research in this area. It will be interesting to investigate further proposals in the future.

VII. Conclusion

In this note, we developed a state-space framework for the pricing design in the OSNR game. The classical control theory enables us to view pricing from a control design point of view and helps us develop insights into the pricing controllability of the network. As a result, we use regulator equations to find an analytical pricing policy so that the network is able to attain a given set of OSNR targets. We hope this study will initiate further investigations and extensions of this model in this area, in particular, the nonlinear case and the game with constraints.

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