# Theory of Linear Games with Constraints and Its Application to Power Control of Optical Networks

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**Abstract**

In this paper, we introduce a class of linear non-cooperative games with linearly coupled constraints. It bears striking connections with classical linear systems theory and finds itself pervasively used in network engineering applications. We characterize its Nash equilibrium (NE) and provide iterative algorithms in a general form to achieve the equilibrium. In addition, we analyze the issue of efficiency in constrained Nash games. In the second part of the paper, we will illustrate this type of games by an application from OSNR-based power control in optical networks, where we can view the slack variables as fictitious players. This powerful interpretation allows us to bridge over the theory and the issue of implementation in engineering.

**Index Terms**

Power control, Game theory, Optical networks, OSNR game, Constrained optimization, Linear system.

## I. Introduction

Recent academic interest in game theory has spawned the application of optimization theory to Nash games, where each player seeks to optimize her own utility in a non-cooperative manner. Solving noncooperative games is largely dependent on optimization tools, such as nonlinear programming for continuous strategy sets. However, Lagrangian methods are not always directly applicable. Extensions are usually made to accommodate such needs. One of the major extensions is on Nash games with constraints. A common practice in solving a constrained game involves the process of embedding the constraints as a penalty into the utility function. This approach offers a way to circumvent the constrained problem by turning it into an unconstrained game [1]–[3]. However, the problem becomes untractable when utilities are nonlinear and strongly coupled. Furthermore, it is extremely challenging to find analytical solutions as the complexity of

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the functions grows. On the other hand, this approach is strongly problem-dependent. Different problems may need a different construction of modified utility functions. As a result, this approach doesn't provide a unifying theory to solve a class of problems.

The recent work of Pavel in [4] develops an extension of duality theory to solve a general class of constrained games. The idea centers around the hierarchical decomposition of the original constrained problem into two unconstrained sub-problems. This work gives a general approach in dealing with constraints for convex cost functions. However, in practice, the theory needs to be further reduced to a certain form for any particular application. As a result, there exists a gap between engineering application and the theory. It will be more useful if we can find a theory that is tailored for a special class of functions, i.e., such as affine functions, commonly appearing in wireless and optical networks.

Motivated by this, we define a class of linear games with linearly coupled constraints and design a novel approach to solve for its Nash equilibrium. Linear games without constraints have been seen in classical Cournot games [5], [6], wireless power control games [7] and OSNR games [8]. However, it is a challenge when it comes to games with coupled constraints. In this regard, we develop a theoretical framework to characterize a solution for this specific type of games and study its efficiency. In addition, we make use of linearity to develop a general algorithm that can be applied to any game that falls into this category.

In the second part of the paper, we discuss an OSNR game with coupled capacity constraints. This problem was first formulated in [8] towards optical signal-to-noise ratio (OSNR) optimization without constraints as part of recent efforts to address dynamical aspects in optical networks. The extended problem with constraints was later addressed in [1], [4], and [9]. We use the linear game theory and directly find an iterative algorithm that bears similarity with the one used in [9]. In this application, we are also able to interpret the slack variables as fictitious players that can be implemented as service channels. This idea gives us a powerful and intuitive understanding of constraints in games.

Contribution of this work is two-fold. The main contribution is to first propose the notion of linear games and characterize it as an important class of games in game theory. In comparison to general games, it is relatively easy to tackle with and has several nice analytical properties. Filling the gap between game theory and engineering, we develop applicable results for engineering applications for this particular class of games. Secondly, based on the theory, we provide a novel viewpoint to implement constraints as fictitious players in the network and solve capacity constrained OSNR Nash games by introducing game theoretical behavior on the optical service channel (OSC).

This paper is organized as follows. In section 2, we introduce the theory of linear games; specifically, we characterize Nash equilibria, derive of iterative algorithms and study efficiency, etc. In section 3, we review the OSNR model and unconstrained game model. In section 4, we will apply the theory to the OSNR game with linearly coupled capacity constraints to show an iterative algorithm that directly arises from the theory. In addition, we take into account the issue of implementation and see the imposed constraint effected by a fictitious player. In section 5, we illustrate the constrained OSNR game with a numerical example. We will point out the direction of future research and conclude the paper in section 6 and 7, respectively.

## II. Linear Games with Coupled Constraints

In recent research of network routing, power control and protocol design, game theory is becoming a useful tool to model multi-agent competition for limited resources and decentralized behavior of coupled dynamics. Its distributed nature first lends its power to understand complicated behaviors such as bargaining and markets [10] and auction theory [11] in microeconomics. Only in recent years has the research community in engineering given attention to this tool to design algorithms and protocols for the purpose of control and management of complex systems. However, gaps exist between the well-known theory in economics and its applicability in engineering systems. With increasing literature on game theory in engineering, it has been observed that this gap is being filled and this is becoming a necessary tool for engineers to acquire in order to understand complicated system behavior. The first part of this paper essentially bears this goal in mind and proposes a theory on linear games. In particular, we consider games with coupled constraints, which are usually hard to deal with in a direct fashion. We will first review basic concepts from noncooperative game theory [5], [12], [13] and then define our notion of linearity in games.

### *A. Preliminaries and Definitions*

Let's consider a non-cooperative game defined by a triplet $\langle \mathcal{N}, A_i, J_i \rangle$, where $A_i = [u_{i,\min}, u_{i,\max}]$ is the continuous strategy set, $J_i : \Omega = \prod_i^N A_i \to \mathcal{R}$ is the cost function and $\mathcal{N} = \{1, 2, \cdots, N\}$ is the index set of players. Each player behaves according to its best response function $BR_i(\mathbf{u}_{-i})$ to minimize its cost, without knowing other player's strategy or behavior.

*Definition 2.1: Best response function* $BR_i(\mathbf{u}_{-i}), i \in \mathcal{N}$ of a non-cooperative game $\langle \mathcal{N}, A_i, J_i \rangle$ is defined by $BR_i(\mathbf{u}_{-i}) = \{u_i \in A_i \mid J_i(u_i, \mathbf{u}_{-i}) \leq J_i(u_i', \mathbf{u}_{-i}), \forall u_i' \in A_i\}$, where $\mathbf{u}_{-i}$ denotes a set of actions other than oneself, i.e., $\mathbf{u}_{-i} = [u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N]^T$.

The function $BR_i$ is set-valued; i.e., it associates a set of actions with other players' actions. The definition of Nash equilibrium can be based on the definition of best response functions.

*Definition 2.2:* Consider an $N$-player game, in which each player minimizes the cost functions $J_i : \Omega \to \mathcal{R}$. A vector $\mathbf{u}^* = [u_i^*]$ or $\mathbf{u}^* = (\mathbf{u}_{-i}^*, u_i^*) \in \Omega$ is called a Nash equilibrium (NE) of this game if $u_i^* \in BR_i(\mathbf{u}_{-i}^*), \forall i \in \mathcal{N}$, or equivalently, $J_i(u_i^*, \mathbf{u}_{-i}^*) \leq J_i(u_i', \mathbf{u}_{-i}^*), \forall u_i' \in A_i, \forall i \in \mathcal{N}$

*Proposition 2.1:* Suppose utility functions $J_i$ are continuously differentiable, strictly convex in $u_i$, $A_i = [u_{i,\min}, u_{i,\max}]$ and $u_{i,\min}, u_{i,\max}$ are sufficiently small and large respectively, so that they are not NEs of the game. The unique best response function of the non-cooperative game is given explicitly by

$$BR_i(\mathbf{u}_{-i}) = \arg\min_{u_i} J_i(u_i, \mathbf{u}_{-i}), i \in \mathcal{N}; \tag{1}$$

and implicitly by

$$\frac{\partial J_i(u_i, \mathbf{u}_{-i})}{\partial u_i} = 0, i \in \mathcal{N}. \tag{2}$$

*Proof:* Since $A_i$ is a compact set, and $J_i$ is continuous differentiable and strictly convex in $u_i$, there exists a minimizing $u_i^*$, for any given $\mathbf{u}_{-i}$ such that $J_i(u_i^*, \mathbf{u}_{-i}) < J_i(u_i, \mathbf{u}_{-i}), \forall u_i \neq u_i^*$. Since $u_{i,\min}$ and $u_{i,\max}$ is chosen to be small and large enough, therefore there exists a slater point $\mathbf{u}^s$ such that $u_i^s$ is in the interior of $A_i$ and $u_{i,\min}$ and $u_{i,\max}$ are not active constraints. Due to the convexity, a unique global minimum is achieved by the necessary condition $\frac{\partial J_i}{\partial u_i} = 0$. As a result, the best response function $BR_i$ is given in (1) and (2). ■

*Definition 2.3:* A non-cooperative game $\langle \mathcal{N}, A_i, J_i \rangle$ is *linear* if the best response function $BR_i(\mathbf{u}_{-i})$ is given by an implicit affine function in $\mathbf{u}$, where $u_i \in A_i$. A linear non-cooperative game (LNG) is *unconstrained* when $A_i = [u_{i,\min}, u_{i,\max}]$ and $u_{i,\min}, u_{i,\max}$ are sufficiently small and large respectively, so that they are not Nash solutions of the game.

In this case, best response function $BR_i(\mathbf{u}_{-i})$ can be given explicitly by $u_i = BR_i(\mathbf{u}_{-i}) = \mathbf{c}_{-i}^T \mathbf{u}_{-i} + d_i, \forall i$, where $\mathbf{c}_{-i}^T = [c_{i,1}, \cdots, c_{i,i-1}, c_{i,i+1}, \cdots, c_{i,N}]$ or implicitly given by $\mathbf{c}_i^T \mathbf{u} - d_i = 0$, where $\mathbf{c}_i^T = [c_{i,1}, c_{i,2}, \cdots, c_{i,N}]$. An explicit expression of $BR_i$ can always be written in an implicit way.

It is important to point out that our assumption of linearity does not imply loss of generality. In most engineering applications, it is often at our disposal to choose a utility that represents the system. This latitude makes our assumption reasonable rather than confining. It should also be noted that linearity in best response function doesn't imply separability or quadratic form of cost functions. A counter-example of non-separable and non-quadratic cost function that results in a linear best response function is given by (25) in section III. A classification based on best response functions allows us to study a broader class of cost functions, for example, non-separable and non-linear cost functions, which can be traditionally difficult to tackle with directly from the cost functions.

*Definition 2.4:* A linear non-cooperative game with *linearly coupled constraints* (LCCG) is a linear non-cooperative game (LNG) with $\mathbf{u} \in \Omega \cap \overline{\Omega}$, where $\overline{\Omega}$ is given by $\overline{\Omega} = \{\mathbf{u} \mid \mathbf{g}(\mathbf{u}) \leq 0\}$, $\mathbf{g}(\mathbf{u})^T = [g_1(\mathbf{u}), g_2(\mathbf{u}), \cdots, g_M(\mathbf{u})]$ and $g_i(\mathbf{u}) = \mathbf{b}_i^T\mathbf{u} - v_i, \mathbf{b}_i \in \mathcal{R}^N, v_i \in \mathcal{R}, i = 1, \cdots, M$. Therefore, we can let $\mathbf{g}(\mathbf{u}) = \mathbf{B}\mathbf{u} - \mathbf{v} \leq 0$ ,where $\mathbf{B} = [\mathbf{b}_1, \cdots, \mathbf{b}_M]^T \in \mathcal{R}^{M\times N}$.

*Definition 2.5:* With the above definition, the Nash equilibrium to LCCG is defined as a point $\mathbf{u}^*$ such that $J_i(u_i^*, \mathbf{u}_{-i}^*) \leq J_i(u_i', \mathbf{u}_{-i}^*), \forall u_i' \neq u_i^*, \forall i, \forall u_i' \in \overline{\Omega}_i(\mathbf{u}_{-i})$ where $\Omega_i$ is the projection set defined in [4] as $\Omega_i(\mathbf{u}_{-i}^*) := \{u_i' \in A_i \mid \mathbf{g}(u_i', \mathbf{u}_{-i}^*) \leq 0\}$.

### B. Characterization of Nash Equilibrium

To find points that satisfy Definition 2.5, we need to make use of the implicit best response functions and the Lagrangian method outlined in [4], especially Lemma 2. In the characterization, we embed slacked variables into the constraints to study the issue of efficiency in constrained games.

*Theorem 2.2:* Suppose the best response function of a corresponding unconstrained linear game (LNG) is uniquely determined by (2). A Nash equilibrium $\mathbf{u}^*$ to LCCG exists if and only if there exists an $\mathbf{x}^T = [\mathbf{u}^T, \mu^T]$ with $\mu_i, \nu_i, i \in \mathcal{N}$ being nonnegative, such that

$$\mathcal{L}(\mathbf{u}, \mu) := \mathbf{\Psi}\mathbf{x} = \mathbf{l}, \tag{3}$$

or equivalently, $\mathbf{l} \in \mathbf{R}(\mathbf{\Psi})$; and the slackness condition

$$\nu^T\mathbf{g}(\mathbf{u}) = 0. \tag{4}$$

where $\mathcal{L}(\mathbf{u}, \mu)$ is defined as a quasi-Lagrangian as it is composed of best response functions and constraints. $\mathbf{\Psi} \in \mathcal{R}^{(M+N)\times(M+N)}$ is called quasi-Lagrangian matrix and is given by $\mathbf{\Psi} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{pmatrix}$.

$\mathbf{l} = \begin{pmatrix} \mathbf{d}^T - \mathbf{B}^T\nu \\ \mathbf{v} \end{pmatrix}$, with $\mathbf{d} \in \mathcal{R}^N$, $\mathbf{v} \in \mathcal{R}^M$ and $\mathbf{C} = [\mathbf{c}_1, \cdots, \mathbf{c}_N]^T \in \mathcal{R}^{N\times N}$ defined earlier. $\mathbf{R}$ denotes the range space.

*Proof:* Suppose we have implicit best response functions $BR_i(\mathbf{u}_{-i})$ described by $\mathbf{c}_i^T\mathbf{u} - d_i = 0, \forall i \in \mathcal{N}$ for the corresponding unconstrained LNG and the slacked constraint condition $\mathbf{g}(\mathbf{u}) + \mu = 0$, where $\mu = [\mu_1, ..., \mu_M]^T$ is a vector with slack variables $\mu_i \geq 0$. If $\mathbf{u}^*$ is an NE, then $\mathbf{u}^* \in \overline{\Omega}$ and there exists nonnegative $\mu^* \in \mathcal{R}^M$ such that the slacked condition (5) holds.

$$\mathbf{B}\mathbf{u} + \mathbf{I}_M\mu = \mathbf{v}, \tag{5}$$

where $\mathbf{I}_M \in \mathcal{R}^{M\times M}$. In addition, from the unconstrained linear best response functions, we can use (23) in Lemma 2 from [4], i.e.,

$$\frac{\partial J_i}{\partial u_i}(u_i, \mathbf{u}_{-i}) + \nu^T\frac{\partial \mathbf{g}}{\partial u_i}(u_i, \mathbf{u}_{-i}) = 0, \forall i, \tag{6}$$

and form a necessary condition in component-wise form given by $\mathbf{c}_i^T\mathbf{u} - \mathbf{d}_i + \mathbf{b}_i^T\nu = 0, \forall i.$ where $\nu_i \geq 0 \in \mathcal{R}^M$ denotes the Lagrangian multiplier such that $\nu^T g(\mathbf{u}) = 0$. Expressed in matrix form, we obtain

$$\mathbf{C}\mathbf{u} = \mathbf{d} - \mathbf{B}^{\mathbf{T}}\nu. \tag{7}$$

Therefore, we can augment the two matrix equalities (5) and (7) into

$$\mathbf{\Psi}\mathbf{x} = \mathbf{l}. \tag{8}$$

where $\mathbf{\Psi}$, $\mathbf{x}^T$ and $\mathbf{l}^T$ are defined in Theorem 2.2. From linear algebra theory [14], a solution exists for the linear system (3) if and only if $\mathbf{l}$ is in the range space of $\mathbf{\Psi}$.

To show the sufficiency, we resort to arguments used in Lemma 2 in [4]. Firstly, if (5) holds, $\mathbf{u} \in \Omega \cap \overline{\Omega}$. In addition, since the best response functions to LNG is uniquely determined from the cost functions, $\mathbf{u}^*$ that satisfies (7) also minimizes the component-wise Lagrangian $L_i(u_i, \mathbf{u}_{-i}, \nu) = J_i(u_i, \mathbf{u}_{-i}) + \nu^T\mathbf{g}(\mathbf{u})$ from (6), i.e., $L_i(u_i^*, \mathbf{u}_{-i}^*, \nu) \leq L_i(u_i', \mathbf{u}_{-i}^*, \nu), \forall u_i' \in \Omega_i(\mathbf{u}_{-i}^*)$. Using (4), we obtain $J_i(u_i^*, \mathbf{u}_{-i}^*) \leq J_i(u_i', \mathbf{u}_{-i}^*), \forall u_i' \in \Omega_i(\mathbf{u}_{-i}^*)$, so, according to Definition 2.5, $\mathbf{u}^*$ is an NE. ■

*Remark 2.1:* We use a two-person Nash game to illustrate Theorem 2.2. Let $A_i = [\epsilon_i, 10]$, where $\epsilon_i \in \mathcal{R}$ is a sufficiently small positive number, $i = 1, 2$. Suppose the two players have cost functions $J_1 = u_1 - 2\ln\left(1 + \frac{1}{2}\frac{u_1}{u_2}\right)$, and $J_2 = u_2 - \ln\left(1 + 2\frac{u_2}{u_1}\right)$, which are convex in $u_1, u_2$ respectively. The best response functions of the unconstrained LNG are uniquely given by $2u_1 + u_2 = 2$ and $u_1 + 2u_2 = 2$, respectively. Suppose also the actions of two players are constrained by $u_1 + u_2 \leq C_0$ in a coupled way. Therefore, $\mathbf{\Psi x} = \mathbf{l}$ is given by

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \mu \end{pmatrix} = \begin{pmatrix} 2-\nu \\ 2-\nu \\ C_0 \end{pmatrix}.$$

We look at three specific cases: (1) $C_0 = \frac{1}{2}$, (2) $C_0 = \frac{4}{3}$, and (3) $C_0 = 2$. With the slack condition $\nu(u_1 + u_2 - \frac{1}{2}) = 0$, we can find a unique solution to the game with the coupled constraint of case (1) given by $u_1^* = u_2^* = \frac{1}{4}, \nu^* = \frac{5}{4}, \mu^* = 0$. The constraint is active. The solution is obviously Pareto efficient but in a win-lose situation, i.e., for one person to gain more, another has to lose. For the second case, $u_1^* = u_2^* = \frac{2}{3}$, $\nu^* = \mu^* = 0$. Again, the constraint is active but the solution is given in an efficient and win-win situation. For case (3), a unique solution is given by $u_1^* = u_2^* = \frac{2}{3}$ and $\nu^* = 0, \mu^* = \frac{2}{3}$. The constraint becomes inactive and not efficient in the Pareto sense.

*Remark 2.2:* Apparently, a Nash equilibrium doesn't exist if the condition in Theorem 2.2 doesn't hold. We also can observe that the slack variable $\mu_i$ gives an interpretation of the efficiency with respect to the constraints while the Lagrangian variable $\nu_i$, usually interpreted as shadow price in linear programming, is related to the sensitivity of the cost to constraints. In this sense, theorem 2.2 allows us to study efficiency and sensitivity at the same time. From (4) and (5), we also note that $\mu_i$ and $\nu_i$ are interrelated in a trade-off manner such that $\nu^T \mu = 0$.

We further describe an NE by separating into two cases when the existence condition holds: (1) $\mathbf{\Psi}$ is nonsingular, which leads to a unique NE; (2) $\mathbf{\Psi}$ is singular, which gives multiple NEs.

*1) Results for Nonsingular Quasi-Lagrangian Matrix:*

*Proposition 2.3:* If matrix $\mathbf{C}$ is diagonally dominant and $C_{i,i} > 0$, then the following equivalent statements hold.

1) $\mathbf{C}$ is a nonsingular matrix.
2) $\mathbf{\Psi}$ is a nonsingular matrix.
3) 0 is not an eigenvalue of matrix $\mathbf{C}$.

*Proof:* Using Gershgorin Theorem [15], we can show that if matrix $\mathbf{C}$ is diagonally dominant, and the diagonal entries are nonzero, then $\mathbf{C}$ is non-singular. Now we show that statement (1) is equivalent to

(2). If $\mathbf{C}$ is nonsingular, then $\det(\mathbf{C}) \neq 0$. Since $\det(\mathbf{\Psi}) = \det(\mathbf{C})\det(\mathbf{I}_M)$ and $\det(\mathbf{I}_M) = 1$,$\det(\mathbf{\Psi}) = \det(\mathbf{C}) \neq 0$. The reverse also holds. From [14], it follows immediately that that statement (3) is equivalent to statement (1). ∎

If $\mathbf{\Psi}$ is nonsingular from Proposition 2.3, we can invert the matrix and obtain

$$\mathbf{u}^* = \mathbf{C}^{-1}\left(\mathbf{d} - \mathbf{B}^T\nu\right), \tag{9}$$

$$\mu^* = -\mathbf{B}\mathbf{C}^{-1}\left(\mathbf{d} - \mathbf{B}^T\nu\right) + \mathbf{v}. \tag{10}$$

*Theorem 2.4:* Suppose $v_i > 0$ in the constraints of LCCG and $\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ is nonnegative. If $v_i > \frac{\sqrt{N}\|\mathbf{B}\|_\infty\|\mathbf{d}\|_\infty}{\rho(\mathbf{C}^T\mathbf{C})}, \forall i$, where $\rho(\cdot)$ denotes the spectral radius, then $\mu_i^* > 0$ and all the constraints are inactive. A Nash equilibrium is thus given by

$$\mathbf{u}^* = \mathbf{C}^{-1}\mathbf{d}, \tag{11}$$

$$\mu^* = -\mathbf{B}\mathbf{C}^{-1}\mathbf{d} + \mathbf{v}. \tag{12}$$

*Proof:* Let $\mathbf{e}_i$ be the elementary basis vector, i.e., $\mathbf{e}_i = [0, \cdots, 0, 1, 0, \cdots, 0]^T \in \mathcal{R}^M$ with all the entries of the vector zeroes except $i$th element being 1. Therefore, $\mathbf{e}_i^T\mathbf{v} = v_i$ and $\mathbf{e}_i^T\mu^* = \mu_i^*$. Using $|v_i| \geq \frac{\sqrt{N}\|\mathbf{B}\|_\infty\|\mathbf{d}\|_\infty}{\rho(\mathbf{C}^T\mathbf{C})}$, we obtain

$$\begin{aligned} v_i &= |v_i| > \frac{\sqrt{N}\|\mathbf{B}\|_\infty\|\mathbf{d}\|_\infty}{\rho(\mathbf{C}^T\mathbf{C})} && (13)\\ &\geq \frac{\|\mathbf{B}\|_\infty\|\mathbf{d}\|_\infty}{\|\mathbf{C}\|_\infty} = \frac{\|\mathbf{e}_i^T\|_\infty\|\mathbf{B}\|_\infty\|\mathbf{d}\|_\infty}{\|\mathbf{C}\|_\infty} && (14)\\ &\geq \|\mathbf{e}_i^T\mathbf{B}\mathbf{C}^{-1}\mathbf{d}\|_\infty = |(\mathbf{B}\mathbf{C}^{-1}\mathbf{d})_i|, \forall i && (15) \end{aligned}$$

For the case where $(\mathbf{B}\mathbf{C}^{-1}\mathbf{d})_i \geq 0$, knowing $\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ is nonnegative and $\nu_i > 0$, we use (10) to obtain $\mu_i^* = v_i - (\mathbf{B}\mathbf{C}^{-1}\mathbf{d})_i + (\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\nu)_i > 0$. And when $(\mathbf{B}\mathbf{C}^{-1}\mathbf{d})_i < 0$, $\mu_i^* \geq -2(\mathbf{B}\mathbf{C}^{-1}\mathbf{d})_i + (\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\nu)_i > 0$. Therefore, it follows that $\mu_i^* > 0, \forall i$, i.e., all the constraints are inactive. Therefore, $\nu_i = 0, \forall i$. Following (10) and (9), we obtain (11) and (12). ∎

Theorem 2.4 gives a condition so that all the constraints are inactive and the NE can be reduced to the form of (11). In the case of a single capacity constraint, as will be investigated later in section IV, we can simplify the assumption of nonnegativity of $\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ into the semi-positive definiteness of $\mathbf{C}$. Theorem 2.4 also has implications in admission control. Specifically, it gives a good estimate on a sufficient allocation capacity for a fixed amount of users. Since $\mu_i^*$ quantitatively reveals the efficiency of the solution, such more-than-enough type of allocation will result in loss of efficiency. Any deviation from $\mu_i^* = 0$ will result in loss of efficiency. This notion will be crystalized later in section 2.3.

*2) Results for Singular Quasi-Lagrangian Matrix:* In the case where $\mathbf{\Psi}$ is singular but $\mathbf{l} \in \mathbf{R}(\mathbf{\Psi})$, we can formulate the following linear program (GLP) to solve for an appropriate Nash equilibrium.

$$\begin{aligned} &\text{(GLP)} && \min_{\mathbf{u}} \mathbf{a}^T\mathbf{x} \\ &\text{s.t.} && \mathbf{\Psi}\mathbf{x} = \mathbf{l}, \quad \nu^T\mathbf{g}(\mathbf{u}) = 0 \\ &&& \nu_i \geq 0, \mu_i \geq 0, u_i \in A_i, i \in \mathcal{N}. \end{aligned} \tag{16}$$

where $\mathbf{a}^T = [a_1, \cdots, a_N, 0, \cdots, 0] \in \mathcal{R}^{(N+M)}$ are design parameters. For example, in engineering application, $u_i$ may be commonly referred to as power consumption. GLP selects a Nash solution that will minimize the weighted total power consumption. However, this choice of $\mathbf{a}$ may not be ideal in applications where we wish to fully allocate the resources. Later, we will take the issue of efficiency into account and revise the objective function.

When $\mathbf{\Psi}$ is nonsingular, we can still use GLP to solve for the unique Nash equilibrium. In this case, the feasible set is simply a singleton.

### *C. Efficiency of Nash Equilibrium*

A well-known fact about Nash equilibrium is that it may not be an efficient solution [16]. A celebrated simple example is a matrix strategic game: Prisoners' Dilemma, in which the noncooperative outcome is worse than the cooperative gain. In addition, recent active research [17], [18] in the community of computer science and operations research strives to quantify the notion of efficiency by price of anarchy, the ratio between social utility and sum of individual utilities from the competition. Work done by Perakis in [19], [20] extends the result of price of anarchy from [17] to general classes of cost functions, i.e., linear and separable functions in [19]; nonlinear and asymmetric functions in [20]. Analysis based on cost/utility functions can become untractable when the function becomes nonseparable and nonlinear. It is worth to notice that in engineering applications dealing with real physical situations, e.g. power control, utility functions naturally arise as a function of signal-to-noise ratio. Therefore, it is challenging to apply directly the results developed in [17]–[20]. In addition, most of the models considered in literature make the results less than practical because they commonly assume no capacity constraints. In this section, we study the efficiency for games with linearly coupled constraints, and develop a quantitative metric to measure the efficiency loss of resources.

An allocation is said to be Pareto efficient if Nash equilibrium $\mathbf{u}^*$ is achieved on the Pareto front. From (5), we observe that an allocation is not Pareto efficient if $\mu_i^* > 0, \forall i$ and thus, $\mu_i^*$ gives us a sense of how much resource is lost.

*Proposition 2.5:* If an allocation is Pareto efficient then $\mu_i^* = 0$ for some $i$.

*Proof:* Since we only consider linear constraints, if an allocation is Pareto efficient, it will be on the Pareto front by fundamental theorem of linear programming [21]. Therefore, there must be an $i$ such that $g_i(\mathbf{u}) = 0$, and thus $\mu_i^* = 0$. ■

*Proposition 2.6:* Suppose $\mathbf{\Psi}$ is a non-singular matrix and $M$ linearly coupled constraints, where $v_i > 0, i \in \{1, 2, \cdots, M\}, u_i \in A_i$. The efficiency of the linear game is measured by $\eta_E$, which is defined as follows

$$\eta_E = \begin{cases} \max_i \left( (\mathbf{B}\mathbf{C}^{-1}\mathbf{d})_i / v_i \right), & \mu_i^* \neq 0, \forall i; \\ 1, & \mu_i^* = 0 \text{ for some } i. \end{cases} \tag{17}$$

For the case where $\mu_i^* \neq 0, \forall i$, $\eta_E$ is bounded from above as follows.

$$\eta_E \leq \frac{\gamma_B d_{\max}}{\gamma_C v_{min}}. \tag{18}$$

where $v_{\min} = \min_i v_i, d_{\max} = \max_i d_i, \gamma_B = \|\mathbf{B}\|_\infty$ and $\gamma_C = \|\mathbf{C}\|_\infty$.

*Proof:* Start with the definition of $\eta_E$ and we obtain for the case where $\mu_i^* \neq 0, \forall i$:

$$\eta_E \leq \frac{\|\mathbf{B}\mathbf{C}^{-1}\mathbf{d}\|_\infty}{v_{\min}} \leq \frac{\|\mathbf{B}\|_\infty \|\mathbf{d}\|_\infty}{\|\mathbf{C}\|_\infty v_{\min}} = \frac{\gamma_B d_{\max}}{\gamma_C v_{\min}}.$$

■

And the loss of efficiency is given by $\zeta_E = 1 - \eta_E$. As the definition suggests, $\eta_E$ measures the worst case of efficiency loss among all non-active constraints. The solution is Pareto efficient if one of the coupled linear constraints is active, as shown in Proposition 2.5.

A common situation might be with only one capacity constraint given by $\sum_{i \in \mathcal{N}} u_i \leq C_0, C_0 > 0, u_i \in A_i$. Then, the efficiency of the linear game can be simply reduced to

$$\eta_E = \frac{C_0 - \mu^*}{C_0} \leq \frac{d_{\max}}{C_0 \max_i \left( \sum_j C_{ij} \right)} \tag{19}$$

The above results provide an estimate of the efficiency of resulting NE. We still need a mechanism that can allow us to minimize the efficiency loss in the constrained game. We know that we can always embed variables such as pricing [22] or arbitrating parameters $\alpha \in \mathcal{R}^N$ into the problem and form $\mathbf{\Psi}(\alpha)$. Consequently, we formulate a general parametric optimization problem (GPLP) similar to (GLP) for nonsingular $\mathbf{\Psi}(\alpha)$, for all $\alpha$.

$$\begin{aligned} \text{(GPLP)} \quad & \min_{\alpha} \|\mathbf{a}^T\mathbf{x}\| \\ \text{s.t.} \quad & \mathbf{\Psi}(\alpha)\mathbf{x} = \mathbf{l}, \quad \nu^T\mathbf{g}(\mathbf{u}(\alpha)) = 0 \\ & \nu_i \geq 0, \mu_i \geq 0, u_i \in A_i, i \in \mathcal{N}. \end{aligned} \tag{20}$$

If matrix $\mathbf{\Psi}(\alpha)$ may be singular, we also need to minimize over $\mathbf{x}(\alpha)$ as in (SGPLP) .

$$\begin{aligned} \text{(SGPLP)} \quad & \min_{\alpha,\mathbf{u}(\alpha)} \|\mathbf{a}^T\mathbf{x}\| \\ \text{s.t.} \quad & \mathbf{\Psi}(\alpha)\mathbf{x} = \mathbf{l}, \quad \nu^T\mathbf{g}(\mathbf{u}(\alpha)) = 0 \\ & \nu_i \geq 0, \mu_i \geq 0, u_i \in A_i, i \in \mathcal{N}. \end{aligned} \tag{21}$$

Since the slack variables $\mu_i$ determine the efficiency loss, we can design parameter $\mathbf{a}^T$ as $\mathbf{a}^T = [0, \cdots, 0, \hat{\mathbf{a}}^T]$, where $\hat{\mathbf{a}}^T \in \mathcal{R}^M, \hat{a}_i \geq 0$. Parametric programming is a relatively mature field in mathematics [23]. We can again use the sensitivity analysis to solve GPLP and SGPLP by seeing $\alpha$ as parametric disturbances. These can be done numerically by MATLAB optimization packages.

### D. Iterative Algorithms

In engineering applications, it is desirable to find an algorithm that can iteratively lead to the optimal solution rather than a static way to find the solution. To simplify the derivation, we assume an appropriate admission control to ensure a sufficient allocation as in Theorem 2.4. With reference from [24], we construct a distributed iterative algorithm based on (3). Since $\mathbf{D}\mathbf{x} = \mathbf{l} - (\mathbf{\Psi} - \mathbf{D})\,\mathbf{x}$, for any $\mathbf{D}$, we propose the following algorithm

$$x_i(n+1) = \frac{1}{\Psi_{i,i}} \left( l_i - \sum_j \Psi^a_{i,j} x_i(n) \right) \tag{22}$$

where $\mathbf{\Psi}^a = \mathbf{\Psi} - \mathbf{D}, \mathbf{D} = \text{diag}(\Psi_{1,1}, \Psi_{2,2}, \cdots, \Psi_{M+N,M+N})$.

*Theorem 2.7:* Algorithm (22) converges to the Nash equilibrium provided that $\mathbf{\Psi}$ is strictly diagonally dominant, i.e., $|\Psi_{i,i}| > \sum_{j\neq i} |\Psi_{i,j}|, \forall i$.

*Proof:* Appendix I ■

The rate of convergence in (22) is determined by $\sigma_1 = \max_i \frac{1}{|\Psi_{i,i}|} \sum_{j\neq i} |\Psi_{i,j}|$. It may happen that $\sigma_1$ may be close to unity and result in slow convergence. To boost the convergence rate, we use the idea from successive over-relaxation (SOR) [25] and use $\omega \in \mathcal{R}$ to modify (22) into $x_i(n+1) = \frac{\omega}{\Psi_{i,i}}\left(l_i - \sum_j \Psi^a_{i,j} x_i(n)\right) + (1-\omega)x_i(n)$.

## III. OSNR Game in Optical Networks

In this section, we apply the results of linear games to the OSNR Nash game in optical networks that has been recently formulated in [8]. As recent technological advances have sparked the need for intelligent network management and control, recent efforts address the issue of dynamic network configurations. These are enabled by a new generation of optical wavelength-division multiplexed (WDM) communication networks, in which devices such as optical add/drop MUXes (OADM), optical cross connects (OXC) and dynamic gain equalizer (DGE) provide essential building blocks for smart optical networks [26].

Following [8], the focus of our application is on the end-to-end link level. Channel optical signal-to-noise ratio (OSNR) is an important performance factor at this level as it directly relates to the bit error rate (BER) in the transmission [27]. OSNR optimization is thus crucial to the improvement of network performance. Recent research work on this is making an effort to derive iterative decentralized OSNR optimization algorithms, particularly, based on a game-theoretical approach. However, it has been a challenge to find an analytical solution for an OSNR game with capacity constraints. Research efforts have been made to solve this problem by integrating constraints into utility functions [1], [2]. And, in particular, theoretical work has been done in [4], [9] to deal with such constraints based on classical Lagrangian duality theory. However, complexity of the solution grows in an undesirable way and it is exceedingly difficult to give an analytical solution for OSNR Nash game.

In this section and the following section, we meet those challenges by using the theory of linear games and interpreting the slack variables as power consumption of fictitious players that can be implemented by optical service channels (OSC) in reality. We first review the optical network model and the basic game-theoretical framework. Later, we will see that it is convenient for us to use the established theory to derive problem-specific results.

### A. *Review of Optical Network Model*

Consider a WDM network with a set of optical links $\mathcal{L} = \{1, 2, .., L\}$ connecting the optical nodes, where channel add/drop is realized. A set $\mathcal{N} = \{1, 2, ..., N\}$ of channels are transmitted, corresponding to a set of multiplexed wavelengths. Illustrated in Figure 1, a link $l$ has $K_l$ cascaded optically amplified spans. Let $N_l$ be the set of channels transmitted over link $l$. For a channel $i \in \mathcal{N}$, we denote by $\mathcal{R}_i$

its optical path, or collection of links, from source (Tx) to destination (Rx). Let $u_i$ be the $i$th channel input optical power (at Tx), and $\mathbf{u} = [u_1, ..., u_N]^T$ the vector of all channels' input powers. Let $s_i$ be the $i$th channel output power (at Rx), and $n_i$ the optical noise power in the $i$th channel bandwidth at Rx. The $i$th channel optical OSNR is defined as $OSNR_i = \frac{s_i}{n_i}$. In [28], it is assumed that the dispersion and nonlinearity effects are considered to be limited, the ASE noise accumulation is the dominant impairment in the model. This assumption simplifies the OSNR expression, and thus the OSNR for the $i$th channel is given as

$$OSNR_i = \frac{u_i}{n_{0,i} + \sum_{j \in \mathcal{N}} \Gamma_{i,j} u_j}, i \in \mathcal{N} \tag{23}$$

where $\mathbf{\Gamma}$ is the full $n \times n$ system matrix which characterizes the coupling between channels. $n_{0,i}$ denotes the $i$th channel noise power at the transmitter. System matrix $\mathbf{\Gamma}$ encapsulates the basic physics present in optical fiber transmission and implements an abstraction from a network to an input-output system. This approach has been used in [7] for the wireless case to model CDMA uplink communication. Different from the system matrix used in wireless case, the matrix $\mathbf{\Gamma}$ given in (24) is commonly asymmetric and is more complicatedly dependent on parameters such as spontaneous emission noise, wavelength-dependent gain, and the path channels take.

$$\Gamma_{i,j} = \sum_{i \in \mathcal{R}_i} \sum_{k=1}^{K_l} \frac{G_{l,j}^k}{G_{l,i}^k} \left( \prod_{q=1}^{l-1} \frac{\mathbf{T}_{q,j}}{\mathbf{T}_{q,i}} \right) \frac{ASE_{l,k,i}}{P_{0,l}}, \forall j \in \mathcal{N}_l. \tag{24}$$

where $G_{l,k,i}$ is the wavelength dependent gain at $k$th span in $l$th link for channel $i$; $\mathbf{T}_{l,i} = \prod_{q=1}^{K_l} G_{l,k,i} L_{l,k}$ with $L_{l,k}$ being the wavelength independent loss at $k$th span in $l$th link; $ASE_{l,k,i}$ is the wavelength dependent spontaneous emission noise accumulated across cascaded amplifiers; $P_{0,l}$ is the output power at each span.

It is also shown in [28] that the OSNR model can be further extended to include crosstalk terms due to WDM components at the optical nodes (OADM or OXC), such as optical filters, demultiplexers, add/drop modules, routers or switches [27].

### *B. Non-cooperative Game Approach*

Let's review the basic game-theoretical model for power control in optical networks without constraints. Consider a game defined by a triplet $\langle \mathcal{N}, (A_i), (J_i) \rangle$. $\mathcal{N}$ is the index set of players or channels; $A_i$ is the strategy set $\{u_i \mid u_i \in [u_{i,\min}, u_{i,\max}]\}$; and, $J_i$ is the cost function, chosen such that minimizing the cost

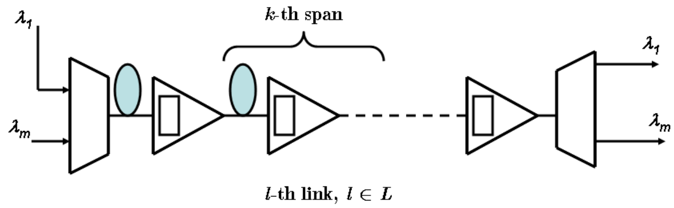


Fig. 1. A Typical Optical Link in DWMW Optical Networks

is related to maximizing OSNR level. In [8], $J_i$ is defined as

$$J_i(u_i, u_{-i}) = \alpha_i u_i - \beta_i \ln\left(1 + a_i \frac{u_i}{X_{-i}}\right), i \in \mathcal{N} \tag{25}$$

where $\alpha_i, \beta_i$ are channel specific parameters, that quantify the willingness to pay the price and the desire to maximize its OSNR, respectively, $a_i$ is a channel specific parameter, $X_{-i}$ is defined as $X_{-i} = \sum_{j\neq i} \Gamma_{i,j} u_j + n_{0,i}$. This specific choice of utility function is non-separable, nonlinear and coupled. However, $J_i$ is strictly convex in $u_i$ and takes a specially designed form such that its first-order derivative takes a linear form with respect to $\mathbf{u}$, i.e. is in the class of linear games defined in section 2.

The solution from the game approach is usually characterized by Nash equilibrium (NE) in Definition 2.2. Provided that $\sum_{j\neq i} \Gamma_{i,j} \leq a_i$ ,the resulting NE solution is given in a closed form by

$$\widetilde{\mathbf{\Gamma}} \mathbf{u}^* = \widetilde{\mathbf{b}}, \tag{26}$$

where $\widetilde{\Gamma}_{i,j} = a_i$, for $j = i$; $\widetilde{\Gamma}_{i,j} = \Gamma_{i,j}$, for $j \neq i$ and $\widetilde{b}_i = \frac{a_i b_i}{\alpha_i} - n_{0,i}$.

Similar to the wireless case [7], we are able to construct iterative algorithms to achieve the Nash equilibrium. A simple deterministic first order parallel update algorithm can be found by $u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{X_{-i}(n)}{a_i}$, or equivalently in terms of $OSNR_i$,

$$u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(n)} - \Gamma_{i,i}\right) u_i(n). \tag{27}$$

As proved in [8], the algorithm (27) converges to Nash equilibrium $\mathbf{u}^*$ provided that $\frac{1}{a_i}\sum_{j\neq i} \Gamma_{i,j} \leq 1, \forall i$.

## IV. Constrained Non-cooperative OSNR Game

In optical networks, a saturation power level exists in each link of channel paths [2]. A launched power has to be below or equal to this threshold so that the nonlinear effects in the span following each amplifier are kept minimum [29]. We can easily interpret this effect as a capacity constraint on an optical link in the network. In this section, we consider the game described in section 3 with such constraint. Before applying the linear game theory, we consider implementing the slack variable as a fictitious player, labeled $F$. The fictitious player can be regarded as an additional player implemented via a channel that doesn't participate in the game for its need for quality of transmission. An example is the optical service channel (OSC) in optical networks [26]. It only requires a certain amount of power to transmit network information and doesn't aim for OSNR optimization. It rather behaves as a player to regulate the network performance. We will use this interpretation to solve an $(N+1)$-person non-cooperative game with constraint of

$$\sum_{i \in \mathcal{N} \cup \{F\}} u_i \leq C_0. \tag{28}$$

Let the payoff function of user $i \in \mathcal{N}$ given by (25) and we choose the payoff function of user $F$ to be

$$J_F(u_F, u_{-F}) = \alpha_F u_F - \beta_F \left( C_0 - \sum_{j \neq F} u_j \right) \ln a_F u_F. \tag{29}$$

Function $J_F$ is convex when $\sum_{j \neq F} u_j \leq C_0$. Since the fictitious player may not ask for an optimal quality of transmission, we do not design function (29) directly related to OSNR, but in terms of power and capacity constraint instead. It is composed of two parts with the first term describing the cost on power usage $u_F$ and the second term the capacity-dependent utility. With the assumption of convexity, the best response function for $J_F$ is given by an implicit expression as in (30).

$$\omega_F u_F + \sum_{j \neq F} u_j = C_0, \tag{30}$$

where $\omega_F = \frac{\alpha_F}{\beta_F}$. We let $u_i \in [u_{i,\min}, u_{i,\max}]$, where $u_{i,\min} \in \mathcal{R}^+$ and $u_{i,\max} \in \mathcal{R}^+$ can be chosen to be sufficiently small and large respectively, so that they will not be solutions to the minimization of the cost function $J_i, i \in \mathcal{N} \cup \{F\}$.

We can observe that if $\omega_F \geq 1$, then any solution $\mathbf{u}$ that satisfies (30) is within the feasible set described by (28). Furthermore, when $\omega_F = 1$, the best response function of user $F$ will impose an equality capacity constraint, $\sum_{i \in \mathcal{N} \cup \{F\}} u_i = C_0$, and the solution will be efficiently achieved on the boundary of the feasible set. Increasing $\omega_F$ to be strictly greater than 1 will result in a less efficient solution.

The construction of the best response function (30) can be seen as a slacked constraint from (31)

$$\omega'_F u_F + u_F + \sum_{j \neq F} u_j = C_0, \tag{31}$$

where $\omega'_F = \omega_F - 1$, and $\omega'_F u_F > 0$ slacks the capacity constraint (28) .

### A. Characterization of Nash Equilibrium

We use the same approach as in [8] to characterize the equilibrium of the game. By the definition of Nash equilibrium in [5], a Nash equilibrium $\mathbf{u}^F$ with a fictitious player is a point which jointly satisfies the best response functions as follows.

$$a_i u_i^F + X_{-i}^F = \frac{a_i \beta_i}{\alpha_i} \quad , \quad \text{for } i \in \mathcal{N}. \tag{32}$$

$$\omega_F u_F^F + \sum_{j \neq F} u_j^F = C_0 \quad , \quad \text{for } i = F. \tag{33}$$

Expressed in matrix form, they become

$$\overline{\mathbf{\Gamma}} \mathbf{u}^F = \overline{\mathbf{b}}. \tag{34}$$

where $\mathbf{u}^F = [u_1^F, \cdots, u_N^F, u_F^F]^T$, $\overline{\mathbf{b}} = [\frac{a_1\beta_1}{\alpha_1} - n_{0,1}, \cdots, \frac{a_N \beta_N}{\alpha_N} - n_{0,N}, C_0]^T$,

$$\overline{\mathbf{\Gamma}} = \begin{pmatrix} a_1 & \Gamma_{12} & \cdots & \Gamma_{1N} & \Gamma_{1F} \\ \Gamma_{21} & a_2 & \cdots & \Gamma_{2N} & \Gamma_{2F} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \Gamma_{N1} & \Gamma_{N2} & \cdots & a_N & \Gamma_{NF} \\ 1 & 1 & \cdots & 1 & \omega_F \end{pmatrix}.$$

Based on a similar argument as in the sufficiency proof of Theorem 2.2, from (34), we can conclude directly that a necessary and sufficient condition for Nash equilibrium to exist is to require $\overline{\mathbf{b}} \in \mathbf{R}(\overline{\mathbf{\Gamma}})$. To further characterize the Nash equilibrium, we resort to Theorem 2.4. Accordingly, we assume some special features of the game, for example, diagonal dominance of the matrix $\overline{\mathbf{\Gamma}}$ and convexity of the utility functions $J_i$. Theorem 4.1 summarizes these conditions and gives a sufficient condition on the uniqueness and existence of the Nash equilibrium to the OSNR game with fictitious player (GFP).

*Theorem 4.1:* If $\frac{\max_i \overline{\mathbf{b}}_i \sqrt{N+1}}{\sqrt{\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}} \overline{\mathbf{\Gamma}})}} \leq C_0$ and $a_i > \sum_{j \neq i} \Gamma_{ij}$ , $\omega_F > N$, then the game with a fictitious player (GFP) will have a unique Nash equilibrium.

*Proof:* Appendix II. ■

*Remark 4.1:* We can make the connection with Theorem 2.4 by taking $\mathbf{C} = \overline{\mathbf{\Gamma}}, \mathbf{B} = \mathbf{1}^T, \mathbf{I}_M = 1, \mathbf{d} = \overline{\mathbf{b}}$, and $\mathbf{v} = C_0$. In this case, $\mathbf{\Psi}$ has its two last rows coming from the same constraint (28): one is from the fact that we implement the fictitious player via a channel and the other comes from the slacked constraint

(31) as needed in Theorem 2.4. It is obvious from (30) and (31) that $\mu = \omega'_F u_F$. With the inequality on $C_0$, the conditions in Theorem 2.4 are satisfied. As a result, we observe that these conditions result in a game with inactive constraints. Therefore, we can determine a unique NE in a closed form as in (11). In comparison with the approach adopted in the proof, this shows that an interpretation of an unconstrained N-person game with an additional fictitious player with utility function (29) that satisfies the conditions in Theorem 4.1 is equivalent to an (N+1)-person game interpretation with capacity constraint as in Theorem 2.4.

*Remark 4.2:* If we further assume that $C_0 \geq \frac{a_i b_i}{\alpha_i} - n_{0,i}, \forall i$, then the inequality condition on $C_0$ in Theorem 4.1 is reduced to $\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}}) \geq N+1$. This result alludes to the maximum number of channels to be admitted in the network for a fixed capacity.

Though we notice that some portion of the power is allocated to the service channel or the fictitious player, we need to accept that this amount of power is a necessary allocation for the network to operate. Furthermore, this power consumption can be adjusted through parameter $\omega_F$. On the other hand, we should also note that the strong assumption of diagonal dominance, in particular, $\omega_F > N > 1$ may not lead to an efficient solution as mentioned earlier. However, due to the sufficiency of Theorem 4.1, a unique and efficient solution may still exist when $\omega_F = 1$.

## B. Iterative Algorithm

Following (22) and (27), the algorithm for the game with a fictitious player is given by a synchronous algorithm given in (35). An update includes two sub-steps: an initial update on $u_i(n+1), i \in \mathcal{N}$ and one on $u_F$.

$$\begin{cases} u_i(n+1) = \frac{\beta_i}{\alpha_i} - \frac{1}{a_i}\left(\frac{1}{OSNR_i(n)} - \Gamma_{i,i}\right) u_i(n), & \forall i \in \mathcal{N}; \\ u_F(n+1) = \frac{1}{\omega_F}\left[C_0 - \sum_{j \neq F} u_j(n)\right]^+, & \text{for } F. \end{cases} \tag{35}$$

where $[z]^+ = \max\{0, z\}$. Since we implement the fictitious player as an OSC, we leave out the algorithm for $\mu$. From Remark 4.1, the $\mu$ algorithm is closely related to $u_F$ by $\mu = \omega'_F u_F$.

*Proposition 4.2:* The algorithm described by (35) converges to $\mathbf{u}^F$ provided that $a_i > \sum_{j \in \mathcal{N}} \Gamma_{ij}$ and $\omega_F > N$.

The proof of convergence can be obtained by modifying the proof of the general algorithm in (22). We note that the use of $[z]^+$ function is to ensure that $u_F$ will not be negative in the process of updates. Parameters $a_i, i \in \mathcal{N}$, and $\omega_F$ determine the rate of convergence. On average, increasing $a_i, i \in \mathcal{N}$ results in a faster convergence for $u_i, i \in \mathcal{N} \cup \{F\}$. And increasing $\omega_F$ will lead to a boost in convergence speed of user $F$'s algorithm.

We also can observe a similarity with the algorithm derived based on duality theory in [4], where $u_F$ in (35) is closely related to the dual variable in [4]. The difference between the two is that we used a fictitious player in the game in the position of the dual variable and the player has it own rule of interactions with other players.

## V. Numerical Example

In this section, we illustrate the linear OSNR Nash game by a MATLAB simulation. We consider an end-to-end link described in Figure 1 with 5 amplified spans. We assume 3 channels are transmitted at wavelengths distributed from 1554nm to 1556nm with channel separation of 1nm. Suppose input noise power is 0.5 percent of the input signal power and the total power constraint is $C_0$=7.0mW. The gain profile for each amplifier is identically assumed to be parabolic as in Figure 2 and gives $G_1 = 29.2$dB, $G_2 = 30.0$dB, and $G_3 = 29.2$dB, respectively. The 3-by-3 $\mathbf{\Gamma}$ matrix is thus given as

$$\mathbf{\Gamma} = \begin{bmatrix} 6.187 \times 10^{-4} & 1.094 \times 10^{-4} & 2.732 \times 10^{-4} \\ 4.063 \times 10^{-4} & 6.786 \times 10^{-4} & 2.206 \times 10^{-4} \\ 2.728 \times 10^{-4} & 3.752 \times 10^{-4} & 2.728 \times 10^{-4} \end{bmatrix}$$

In Figure 3, we show the convergence of channel power evolution in steps with a fictitious channel. This takes 0.5035mW of the remaining power from the capacity and achieves a power allocation efficiency of 92.81% for the link, calculated from (17). Figure 4 shows the resulting OSNR levels in the game.

## VI. Direction of Future Research

This paper deals with linear non-cooperative games, in which interesting results are derived from the inherent property of linearity. Since our interest is to solve power control games in optical networks, we haven't discussed yet a special case where $\mathbf{C}$ is symmetric. This case is commonly found in Cournot games and wireless power control game. With additional property of symmetry and positive definiteness, it is possible that we can examine the method of conjugate gradient iterative algorithm to solve the game. Furthermore, our definition of linear games is based on best response function. Since best response functions are derived from cost functions, we can directly define the quasi-Lagrangian in terms of $J_i$.

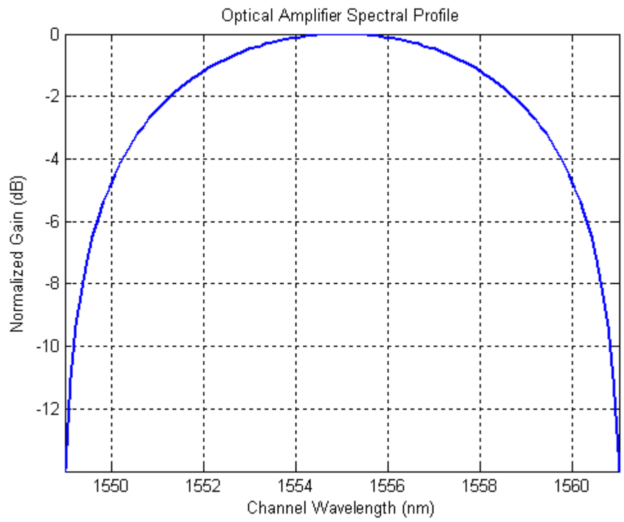


Fig. 2. Optical Amplifier Spectral Profile

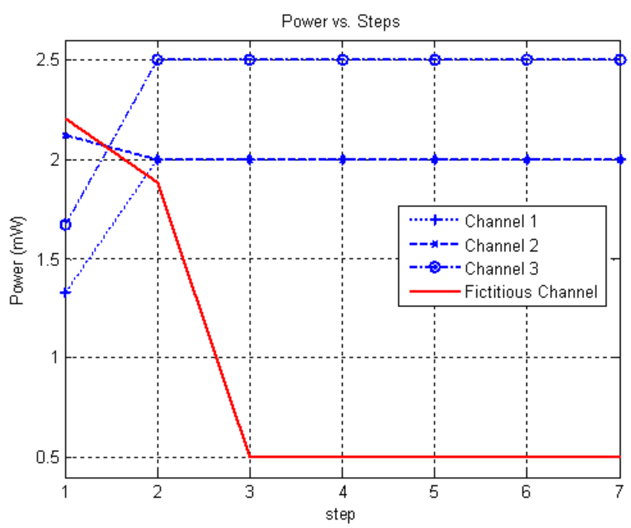


Fig. 3. Channel Power in Time Steps

In addition, it will be more challenging to extend this result to nonlinear problems or a special case of nonlinear problems, where quasi-Lagrangian needs to be otherwise defined.

In this paper, we assumed that the best response function is a static function without any dynamics. We can actually further study a dynamic game with best response functions governed by a first-order differential equation. This study would allow us to include dynamics in the OSNR optical network model and address other practical physical phenomena in optical networks. Furthermore, the feature of linearity could enable an in-depth analysis of pricing design based on classical linear system control theory.

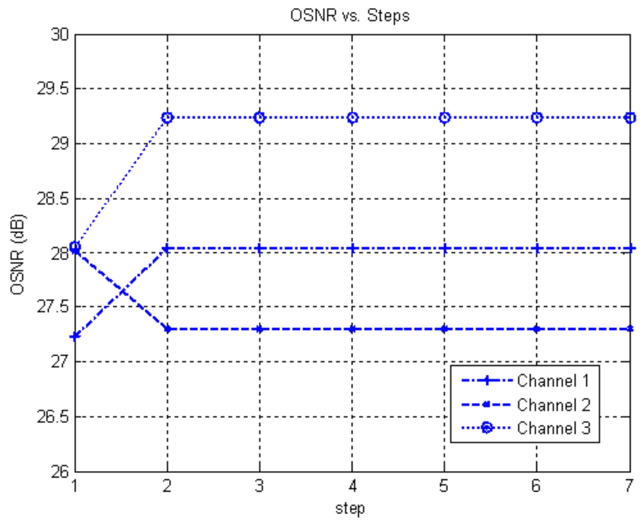


Fig. 4. Channel OSNR in Time Steps

## VII. Conclusion

In this paper, we examined a class of linear games and discussed several issues such as existence and uniqueness of NE, iterative algorithms and efficiency analysis. The appealing feature of this class lies in the freedom to choose cost functions. Thus, a variety of engineering problems with practical concerns of constraints can be solved. As a special case or an extension to the duality theory in [4], linear game theory is a useful tool to solve a class of problems. We illustrate an application of the theory to derive the algorithm, which is very similar to the one described in [9]. In the example of an OSNR game, we can intuitively interpret the slack variables as fictitious players and extend the linear game framework to address a practical issue of implementation.

## Appendix I

Let's define $e_i(n) = x_i(n) - x_i^*$ and $\mathbf{e}(n)$=$[e_i(n)]$. Since $\mathbf{\Psi x}^* = \mathbf{l}$, $\Psi_{i,i}x_i^* + \sum_{j\neq i}\Psi_{i,j}x_j^* = l_i$. Substitute the expression for $x_i^*$ into $e_i(n+1)$, and we obtain $e_i(n+1) = x_i(n+1) - x_i^* = -\frac{1}{\Psi_{i,i}}\left[\sum_{j\neq i}\Psi_{i,j}(x_j(n) - x_j^*)\right]$. Therefore,

$$\begin{aligned}
\|\mathbf{e}(n+1)\|_\infty &\leq \max_i \left| \frac{1}{\Psi_{i,i}} \left[ \sum_{j\neq i} \Psi_{i,j}(e_i(n)) \right] \right| && (36)\\
&\leq \max_i \frac{1}{|\Psi_{i,i}|} \sum_{j\neq i} |\Psi_{i,j}| \|\mathbf{e}(n)\|_\infty . && (37)
\end{aligned}$$

Since we assumed that $\sum_{j\neq i}|\Psi_{i,j}| < |\Psi_{i,i}|, \forall i$, we can conclude that $\|\mathbf{e}(n)\| \to 0$ from the contraction mapping theorem. As a result, we have $x_i(n) \to x_i^*$ as $n \to \infty$.

## APPENDIX II

First of all, we need to show that the utility functions are convex and there exists a minimizing $\mathbf{u}^F$. It has been proved in [8] that functions (25) is convex in $u_i$. We just need to show the convexity of $J_F$ in $u_F$. Knowing that $J_F$ is formed by an addition of two functions and that sum of convex functions results in a convex function, we only need to guarantee the pricing and utility functions are convex. The linear pricing function is already convex. With the condition that $\sum_{j\neq F} u_j \leq C_0$, the convexity of $J_F$ in $u_F$ will follow. Due to the fact that $u_i \in [u_{i,\min}, u_{i,\max}]$ gives a closed compact set, there exists a minimizing $\mathbf{u}^F$, for any given $\mathbf{u}_{-i}$, such that $J_i(u_i^F, \mathbf{u}_{-i}) < J_i(u_i, \mathbf{u}_{-i}), \forall u_i \neq u_i^F, i \in \mathcal{N} \cup \{F\}$.

Secondly, we derive a sufficient condition for convexity of $J_F$ in $u_F$. Starting with the condition $\sum_{j\neq F} u_j \leq C_0$, we use matrix norm inequality $\|\overline{\mathbf{\Gamma}}_{m\times n}\|_2 \leq \sqrt{m}\|\overline{\mathbf{\Gamma}}_{m\times n}\|_\infty$ [14] to obtain an upper bound on $\|\overline{\mathbf{1}}^T\overline{\mathbf{\Gamma}}^{-1}\overline{\mathbf{b}}\|_\infty$, where $\overline{\mathbf{1}}^T = [1, ..., 1, 0]$.

$$\|\overline{\mathbf{1}}^T\overline{\mathbf{\Gamma}}^{-1}\overline{\mathbf{b}}\|_\infty \leq \|\overline{\mathbf{1}}^T\overline{\mathbf{\Gamma}}^{-1}\|_\infty\|\overline{\mathbf{b}}\|_\infty \leq \frac{\|\overline{\mathbf{b}}\|_\infty}{\|\overline{\mathbf{\Gamma}}\|_\infty} = \frac{\max_i \overline{\mathbf{b}}_i}{\|\overline{\mathbf{\Gamma}}\|_\infty} \leq \frac{\max_i \overline{\mathbf{b}}_i\sqrt{N+1}}{\|\overline{\mathbf{\Gamma}}\|_2} = \frac{\max_i \overline{\mathbf{b}}_i\sqrt{N+1}}{\sqrt{\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}})}}.$$

If inequality $\frac{\max_i \overline{\mathbf{b}}_i\sqrt{N+1}}{\sqrt{\rho(\overline{\mathbf{\Gamma}}^{\mathbf{T}}\overline{\mathbf{\Gamma}})}} \leq C_0$ holds, then the condition of convexity of the fictitious player will hold. We note that this inequality corresponds to the condition on $v_i$ in Theorem 2.4.

Lastly, we prove that there exists a unique solution under the assumption of diagonal dominance of matrix $\overline{\mathbf{\Gamma}}$. With $a_i > \sum_{j\neq i}\Gamma_{ij}$ , $\omega_F > N$, matrix $\overline{\mathbf{\Gamma}}$ becomes diagonally dominant. From Gershgorin's Theorem [15], it follows that $\overline{\mathbf{\Gamma}}$ is nonsingular and there exists a unique solution to linear system (34).

## REFERENCES


[1] Y. Pan and L. Pavel, "Global convergence of an iterative gradient algorithm for the Nash equilibrium in an extended OSNR game," *Proc. IEEE INFOCOM*, May 2007.

[2] ——, "OSNR optimization in optical networks: Extension for capacity constraints," *Proceedings of 2005 American Control Conference*, pp. 2379–2385, June 2005.

[3] T. Basar and R. Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," *Proc. IEEE Infocom 2002*, pp. 294 – 301, June 2002.

[4] L. Pavel, "An extension of duality to a game-theoretic framework," *Automatica*, vol. 43, no. 2, pp. 226–237, 2007.

[5] M. Osborne and A. Rubinstein, *A Course in Game Theory*. MIT Press, 1994.

[6] M. Osborne, *Introduction to Game Theory*. MIT Press, 1995.

[7] R. Srikant, E. Altman, T. Alpcan, and T. Basar, "CDMA uplink power control as noncooperative game," *Wireless Networks*, vol. 8, p. 659 690, 2002.

[8] L. Pavel, "A noncooperative game approach to OSNR optimization in optical networks," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 848–852, May 2006.

[9] ——, "Hierarchical iterative algorithm for a coupled constrained OSNR nash game," *Proceedings of IEEE GLOBECOM*, November 2006.

[10] M. Osborne and A. Rubinstein, *Bargaining and Market*. Elsevier, 1990.

[11] V. Krishna, *Auction Theory*, 1st ed. Academic Press, 2002.

[12] J. Nash, "Equilibrium points in N-person games," *Proc. Nat. Acad. Sci.*, vol. 36, pp. 48 –49, 1950.

[13] T. Basar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed. Society for Industrial Mathematics, 1987.

[14] R. Horn and C. Johnson, *Matrix Analysis*. Cambridge University Press, 1990.

[15] A. Berman and R. Plemmons, *Nonnegative Matrices in Mathematical Sciences*. SIAM, 1994.

[16] P. Dubey, "Inefficiency of Nash equilibria," *Mathematics of Operations Research*, vol. 11, no. 1, February 1986.

[17] T. Roughgarden and E. Tardos, "Bounding the inefficiency of equilibria in nonatomic congestion games," *Games and Economic Behavior*, vol. 47, pp. 389–403, 2004.

[18] J. Correa, A. Schulz, and N. Stier-Moses, "On the inefficiency of equilibria in congestion games," *IPCO 2005*, pp. 167–181, 2005.

[19] G. Perakis, "The 'price of anarchy' under nonlinear and asymmetric costs," *Operations Research Center Working Papers,MIT*, 2003.

[20] ——, "The price of anarchy when costs are non-separable and asymmetric," *IPCO 2004*, pp. 46–58, 2004.

[21] D. Bertsekas, *Nonlinear Programming*. Athena Scientific, 2003.

[22] C. Saraydar, N. Mandayam, and D. Goodman, "Efficient power control via pricing in wireless data networks," *IEEE Transactions on Communications*, vol. 50, no. 2.

[23] S. Zlobec, *Stable Parametric Programming*. Springer, 2001.

[24] G. Golub and C. V. Loan, *Matrix Computation*. John Hopkins University Press, 1996.

[25] D. Bertsekas and J. Tsitsiklis, *Parallel and Distributed Computation: Numerica Methods*. Athena Scientific, 1997.

[26] R. Ramaswami and K. Sivarajan, *Optical Networks: A Practical Perspective*. Academic Press, 2002.

[27] G. Agrawal, *Lightwave Technology*. Wiley-Interscience, 2005.

[28] L. Pavel, "OSNR optimization in optical networks: Modeling and distributed algorithms via a central cost approach," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 4, pp. 54–65, April 2006.

[29] R. T. F. Forghieri and D. Favin, "Simple model of optical amplifier chains to evaluate penalties in wdm systems," *Journal of Lightwave Technology*, vol. 16, no. 9, p. 1570 1576, 1998.