ENGR 371 **Fall 2002**

Chapter 2: Probability

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2.1 Sample space and events

- A statistical experiment has unpredictable outcomes, but all possible outcomes form a set. The set of all possible outcomes of a statistical experiment is called **sample space** and is represented by the symbol S. Members of S are sometimes called *sample points*, or *outcomes*.

  Example 1 In the experiment of flipping a coin $S := \{H, T\}$

  Example 2 In the experiment of tossing a die, we could have $S := \{1, 2, 3, 4, 5, 6\}$ if we are interested in the exact value of the number shown by upper face of the die. If we are interested only in the parity of this number then $S := \{\text{even,odd}\}$.

- Tree diagrams are used to help us visualize sample spaces.

  Exercise: Show the sample space of an experiment where 1. a coin is flipped, 2. IF a head shows up a second coin is flipped ELSE (i.e. if a tail shows up) a die is tossed.

- Sample space is a set. A set can be described in two ways: by listing its elements explicitly (this is known as *extensional* description), or by stating a property that characterizes its elements (*intentional* description).

- An **event** is a subset of the sample space. An event is said to be simple if and only if it consists of exactly one outcome (sample point), and is called compound otherwise.

- All operations and notions of set theory apply to events: complement, intersection, union, mutually exclusive (disjoint).

- Venn diagrams are used for illustration only. They can be used for representation of small sets as well. Results maybe verified using Venn diagrams, but not proved!

2.3 Counting

- **Multiplication rule:** if an operation can be performed in n_1 ways, and for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways:

$$n_1 + n_1 + \cdots + n_1 (n_2 \text{ times }) = n_1 n_2.$$

- Generalized multiplication rule: for k independent operations.

- A **permutation** is an arrangement of all or part of a set of objects. The number of possible ways to arrange k objects from a set of n objects is:

$$P(n, k) := n(n-1)(n-2) \dots (n-k+1)$$

- A **circular permutation** is an arrangement in circle of k objects from a set of n objects. The number of possible ways to make such an arrangement is $P(n,k)/k$.

  To see why, suppose $k=4$ and we are arranging a, b, c, and d in a circle. Observe that all permutations that differ only in a shift will look the same in a circular arrangement. For example, *abcd* looks the same as *dabc*, *cdab*, and *bcda*. Thus each circular permutation corresponds to $k=4$ different ordinary permutations. The total number of circular permutations therefore is:

$$\frac{\text{number of ordinary permutations}}{k} = \frac{P(n,k)}{k}$$

- The number of distinct permutations of n things, of which n_1 are of one kind, n_2 of a second kind,. . ., n_k of the k^{th} kind is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

  Theorem 2.7: the number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth is

$$\frac{n!}{(n_1!n_2!\dots n_r!)}$$

  where $n = n_1 + n_2 + \cdots + n_r$.

- A combination of k objects from a set of n objects is simply a partition with two cells: selected (k objects), and rejected ($n-k$ objects).

2.4 Probability

- For simplicity we assume that the sample space is finite.

- To each sample point s we assign a nonnegative number $0 \le P(s) \le 1$ to indicate the likelihood of the simple event s, such that $\sum_{s\in S} P(s) = 1$. Then the probability of event A is defined as:

$$P(A) = \sum_{s\in A} P(s)$$

  Example 3 Suppose a sample space contains N elements, all of which are equally likely to occur, i.e. $P(s) = \text{constant} = w$ (say). Then according to the above property we must have:

$$\sum_{s\in S} P(s) = \sum_{s\in S} w = w \sum_{s\in S} 1 = wN = 1$$

  which gives $w = 1/N$. Now suppose an event A contains n elements. Then $P(A)$ would be:

$$P(A) = \sum_{s\in A} P(s) = \sum_{s\in A} w = w \sum_{s\in A} 1 = \frac{1}{N} n = \frac{n}{N}$$

2.5 Additive rules

The only rules that you need to remember is:

If A and B are disjoint (i.e. $A \cap B = \emptyset$) then $P(A \cup B) = P(A) + P(B)$.

It follows that in general:

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

2.6 Conditional probability

- The probability that an event B occurs given that another event A has already occurred is called *conditional probability of B given A*, denoted by $P(B|A)$. The sample space in this case is reduced from S to A:
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$
- Conditional probability could be larger or smaller than the unconditional probability. Extreme cases:
  1. When A and B are disjoint, $P(B|A) = 0$.
     Example. A: it is rainy, B: it is sunny. What is the likelihood that it is sunny today if it rained?
  2. When $A \subseteq B$, $P(B|A) = 1$.
     Example. A: it is rainy, B: it is cloudy. What is the chance of today being cloudy if it rained?
- Two events A and B are *independent* if
$$P(B|A) = P(B) \quad [\text{or equivalently}, P(A|B) = P(A)].$$

2.7 Multiplicative rules

$P(A \cap B) = P(B)P(A|B)$: probability that both A and B occur is equal to the probability that B occurs, multiplied by the probability that A occurs, given that B has occurred.

2.8 Bayes' rule

Suppose the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space such that $P(B_i) \neq 0$ for all $i = 1, 2, \ldots, k$.

- Theorem of total probability, or rule of elimination:
$$P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$
- Bayes' theorem. For $r = 1, 2, \ldots, k$ we have:
$$P(B_r|A) = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$