Economics 210
Handout # 2 Jointly Distributed Random Variables

Consider the random experiment flip a coin three times. Suppose that with each outcome we associate two numbers X = the number of heads, and Y = the number of changes in the sequence. For example, with the outcome HHH we associate X = 3 and Y = 0, with the outcome HTH we associate X = 2 and Y = 2, and so forth. X and Y are called **jointly distributed random variables.**

Joint Probability Density Function

The joint probability density function of two jointly distributed random variables is defined as follows. **P(X, Y)** is the probability that X takes on the value X and Y takes on the value Y. For example, P(2,1) = the probability that X = 2 and Y = 1. P(2,1) = the sum of the probabilities of the outcomes with which we have associated the pair of numbers X = 2 and Y=1. For the coin flipping example P(2,1) = 2/8 = the sum of the probabilities of the outcomes HHT and THH. We can represent a joint probability density function with a table or a mathematical expression.

For the coin flipping experiment the joint probability density function can be represented by the following table. As an exercise you should verify all the values in this table.

X \ Y	**0**	**1**	**2**
0	1/8	0	0
1	0	2/8	1/8
2	0	2/8	1/8
3	1/8	0	0

Marginal Probability

$$P(X) = \sum_{Y} P(X,Y) \qquad P(Y) = \sum_{X} P(X,Y)$$

This simply means to find the probability that X takes on a specific value we sum across the row associated with that value. To find the probability that Y takes on a specific value we sum the column associated with that value. Note that a marginal probability density function is just a probability density function, a concept with which you are already familiar.

Conditional Probability

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} \qquad P(Y|X) = \frac{P(X,Y)}{P(X)}$$

For example: $P(X = 2|Y = 1) = \dfrac{P(2,1)}{P(Y = 1)} = \dfrac{2/8}{4/8} = 1/2$

We can construct a conditional probability density function. A conditional probability density function is itself a probability density function which means that it has a mean and variance and the probabilities sum to 1.

Conditional Expectation

$$E(X|Y) = \sum_X XP(X|Y)$$

Independence: X and Y are independent if

$$P(X|Y) = P(X) \text{ for all X,Y pairs or equivalently}$$
$$P(X,Y) = P(X)P(Y) \text{ for all X,Y pairs.}$$

(Note: A simple way to check whether two random variables are independent is to check whether the probability in each cell is equal to the product of the associated marginal probabilities)

Expected value of a function of X and Y. Let $G(X,Y)$ be a function of X and Y. Then

$$E[G(X,Y)] = \sum_X \sum_Y G(X,Y)\,P(X,Y)$$

Covariance: The covariance of two random variables X and Y

$$\sigma_{XY} = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$
$$= \sum_X \sum_Y [(X - E(X))(Y - E(Y))P(XY)] = \sum_X \sum_Y XYP(X,Y) - E(X)E(Y)$$

Correlation: The correlation between two random variables X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \qquad -1 \leq \rho_{XY} \leq +1$$

The relationship between the concepts of correlation and independence. Correlation is a measure of linear dependence between two variables. If two variables are uncorrelated, there is no linear dependence between them but there may dependence of another sort. So uncorrelated does **not** imply independence. However if two variables are independent there is no dependence between them, linear or otherwise. So independence does imply uncorrelated. Conversely, if two variables are dependent they are not necessarily correlated, but if they are correlated they are dependent.

Linear combinations of two or more random variables. Suppose that X and Y are two random variables and that Z = a X + b Y.

$$E(Z) = aE(X) + bE(Y)$$
$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$$

These results can readily be extended to linear combinations of three on more random variables. For example if Z = aX + bY + c W, then

$$E(Z) = aE(X) + bE(Y) + cE(W)$$
$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + c^2\sigma_W^2 + 2ab\sigma_{XY} + 2ac\,\sigma_{XW} + 2bc\,\sigma_{YW}$$

In addition **a linear combination of normally distributed random variables is normally distributed.**

Continuous Jointly Distributed Random Variables

The formulas in this handout use summations. If we are dealing with continuous random variables you should substitute the appropriate formula using integrals. For example, if X and Y are continuous jointly distributed random variables.

$$E[G(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(X,Y)\,P(X,Y)\,dX\,dY$$

Double Summations

Many of the formulas in this handout make use of double summations. This sections presents information which you will need to evaluate double summations. There are three principle methods used to calculate double summations. We will refer to these methods as the definitional method (I), the inside-out method (II) and the summation over pairs method (III).

$$(\text{I.})\quad \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m} x_{1j} + \sum_{j=1}^{m} x_{2j} + \ldots + \sum_{j=1}^{m} x_{nj}$$

$$= (x_{11} + x_{12} + \ldots x_{1m}) + (x_{21} + x_{22} + \ldots x_{2m}) + \ldots + (x_{n1} + x_{n2} + \ldots x_{nm})$$

$$(\text{II.})\quad \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n} (x_{i1} + x_{i2} + \ldots + x_{im})$$

$$= (x_{11} + x_{12} + \ldots x_{1m}) + (x_{21} + x_{22} + \ldots x_{2m}) + \ldots + (x_{n1} + x_{n2} + \ldots x_{nm})$$

$$(\text{III.})\quad \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{\text{all ij pairs}} x_{ij}$$

As an example let us evaluate $\sum_{x=1}^{3}\sum_{y=1}^{2} xy^2$

$$\text{I.}\quad \sum_{x=1}^{2}\sum_{y=1}^{3} xy^2 = \sum_{y=1}^{3} 1y^2 + \sum_{y=1}^{3} 2y^2 = (1 + 4 + 9) + (2 + 8 + 18) = 42$$

$$\text{II.}\quad \sum_{x=1}^{2}\sum_{y=1}^{3} xy^2 = \sum_{x=1}^{2} x + 4x + 9x = \sum_{x=1}^{2} 14x = 14 + 28 = 42$$

$$\text{III.}\quad \sum_{x=1}^{2}\sum_{y=1}^{3} xy^2 = \sum_{\text{all xy pairs}} xy^2 = 1\cdot 1^2 + 1\cdot 2^2 + 1\cdot 3^2 + 2\cdot 1^2 + 2\cdot 2^2 + 2\cdot 3^2$$

$$= 1 + 4 + 9 + 2 + 8 + 18 = 42$$