The Bernoulli Random Variable

- Suppose a random experiment can have one of two outcomes which we will label success and failure.

- The sample space of the experiment is then

$$S = \left\{ \text{success}, \text{failure} \right\}$$

- We can consider the random variable X defined by

$$\begin{aligned} X(\text{failure}) &= 0 \\ X(\text{success}) &= 1 \end{aligned}$$

- The random variable X is called a Bernoulli random variable.

- We shall often refer to the experiment giving rise to a Bernoulli random variable as a Bernoulli Trial.

- Suppose that P(success) $= p$.

- The probability mass function of X is then given as

$$\begin{aligned} p(0) &= 1 - p \\ p(1) &= p \end{aligned}$$

- An alternative way to write this which will be useful is

$$p(x) = p^x(1-p)^{1-x} \quad x = 0, 1.$$

Theorem 2.6
If X is a Bernoulli random variable then

$$\mathsf{E}[X^r] = p \quad \textit{for any } r > 0.$$

Corollary 2.6.1
The mean and variance of a Bernoulli random variable are

$$E[X] = p \qquad \mathrm{Var}(X) = p(1-p).$$

The Binomial Random Variable

- Suppose that we repeat the same Bernoulli trial n times independently of each other.

- We may then be interested in counting the number of successes which occur in the n trials.

- This random variable is called a binomial random variable.

- We shall generally write $X \sim \text{binomial}(n, p)$ or $X \sim \text{bin}(n, p)$.

- The quantities n (number of trials) and p (probability of success in each trial) are called the parameters of the distribution.

- n must be a positive integer and p must be in the interval $[0, 1]$.

- The support of a binomial random variable is

$$\mathcal{X} = \{0, 1, \ldots, n\}.$$

Theorem 2.7

Suppose that X is a binomial random variable with parameters n and p. The probability mass function of X is given by

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n.$$

- Note that a Bernoulli random variable is just a binomial random variable with $n = 1$.

- The random variable is called after the Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

- This theorem also shows that the binomial probability mass function sums to 1.

Theorem 2.8

Suppose X is a binomial random variable with parameters n and p. The mean and variance of X are

$$\begin{aligned} \mathsf{E}[X] &= np \\ \mathsf{Var}(X) &= np(1-p) \end{aligned}$$

Theorem 2.9

Suppose that X is a binomial random variable with parameters n and p. Then

$$\mathsf{P}(X = x+1) = \left(\frac{p}{1-p}\right)\left(\frac{n-x}{x+1}\right)\mathsf{P}(X = x) \quad x = 0, 1, \ldots, n-1.$$

- This recursive formula makes it easier to calculate the cumulative distribution function of the binomial random variable.

$$F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} \qquad x = 0, 1, \ldots, n.$$

- We can also see from this formula that the binomial distribution is unimodal.

The Geometric Random Variable

- Suppose that we continue performing independent Bernoulli trials until we observe a success.

- The number of trials needed is called a geometric random variable.

- The support of a geometric random variable is the set of all positive integers.

- A geometric random variable has only one parameter, the success probability p.

- We shall write $X \sim \text{geom}(p)$.

Theorem 2.10

Suppose that X is a geometric random variable with success probability p. The probability mass function of X is

$$p(x) = p(1-p)^{x-1} \qquad x = 1, 2, \ldots$$

Theorem 2.11

The mean and variance of a geometric random variable with success probability p are given by

$$\begin{aligned} \mathsf{E}[X] &= \frac{1}{p} \\ \mathsf{Var}(X) &= \frac{1-p}{p^2} \end{aligned}$$

The Negative Binomial Random Variable

- Suppose we stop conducting independent Bernoulli trials only when we observe r successes.

- The number of trials required is a negative binomial random variable.

- A negative binomial random variable has two parameters: r the number of successes required and p the success probability on each trial.

- The support of such a random variable is $\{r, r+1, r+2, \ldots\}$.

Theorem 2.12

Suppose that X is a negative binomial random variable with parameters r and p. The probability mass function of X is

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}.$$

Theorem 2.13

The mean and variance of a negative binomial random variable with parameters r and p are given by

$$\begin{aligned} \mathrm{E}[X] &= \frac{r}{p} \\ \mathrm{Var}(X) &= \frac{r}{p}\left(\frac{1}{p} - 1\right) = \frac{r(1-p)}{p^2} \end{aligned}$$

The Poisson Random Variable

- Suppose that events occur randomly over time at a constant rate of λ per unit time.

- We want to find the probability distribution of the number of events that occur in a unit of time.

- We can think of dividing the unit time into n small intervals.

- Let us make the intervals so small that at most one event can occur in each interval.

- Then the number of events in a single interval is a Bernoulli random variable.

- Since events occur randomly we would expect λ/n events per interval.

- Thus the probability of success in each interval is λ/n.

- We shall assume that the number of events to occur in non-overlapping intervals are independent.

- The number of events in the whole period of time is then a binomial random variable with parameters n and λ/n.

- If we let the number of intervals $n \to \infty$ and hence $\lambda/n \to 0$ then we get the Poisson random variable.

Definition 2.6

A random variable X is said to be a Poisson random variable with parameter λ if, and only if, it has the probability mass function

$$p(x) = \frac{\lambda^x \mathrm{e}^{-\lambda}}{x!} \quad x = 0, 1, \ldots$$

Theorem 2.14

Suppose that X is a Poisson random variable with parameter λ. The mean and variance of X are

$$\mathrm{E}[X] = \mathrm{Var}(X) = \lambda$$

- We can use the Poisson distribution as an approximation to calculate binomial probabilities when n is large and p is small. The parameter of the Poisson would be $\lambda = np$.

- We can even use the Poisson approximation when the probabilities of successes are not equal but all are small in which case the Poisson parameter would be the sum of the success probabilities.

- Even when there is a form of weak dependence between the trials, the Poisson approximation can still apply.

- If λ is the average number of events per unit time then the number of events in t units of time is Poisson with parameter λt.