CONDITIONAL MEANS AND VARIANCES, PART III: CONDITIONAL VARIANCES M 384G/374G

Marginal Variance: The *definition* of the (population) (marginal) variance of a random variable Y is

$$\mathrm{Var}(Y) = E([Y - E(Y)]^2\ )$$

What does this say in words (and pictures)?

There is another *formula* for Var(Y) that is sometimes useful in computing variances or proving things about them. It can be obtained by multiplying out the squared expression in the definition:

$$\mathrm{Var}(Y) = E([Y - E(Y)]^2\ ) = E(Y^2 - 2YE(Y) + [E(Y)]^2)$$

$$= \underline{\hspace{10cm}}$$

(Fill in details, and say the final result in words!)

Conditional Variance: Similarly, if we are considering a conditional distribution Y|X, we define the *conditional variance*

$$\mathrm{Var}(Y|X) = E([Y - E(Y|X)]^2 \mid X)$$

(Note that *both* expected values here are conditional expected values.)

What does this say in words (and pictures)?

Exercise: Derive another formula for the conditional variance, analogous to the second formula for the marginal variance. (And say it in words!)

Conditional Variance as a Random Variable: As with E(Y|X), we can consider Var(Y|X) as a random variable. For example, if Y = height and X = sex for persons in a certain population, then Var(height | sex) is the variable which assigns to each person in the population the variance of height for that person's sex.

Expected Value of the Conditional Variance: Since Var(Y|X) is a random variable, we can talk about its expected value. Using the formula $\text{Var}(Y|X) = E(Y^2|X) - [E(Y|X)]^2$, we have

$$E(\text{Var}(Y|X)) = E(E(Y^2|X)) - E([E(Y|X)]^2)$$

We have already seen that the expected value of the conditional expectation of a random variable is the expected value of the original random variable, so applying this to Y^2 gives

$$(*) \qquad E(\text{Var}(Y|X)) = E(Y^2) - E([E(Y|X)]^2)$$

Variance of the Conditional Expected Value: For what comes next, we will need to consider the variance of the conditional expected value. Using the second formula for variance, we have

$$\text{Var}(E(Y|X)) = E([E(Y|X)]^2) - [E(E(Y|X))]^2$$

Since $E(E(Y|X)) = E(Y)$, this gives

$$(**)\text{Var}(E(Y|X)) = E([E(Y|X)]^2) - [E(Y)]^2.$$

Putting It Together:

Note that (*) and (**) both contain the term $E([E(Y|X)]^2)$, but with opposite signs. So adding them gives:

$$E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)) = E(Y^2) - [E(Y)]^2,$$

which is just Var(Y). In other words,

$$(***) \quad \text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)).$$

In words: The marginal variance is the sum of the expected value of the conditional variance and the variance of the conditional means.

Consequences:

I) This says that two things contribute to the marginal (overall) variance: the expected value of the conditional variance, and the variance of the conditional means. (See

Exercise) Moreover, Var(Y) = E(Var(Y|X)) if and only if Var(E(Y|X)) = 0. What would this say about E(Y|X)?

II) Since variances are always non-negative, (***) implies

$$\mathrm{Var}(Y) \geq E(\mathrm{Var}(Y|X)).$$

III) Since Var(Y|X) ≥ 0, E(Var(Y|X)) must also be ≥ 0. (Why?). Thus (***) implies

$$\mathrm{Var}(Y) \geq \mathrm{Var}(E(Y|X)).$$

Moreover, Var(Y) = Var(E(Y|X)) if and only if E(Var(Y|X)) = 0. What would this imply about Var(Y|X) and about the relationship between Y and X?

IV) Another perspective on (***) (cf. Textbook, pp. 36 - 37)

i) E(Var(Y|X) is a weighted average of Var(Y|X)

ii) $\mathrm{Var}(E(Y|X) = E([E(Y|X) - E(E(Y|X))]^2)$

$= E([E(Y|X) - (E(Y)]^2)$,

which is a weighted average of $[E(Y|X) - (E(Y)]^2$

Thus, (***) says that Var(Y) is a weighted mean of Var(Y|X) plus a weighted mean of $[E(Y|X) - (E(Y)]^2$ (and is a weighted mean of Var(Y|X) if and only if all conditional expected values E(Y|X) are equal to the marginal expected value E(Y).)