A SUPPLEMENT ON FIXED POINT THEOREMS AND NONLINEAR DISCRETE DYNAMICS

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1. Fixed Point Theorems

1.1. Some definitions.

Definition 1.1. (Upper Semicontinuity) Let f be a function defined on a normed linear space X, and associating with each $x \in X$ a subset $f(x)$ of some other normed linear space Y. Then f is said to be ***upper semicontinuous*** at a point $x_0 \in X$ if , for any sequence $\{x_i\}$ converging to x_0 and any sequence $\{y_i \in f(x_i)\}$ converging to y_0, we have $y_0 \in f(x_0)$. The function f is ***upper-semicontinuous*** if it is upper semicontinuous at each point of X.

Remark 1.2. The major difference of u.s.c. from continuity is that we have $y_0 \in f(x_0)$ in the definition. Semi-continuity is a property of real-valued functions that is weaker than continuity. An example of upper semi-continuous function is $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. The floor function $f(x) = \lfloor x \rfloor$ is another example of upper semi-continuous function.

Definition 1.3. (**Set-valued Functions**): A set-valued function f is a function from X to subsets of X, denoted as 2^X.

Exercise 1.4. this

Definition 1.5. (**Closed Graph**): The graph of a set-valued function is closed if for every $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ we have that if $x_n \to x$ and $y_n \to y$ then $y \in f(x)$.

Remark 1.6. A graph is a set $\{(x, f(x)) : x \in X\}$

Definition 1.7. (**Quasi-concave function**): A function $f : \mathbb{R}^{\ltimes} \to \mathbb{R}$ is a quasi-concave function if $\forall x_1, x_2 \in X$ and $\alpha \in [0, 1]$, we have $f(\alpha x_1 + (1 - \alpha)x_2) \geq \min\{f(x_1), f(x_2)\}$.

Definition 1.8. (Equicontinuous) Let the space of all real-valued bounded continuous functions on S, denoted $C(S)$, be endowed with the *sup* norm. A subset F of $C(S)$ is ***equicontinuous*** if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - y\| < \delta$ implies $\|f(x) - f(y) < \epsilon\|$, $\forall f \in F$

Remark 1.9. Please note that equicontinuity is defined for a set of functions. One compact way to put the definition of equicontinuity is that
Let $\{f_n\}$ be a sequence of functions from $X \subset \mathbb{R} \to \mathbb{R}$. $\{f_n\}$ is equicontinuous if $\forall \epsilon > 0$, and $x \in X, \exists \delta > 0 (\forall n$, and $x' \in \mathcal{B}^n_\delta, |f(x)_f(x')| \leq \epsilon)$.

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Definition 1.10. **(Lefschetz number)** Let $f : X \to X$ be a continuous map from a compact triangulable space X to itself. A point x of X is a fixed point of f if $f(x) = x$. Define the Lefschetz number Λ_f of f by $\Lambda_f = \sum_{k\geq 0}(-1)^k \mathrm{Tr}(f_*|H_k(Z,\mathbb{Q}))$.

1.2. **Some versions of fixed point theorems.** Fixed-point theorem is one of the most important mathematical tools for economists and mathematicians to prove existence of a solution. One of the oldest fixed-point theorems, Brouwer's Fixed-Point theorem, was developed in 1910 and by 1928, Jon von Neumann used it to prove the existence of a 'minimax' solution to two-agent games. von Neumann in 1937 generalized Brouwer'ss theorem, which later was simplified by Kakutani. John Nash was among the first to use Kakutani's fixed point theorem.

Theorem 1.11. **(Intermediate value Theorem)** *If f is continuous on a closed interval $[a, b]$, and c is any number between $f(a)$ and $f(b)$ inclusive, then there is at lease one number x in the closed interval such that $f(x) = c$.*

Remark 1.12. This theorem can be found in most undergraduate calculus textbook, for example, *Stewart, Single Variable Calculus, Fourth Edition.*It is proven by observing $f([a, b])$ is a connected set because the image of a connected set under a continuous function is connected.

Theorem 1.13. **(Fixed Point Theorem)** *If g is a continuous function $g(x) \in [a, b] \forall x \in [a, b]$, then g has a fixed point in $[a, b]$*

Remark 1.14. This fixed point theorem directly results from the intermediate value theorem above.
Proof:
Suppose that $g(a) \geq a, g(b) \leq b$, i.e.,$g(a)-a \geq 0, g(b)-b \leq 0$. Since g is continuous, the intermediate value theorem guarantees that $c \in [a, b]$ such that $g(c) - c = 0$, or $g(c) = c$.
Reference: http://mathworld.wolfram.com/FixedPointTheorem.html

Theorem 1.15. **(Weiestrass Intermediate Value Theorem)** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, where $[a, b]$ is a non-empty, compact, convex, subset of $\mathbb{R}$ and $f(a) \cdot f(b) < 0$, then there exists a $x^* \in [a, b]$ such that $f(x^*) = 0$.*

Theorem 1.16. **(Banach Fixed Point Theorem, a.k.a., Contraction Mapping Theorem)** *Let (X, d) be non-empty complete metric space. Let $T : X \to X$ be a contraction mapping on X, i.e., there exists a nonnegative real number $q < 1$ such that $d(Tx, Ty) \leq q \cdot d(x, y) \forall x, y \in X$. Then the map T admits **one and only one** fixed point $x^* \in X, Tx^* = x^*$.*

Furthermore, this fixed point can be found as follows:

1. *Start with an arbitrary element $x_0 \in X$.*
2. *Define an iterative sequence by $x_n = Tx_{n-1}$ for $n \in \mathbb{Z}^+$.*
3. *This sequence converges and it limit is x^*.*
4. *The speed of convergence is described by $d(x^*, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0)$.*
5. *The smallest such value of q is sometimes called the **Lipschitz** constant.*

Remark 1.17. It guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces, and provide constructive method to find those fixed points. It was first stated in 1922.

Theorem 1.18. (**Brouwer's Fixed Point Theorem**) *Suppose that $A \subset \mathbb{R}^n$ is a non-empty, compact, convex set, and that $f : A \longrightarrow A$ is a continuous function from A into itself. Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x = f(x)$.*

Remark 1.19. An intuition behind Brouwer's theorem is the following: Consider we have two pieces of paper on top each other. We first take the paper on the top and crush it in a 'continuous' fashion and the if we make the crushed paper flat and put it on the top of the other paper, then Brouwer's fixed point theorem tells that there is at least one point on that crushed paper which remains in the same location as it was before. This example can be extended to 3-D by considering a bottle of water and stirring the water.

Remark 1.20. There are several ways to prove Brouwer's theorem and one of the approaches is to invoke Sperner's Lemma, which is a combinatorial result about labeled simplical subdivisions. This approach also provides insights into computational algorithms for finding approximations to fixed points.
Reference: http://www.math.hmc.edu/funfacts/

Lemma 1.21. (Sperner's Lemma in 2D) *Given a triangle ABC and a triangulation T of the triangle. The set S of vertices of T is colored with three colors in such a way that*

1. *A, B, C are colored 1,2,3 respectively*
2. *Each vertex on an edge of ABX is to be colored only with one of the two colors of the ends of its edge. For example, each vertex AC must have a color either 1 or 3.*

Then, there exists a triangle from T, whose vertices are colored with three different colors.

Theorem 1.22. (An application of Fixed-Point theorem: **Minimax Theorem**)*Let X and Y be mixed strategies for players A and B. Let A be the payoff matrix. Then $\max_X \min_Y X^T AY = \min_Y \max_X X^T AY = v$, where v is called the value of the game and X and Y are called the solutions. If there is more than one optimal mixed strategy, then there are infinitely many.*

Proof: (More to come).

Theorem 1.23. (**Kakutani's Fixed Point Theorem**) *Suppose that $A \subset \mathbb{R}^n$ is a non-empty, compact, convex set and that $f : A \longrightarrow A$ is an upper semi-continuous correspondence from A into itself with the property that the set $f(x) \subset A$ is non-empty and convex for every $x \in A$. Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x = f(x)$*

Remark 1.24. Kakutani's Fixed Point Theorem was first applied to show the existence of Nash equilibrium and the proof of the following theorem is the direct application of Kakutani's Fixed Point Theorem.

Theorem 1.25. (Existence of Nash Equilibrium) *The strategic game $\langle N, (A_i), (u_i)_{i \in N} \rangle$ has a Nash equilibrium if $\forall i$:*

1. *The set A_i if actions is nonempty convex, compact, subsets of Euclidean space.*
2. *The preference relation $\succeq_i$ is continuous.*

(3) *The preference relation* $\succeq_i$ *is quasi-concave on* A_i.

Theorem 1.26. (**Tarsky's Fixed Point Theorem**) *Suppose that* $f : [0,1]^n \longrightarrow [0,1]^n$ *is a nondecreasing function. Then* $f(\cdot)$ *has a fixed point, that is, there exists an* $x \in A$ *such that* $x = f(x)$.

Theorem 1.27. *(*__Schauder's Fixed Point Theorem__*) Let* S *be a bounded subset of* $\mathbb{R}^n$*, and let* $C(S)$ *be the space of real-valued bounded continuous function on* S*, endowed with the* sup *norm. Let* $F \subset C(S)$ *be non-empty, closed, bounded and convex. Then if the mapping* $T : F \to F$ *is continuous and the family* $T(F)$ *is equicontinuous,* T *has a fixed point in* F.

Theorem 1.28. (**Lefschetz Trace Formula**) *Let* K *be a finite complex, let* $h : [K] \to [K]$ *be a continuous map. If* $\Lambda(h) \neq 0$*, then* h *has a fixed point.*

Remark 1.29. It is used when we can't conclude anything about convexity of a function.

1.3. **Application in Systems and Control.**

Lemma 1.30. *Suppose* $\mathcal{B}$ *is a nonempty, closed, bounded, convex set in* $\mathbb{R}^n$ *that is invariant with respect to the system* $\dot{x} = f(x)$*, where f is smooth. Then this system has an equilibrium point in* $\mathcal{B}$.

Remark 1.31. This lemma is taken from lecture notes by B.Francis. It illustrates how Brouwer's Fixed Point theorem can be used in dynamical systems. The set $\mathcal{B}$ is said to be (positively) invariant with respect to a system $\dot{x} = f(x)$ because if $x(0)$ is in $\mathcal{B}$ then $x(t) \in \mathcal{B} \forall t > 0$. The proof starts by letting $\phi(t, x(0)) = x(0) + \int_0^t f(\phi(\tau, x(0)))d\tau$, and then uses the fixed-point theorem to show that $\int_0^t f(\phi(t, q_t)) = 0$ for some q_t. Finally to show that $f(\overline{x}) = 0$ for some $\overline{x}$.

2. Nonlinear Discrete Dynamics

Nonlinear difference equations are often seen in the distributed algorithm to compute for complex behavior of a dynamical systems. Wide applications are found in engineering, economics and computer science. The first-order single difference equations system in the one time variable y_t can be written in the form of $y_{t+1} = f(y)$, where f represents a function that acts as an updating rule for the feasible y in the relevant range. Similar to continuous time trajectories, modeled by ordinary differential equations, the discrete dynamical systems can also be studied or characterized by its phase curves or time path.

2.1. **The Four Types of Time Path.** The four types are:

Type (1) Damped without Oscillations when $0 < \frac{df(y_t)}{dy_t} < 1$.

Type (2) Explosive without Oscillations when $\frac{df(y_t)}{dy_t} > 1$.

Type (3) Damped Oscillations when $-1 < \frac{df(y_t)}{dy_t} < 0$.

Type (4) Explosive Oscillations when $\frac{df(y_t)}{dy_t} < -1$.

There is a second type of phase line forming a limit cycle, which can produce cycles that either explode nor disappear.

2.2. **Notes on Difference Equation.** In set theory, recursion theorem, guarantees that recursively defined functions exist.

Theorem 2.1. *(Recursion Theorem)Given a set X, an element a of X and a function $f : X \to X$, the theorem states that there is a unique function $F : N \to X$ such that $F(0) = a, F(n+1) = f(F(n))$.*

2.3. **On Discrete Lyapunov Equation.**

2.4. **Notes on Stochastic Difference Equations.**

3. Some Results from Matrix Analysis

The reference for this section comes from R. Horn, C.Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

3.1. **On Eigenvalues.**

Theorem 3.1. *(Ostrowski) Let $A = [a_{ij}] \in M_n$, let $\alpha \in [0,1]$ be given and let R'_i and C'_i denote the deleted row and column sums of A, respectively, that is, $R'_i = \sum_{j=1,j\neq i}^n \|a_{ij}\|$,$C'_i = \sum_{j=1,j\neq i}^n \|a_{ji}\|$. Then all the eigenvalues of A are located in the union of n discs $\cup_{i=1}^n \{z \in \mathbb{C} : \|z - a_{ij}\| \leq R_i^{'\alpha} C_i^{'(1-\alpha)}\}$.*

Exercise 3.2. Let's try it on a 2×2 matrix, say $\begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix}$, and compare it with gershgorin disc.

Theorem 3.3. (Courant-Fischer Minimax Theorem) *If $A \in \mathbb{R}^{n\times n}$ is symmetric, then $\lambda_k(A) = \max_{dim(S)=k} \min_{y\neq 0, y\in S} \frac{y^T Ay}{y^T y}$ for $k = 1..n$.*

3.2. **Function of Matrices.** There are a lot of rigorous ways to establish the notion of a matrix function. One of the most elegant approach is in terms of line integral.

Definition 3.4. Suppose $f(z)$ is analytic inside on a closed contour Γ which encircles $\lambda(A)$. We define $f(A)$ be the matrix

$$f(A) = \frac{1}{2\pi i}\oint_\Gamma f(z)(zI - A)^{-1}dz, \text{ or equivalently,}$$
$$[f(A)]_{ij} = \frac{1}{2\pi i}\oint_\Gamma f(z)e_k^T(zI - A)^{-1}e_j dz.$$

We need result from complex variables to calculate this integral, namely, residue calculus.

Exercise 3.5. (**Jordan Block Characterization**) Let $J_i \in \mathbb{C}^{m_i\times m_i}$ to be a Jordan block of the form:$J_i = \begin{pmatrix} \lambda_i & 1 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda_i \end{pmatrix}$, then verify that $f(J_i) =$

$$\begin{pmatrix} f(\lambda_i) & f^{(1)}(\lambda_i) & \cdots & \cdots & \frac{f^{m_i-1}(\lambda_i)}{(m_i-1)!} \\ 0 & f(\lambda_i) & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & f^{(1)}(\lambda_i) \\ 0 & \cdots & \cdots & \cdots & f(\lambda_i) \end{pmatrix}.$$ Use this result to compute e^{At} for $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

This is an alternative way from the one in lecture notes for ECE557.

4. Linear System

4.1. Supplementary Results from Linear Systems.

Theorem 4.1. *(Abel-Jacobi-Liouville Theorem) If Φ is the transition matrix for $\dot{x}(t) = A(t)x(t)$, then* $\det \Phi(t, t_0) = \exp \int_{t_0}^{t} \mathbf{tr} A(\sigma) d\sigma$.

4.2. Controllability and Observability.

The system (A,B) is controllable if every state is reachable. If the controllability matrix $Q_c = [BABA^2B...A^{n-1}B]$ is of rank n, then the system (A,B) is controllable. If the observability matrix$Q_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is rank n ,then the system (C,A) is observable.

If the system is not controllable or observable, then we can use the PBH test to see which eigenvalue/pole give rise to the uncontrollability or unobservability. For a particular λ to be controllable, $rank([A - \lambda IB]) = n$; otherwise it is not. For a particular λ to be observable, $rank\left(\begin{bmatrix} A - \lambda I \\ B \end{bmatrix}\right) = n.$.

Theorem 4.2. (PBH Test) *(A, B) is controllable if and only if $rank([A-\lambda IB]) = n$ for all eigenvalues λ of A. (A, C) is observable if and only if $rank\left(\begin{bmatrix} A - \lambda I \\ B \end{bmatrix}\right) = n$ for eigenvalues λ of A.*

Exercise 4.3. Let SISO system transfer function be $G(s) = \frac{s-1}{s^2-1}$. Put it in the state space form and determine whether the system is controllable or observable for each λ.

Hint: Let $\frac{s-1}{s^2-1} = \frac{v}{u}\frac{y}{v}$, where v is an intermediate variable.

4.3. Riccati Equation.

Theorem 4.4. *If the inner product between two matrices X and P is $\mathbf{tr}P^TX$, then the adjoint differential equation associated with $\dot{X}(t) = A_1(t)X(t) + X(t)A_2(t)$ (LM)is $\dot{P}(t) = -P(t)A_2^T(t) - A_1^T(t)P(t)$ (MA).*

Theorem 4.5. *If the eigenvalues of A have negative real parts, then $A^TP + PA = -Q$ can be solved for Q and the solution will be unique. Moreover, under this same hypothesis, P^* is given by the convergent integral $P^* = \int_0^\infty e^{A^Tt}Qe^{At}dt$.*

Proof: Let $\frac{d}{dt}(e^{A^Tt}Qe^{At}) = A^Te^{A^Tt}Qe^{At} + e^{A^Tt}Qe^{At}A$ and notice that $A^TP^* + P^*A = \int_0^\infty A^Te^{A^Tt}Qe^{At}dt + \int_0^\infty e^{A^Tt}Qe^{At}Adt = \int_0^\infty \frac{d}{dt}(e^{A^Tt}Qe^{At})dt = e^{A^Tt}Qe^{At}|_0^\infty = -Q$. The integral converges as the matrix A is Hurwitz, all the eigenvalues are negative in real parts. Next, to show the uniqueness of the solution, let's define an operator $L : \mathbb{R}^{(n^2)} \to \mathbb{R}^{(n^2)}$ such that $L(P) = A^TP + PA$. For every Q, there is a P^*; thus the range space of this mapping is n^2 and the null space is $\{0\}$.

In undergraduate level differential equation class, we might have acquainted ourselves with the Riccati types of equations. One common type of Riccati equation is given by $\frac{dy}{dx} = f(x, y)$. If we approximate $f(x, y) = P(x) + Q(x)y + R(x)y^2 + ...$, then we have an equation of the following: $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$.Suppose we know one of its particular solution y_1 and let the new function $z = \frac{1}{y-y_1}$,the ODE can be reduced to $\frac{z}{x} = -(Q(x) + 2y_1R(x))z - R(x)$, which is a first order and we can solve easily.

Exercise 4.6. Evaluate the integral $\int_0^{2\pi} \sin(\sin(t))\cos(\sin(t))dt$.

4.4. **Linear Time-Varying System.** The general phase flow of linear time-varying (LTV) system of the form $\dot{x}(t) = A(t)x(t)$ is given by $\Phi(t, t_0) = I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2d\sigma_1 + ...$ This expression is called the Peano-Baker series and, for linear time invariant system, it can simply reduced to e^{At}.

Theorem 4.7. *All solutions of the periodic equation* $\dot{x}(t) = A(t)x(t); A(t + T) = A(t)$*; approach zero as t approaches infinity if the zeros* $\det[Is - \Phi(t_0 + T, t_0)]$ *lie in the disk* $|s| < 1$*. Bounded if* $|s| \le 1$*.*

Exercise 4.8. Find the phase flow for the system $\dot{\mathbf{x}}(t) = \begin{pmatrix} t & 1 \\ 1 & t^2 \end{pmatrix} \mathbf{x}(t)$, and characterize its stability.

Definition 4.9. A transformation $z(t) = L(t)x(t)$ is a Liapunov transformation if

(1) L has a continuous derivative on the interval $(-\infty, \infty)$.
(2) L and $\dot{L}$ are bounded on the interval $(-\infty, \infty)$.
(3) there exists a constant m such that $0 < m \le |\det L(t)|, \forall t$.

If time varying linear system can be transformed into time invariant systems by means of Liapunov transformation, then it is called reducible.

Exercise 4.10. Show that $\dot{x}(t) = e^{At}Be^{At}x(t)$ is reducible.

4.5. **Periodic Homogeneous System.** We will look at a special type of time-varying system with periodicity. Consider $\dot{x}(t) = A(t)x(t), A(t + T) = A(t)$ and let's define matrix $P^(-1) = \Phi(t, 0)e^(-Rt)$, where R is a solution to $\Phi(T, 0) = e^{RT}$.

Theorem 4.11. *(Floquet-Liapunov) If* $A(t + T) = A(t)$*, then the associated transition matrix can be written as* $\Phi(t, t_0) = P^{-1}(t)e^{R(t-t_0)}P(t_0)$*.*

Exercise 4.12. Show that $P^{-1}(t + T) = P^{-1}(t)$ where P is what defined above.

Theorem 4.13. *A linear time varying periodic system is reducible in the sense of Lyapunov.*

In class, we have seen the application of a low pass filter by averaging the time-varying matrix over the entire period. Please consult the reference: V.Solo, O*n the stability of Slowly Time-Varying Linear Systems*, Math. Control Signals Systems, 1994, page 331-350.

5. Notes on Real and Function Analysis

5.1. **Topological Spaces.** A collection of all open subsets of X, denoted as $\mathcal{T}$, has the following properties:

(1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
(2) The union of any members of $\mathcal{T}$ is a member of $\mathcal{T}$
(3) The intersection of **finitely** many members of $\mathcal{T}$ is a member of $\mathcal{T}$.

With with, one can define a topological space $(X, \mathcal{T})$ to be a set X and a collection $\mathcal{T}$ of subsets of X, satisfying the aforementioned axioms. The set $\mathcal{T}$ is a topology for X.

Definition 5.1. (Metric Space) A set X whose elements we shall call points, is said to be metric space if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q, such that

(1) $d(p, q) > 0$ if $p \neq q; d(p, p) = 0$.
(2) $d(p, q) = d(q, p)$.
(3) $d(p, q) \leq d(p, r) + d(r, q), \forall r \in X$

Exercise 5.2. Show that a metric space is a topological space.

Definition 5.3. (Function space $C[a, b]$) X is taken as the set of all real-valued functions $x, y, ...$ of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$.

Definition 5.4. (Hilbert Space l^2) with the metric defined by $d(x, y) = \sqrt{\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2}$ (To be completed)

Definition 5.5. (Banach Space) Banach space is a **complete** vector space with a norm defined on it.

5.2. **Padé Appromixation.** Padé approximant is the 'best' approximation of a function by a rational function of given order. It often gives better approximation of the function than truncating its Taylor series, and it my still work where the Taylor series does not converge. Given a function $f(x)$ and two integers $m \geq 0$ and $n \geq 0$, the Padé appromixmant of order (m, n) is the rational function: $R(x) = \frac{p_0 + p_1 x + p_2 x^2 + ... + p_m x^m}{1 + q_1 x + q_2 x^2 + ... + q_n x^n}$,and we have the conditions that $f(0) = R(0), f'(0) = R'(0), f''(0) = R''(0), ... f^{(m+n)}(0) = R^{(m+n)}(0)$ to determine the coefficients.

Exercise 5.6. Find the Padé approximation of $H(s) = e^{-\tau s}$of order (1,1). (Notes: $H_{Pade}(s) = \frac{-\tau s/2+1}{\tau s/2+1}$ by solving a system of equations of (1) $p_1 - q_1 = -\tau$; (2) $2q_1^2 - 2p_1 q_1 = \tau^2$).

6. Brockett's Theorem

The following is taken from the reference of RW Brockett, "Asymptotic stability and feedback stabilization" *Differential Geometric Control Theory*, 1983

Theorem 6.1. *(Brockett's Theorem)*$\dot{x} = f(x, u)$ *be given with* $f(x_o, 0) = 0$ *and* f *is continuously differentiable in a neighbourhood of* $x(x_o, 0)$. *A necessary condition for the existence of a continuously differentiable control law which makes* $(x_o, 0)$ *asymptotically stable is that:*

(1) *The linearized system should have no uncontrollable modes associated with eigenvalues whose real part is positive.*
(2) *There exists a neighbourhood* N *of* $(x_o, 0)$ *such that for each* $\xi \in N$ *there exists a control* u_ξ *defined in* $[0, \infty)$ *such that this control steers the solution of* $\dot{x} = f(x, u_\xi)$ *form* $x = \xi$ *at* $t = 0$ *to* $x = x_o$ *at* $t = \infty$.
(3) *The mapping* $Y : (x, u) \to f(x, u)$ *should be onto an open set containing 0.*

We can use the Brockett's theorem and check at each equilibrium point to see whether its linearization with the input can give the eigenvalue whose real part is smaller than 0. An example will be brockett's integrator.

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