Summary from Saturday Reading Group on Optimization
January 2007

1 Summary of Necessary/Sufficient Conditions

1.1 Lagrange Multiplier Theorem: Necessary Conditions

Nonlinear Program:

$$\min_{x \in F} f(x) s.t. h_i(x) = 0, i = 1, ..., m$$

Assume that the constraint gradients are $\nabla h_1(x^*), \nabla h_2(x^*), ..., \nabla h_m(x^*)$ are linearly **independent**. If x^* is a local minimum of f subject to $h(x) = 0$, then there exists a **unique** vector $\lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_m^*)$,called Lagrange multiplier vector such that

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) = 0.$$

In addition, if f and h are twice continuously differentiable, we have

$$y^T(\nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(x^*))y \geq 0 \forall y \in V(x^*),$$

where

$$V(x^*) = \{y | \nabla h_i(x^{*T} y = 0, i = 1, ..., m\}$$

From now on, the following function is defined as $L(x, \lambda)$, the Lagrangian function:

$$L(x, \lambda) = f(x^*) + \sum_{i=1}^{m} \lambda_i^* h_i(x^*)$$

1.2 Lagrange Multiplier Theorem: Second-order Sufficient Condition

Nonlinear Program:

$$\min_{x \in F} f(x) s.t. h_i(x) = 0, i = 1, ..., m$$

Assume that f and h are twice continuously differentiable, and let $x^* \in \mathcal{R}^n and \lambda^* \in \mathcal{R}^m$ satisfy

$$\nabla_x L(x^*, \lambda^*) = 0,$$

$$\nabla_\lambda L(x^*, \lambda^*),$$

$$y^T \nabla^{xx L(x^*,\lambda^*) y>0,}$$

$$\forall y \neq 0 with \nabla h(x^*)^T y = 0.$$

Then x^* is a strict local minimum of f subject to $h(x) = 0$. In fact, there exists scalars $\gamma > 0$ and $\epsilon > 0$ such that

$$f(x) \geq f(x^*) + \frac{\gamma}{2}\|x - x^*\|^2, , \forall x with h(x) = 0 and \|x - x^*\| < \epsilon.$$

References

[1] D.P.Bertsekas, *Nonlinear Programming*, second edition, Athena Scientific(2004).

[2] Reading Group on Optimization `http://individual.utoronto.ca/quanyan/`.

Updated 29 January 2007.