Partial Differential Equations, Winter 2015

Homework #2

Due: Thursday, February 12th, 2015

1. (Chapter 2.1)
   Solve
   \[ u_{xx} + u_{xt} - 20u_{tt} = 0, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \]

   Hint: Factor the operator as we did for the wave equation.

   **Solution:**
   Rewrite the equation in operator form and factor
   \[
   \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial xt} - 20\frac{\partial^2 u}{\partial t^2} = 0
   \Rightarrow \left( \frac{\partial}{\partial x} + 5 \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} \right) u = 0
   \]

   Change variables, let \( \zeta = ax + bt, \eta = cx + dt. \) Then
   \[
   \frac{\partial}{\partial x} = a \frac{\partial}{\partial \zeta} + b \frac{\partial}{\partial \eta},
   \frac{\partial}{\partial t} = c \frac{\partial}{\partial \zeta} + d \frac{\partial}{\partial \eta}.
   \]

   We will choose \( a, b, c, d. \) Add the two above equations as in the factorization,
   \[
   \frac{\partial}{\partial x} + 5 \frac{\partial}{\partial t} = (a + 5b) \frac{\partial}{\partial \zeta} + (c + 5d) \frac{\partial}{\partial \eta},
   \frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} = (a - 4b) \frac{\partial}{\partial \zeta} + (c - 4d) \frac{\partial}{\partial \eta}
   \]

   To simplify our equations take \( b = 1, a = -5 \) and \( d = 1, c = 4. \) \( (\zeta = -5x + t, \eta = 4x + t.) \) Then,
   \[
   \frac{\partial}{\partial x} + 5 \frac{\partial}{\partial t} = 9 \frac{\partial}{\partial \eta},
   \frac{\partial}{\partial x} - 4 \frac{\partial}{\partial t} = -9 \frac{\partial}{\partial \zeta}.
   \]
So our original equation becomes $u_{\zeta,\eta} = 0$. Integrating first by $\zeta$, then $\eta$ we get,

$$u(\zeta,\eta) = f(\zeta) + g(\eta) \tag{1}$$
$$\implies u(x,y) = f(t-5x) + g(4x+t), \tag{2}$$

where $f$ and $g$ are arbitrary functions of one variable. This is the general solution to the problem. To find the particular solution we have to plug in the initial conditions to find formulas for $f$ and $g$ in terms of $\phi$ and $\psi$

$$\phi(x) = u(x,0) = f(-5x) + g(4x) \tag{3}$$
$$\psi(x) = u_t(x,0) = f'(-5x) + g'(4x). \tag{4}$$

Notice that we differentiated equation (2) with respect to $t$ and then evaluated at $t = 0$ to get equation (4). Now differentiating equation (3) with respect to $x$ we get

$$\phi'(x) = -5f'(-5x) + 4g'(4x). \tag{5}$$

Adding and subtracting the right multiples of (5) and (4) gives the equations

$$9g'(4x) = 5\psi(x) + \phi'(x) \tag{6}$$
$$9f'(-5x) = 4\psi(x) - \phi'(x)$$

We integrate each equation with respect to $x$ to get

$$\frac{9}{4}g(4x) = 5 \int_0^x \psi(s)ds + \phi(x) + c_1$$
$$-\frac{9}{5}f(-5x) = 4 \int_0^x \psi(s)ds - \phi(x) + c_2.$$ 

Changing variables gives

$$g(u) = \frac{20}{9} \int_0^{u/4} \psi(s)ds + \frac{4}{9}\phi(u/4) + c_1$$
$$f(u) = -\frac{20}{9} \int_0^{-u/5} \psi(s)ds + \frac{5}{9}\phi(-u/5) + c_2.$$ 

Finally if you plug our results for $g$ and $f$ back into equation (3) you will find that $c_1 + c_2 = 0$. And so we have completely determined the solution,

$$u(x,y) = f(t-5x) + g(4x+t) \tag{6}$$
$$= -\frac{20}{9} \int_0^{-(t-5x)/5} \psi(s)ds + \frac{5}{9}\phi(-(t-5x)/5)$$
$$\quad + \frac{20}{9} \int_0^{(4x+t)/4} \psi(s)ds + \frac{4}{9}\phi((4x+t)/4). \tag{7}$$
2. (Chapter 2.2)
If we take account of air resistance in the wave equation, we have an extra term proportional to the speed,
\[ \rho u_{tt} - Tu_{xx} + ru_t = 0, \quad \text{where } r > 0. \]
Show that the energy of this system,
\[ E = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2) dx \]
is decreasing in time.
\[ \textbf{Solution:} \]
Take \( \frac{\partial}{\partial t} \) inside the integral and use the chain rule to get
\[ \frac{dE}{dt} = \frac{1}{2} \int_{-\infty}^{\infty} dx (2\rho u_t u_{tt} + T u_x u_{xt}) \]
\[ = \int_{-\infty}^{\infty} dx (u_t (Tu_{xx} - ru_t) + Tu_x u_{xt} dx) \]
\[ = \int_{-\infty}^{\infty} Tu_t u_{xx} dx + Tu_x u_{xt} dx - r(u_t^2) dx. \]
Integrate \( u_t u_{xx} \) by parts,
\[ \int_{-\infty}^{\infty} u_t u_{xx} dx = u_t u_x |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{tx} u_x dx. \]
The first term on the right side is zero because we assume \( u(x, t) = 0 \) for \( |x| \geq R \) for some \( R > 0 \). And so using this identity we get
\[ \frac{dE}{dt} = -\int_{-\infty}^{\infty} r(u_t^2) dx < 0. \]

3. (Chapter 2.3)
The purpose of this exercise is to show that the maximum principle is not true for the equation \( u_t = xu_{xx} \) because it has a variable coefficient.
\( (a) \) Verify that \( u(x, t) = -2xt - x^2 \) is a solution.
\( (b) \) Find the location of its maximum in the closed rectangle \( \{-2 \leq x \leq 2, 0 \leq t \leq 1\} \)
(Hint: it’s not on the bottom, left, or right sides of the boundary.)
\[ \textbf{Solution:} \]
\( (a) \) \( u_t = -2x \) and \( u_{xx} = -2 \) and so \( u_t = xu_{xx} \).
(b) To find global max and min, find critical points. \( u_t = 0 = u_x \), which gives \( x = 0, t = 0 \). Apply the second derivative test. \( u_{xx} = -2 < 0 \), and

\[
\text{det} \begin{pmatrix} u_{tt} & u_{tx} \\ u_{xt} & u_{xx} \end{pmatrix} \bigg|_{x=0,t=0} = -4 < 0. \tag{8}
\]

So \( t = 0, x = 0 \) is a saddle point. There are no other critical points so the maxima and minima of \( u \) on the closed rectangle must occur on the boundary.

i. Bottom side: \( -2 \leq x \leq 2, t = 0 \). \( u(x,0) = -x^2 \). Max occurs at \( x = 0 \) and is \( u(0,0) = 0 \).

ii. Left side: \( x = -2, 0 \leq t \leq 1 \). \( u(-2,t) = 4t - 4 \). Increasing so max occurs at \( t = 1 \) and is \( u(-2,1) = 0 \)

iii. Right side: \( x = 2, 0 \leq t \leq 1 \). \( u(2,t) = -4t - 4 \). Decreasing so max occurs at \( t = 0 \) and is \( u(2,0) = -4 \)

iv. Top side: \( -2 \leq x \leq 2, t = 1 \). \( u(x,1) = -x^2 - 2x \). Max occurs at \( x = 1 \) and is \( u(-1,1) = 1 \).

So indeed the max occurs on the top side.

4. (Chapter 2.3)

Prove the comparison principle for the heat equation: if \( u = u(x,t) \) and \( v = v(x,t) \) are two solutions to

\[
w_t = kw_{xx}, \quad \text{on } 0 \leq x \leq l, 0 \leq t \leq T,
\]

and if \( u(x,t) \leq v(x,t) \) for \( t = 0, x = 0, \) and \( x = l \), then \( u(x,t) \leq v(x,t) \) for all \( 0 \leq x \leq l \) and \( 0 \leq t \leq T \). (Hint: maximum principle)

Solution:
Let \( w = u - v \). \( w \) satisfies the diffusion equation. Moreover, \( w(x,0) \leq 0, w(0,t) \leq 0, \) and \( w(l,t) \leq 0 \). So \( w \leq 0 \) on the boundaries \( t = 0, x = 0, x = l \). By the maximum principle \( w \leq 0 \) \( \implies \) \( u \leq v \) on the whole closed rectangle.

5. (Chapter 2.4)

Solve the diffusion equation \( u_t = u_{xx} \) with the initial condition

\[
u(x,0) = \phi(x) = \begin{cases} 1, & |x| < l \\ 0, & |x| \geq l \end{cases}
\]

Write your answer in terms of \( \text{Erf}(x) \).

Solution:
The solution is

\[
u(x,t) = \int_{-\infty}^{\infty} S(x-y)\phi(y)dy,
\]
with $S(u) = \exp \left( -\frac{x^2}{4t} \right) / (2\sqrt{\pi} t)$ since $\kappa = 1$ in the equation. Using the definition of $\phi$,

$$u(x, t) = \frac{1}{2\sqrt{\pi} t} \int_{-\ell}^{\ell} dy \exp \left( -\frac{(x-y)^2}{4t} \right)$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{2\sqrt{t}}}^{\frac{x}{2\sqrt{t}}} du \exp \left( -\frac{u^2}{t} \right),$$

where we have used the change of variable $u = (y-x)/(2\sqrt{t})$. Split the integral into two pieces,

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{\ell-x}{2\sqrt{t}}} du \exp \left( -\frac{u^2}{t} \right) + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\frac{x}{2\sqrt{t}}} du \exp \left( -\frac{u^2}{t} \right).$$

For the second term on the right hand side use the change of variable $s = -u$ and then use the definition of Erf to get,

$$u(x, t) = \frac{1}{2} \left( \text{Erf} \left( \frac{\ell-x}{2\sqrt{t}} \right) + \text{Erf} \left( \frac{\ell+x}{2\sqrt{t}} \right) \right)$$

6. (Chapter 2.4)

Solve the equation $u_t = \kappa u_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that $u_{xxx}$ satisfies the heat equation with initial condition zero. Therefore, $u_{xxx}(x, t) = 0$ for all $x$ and $t$. Integrating this result three times gives $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it is easy to solve for $A$, $B$, and $C$ by plugging into the original problem.

**Solution:** Let $v = u_{xxx}$. Then

$$v_t = (u_{xxx})_t$$

$$= (u_t)_{xxx} \quad \text{by exchange of partials}$$

$$= (\kappa u_{xx})_{xxx}$$

$$= \kappa (u_{xxx})_{xx}$$

$$= \kappa v_{xx}.$$  

So $v$ satisfies the diffusion equation. Moreover, $v(x, 0) = u_{xxx}(x, 0) = 0$. The only solution of the diffusion equation with initial condition zero is the zero function (check). So integrating with respect to $x$ three times,

$$u_{xxx}(x, t) = 0 \implies u(x, t) = A(t)x^2 + B(t)x + C(t),$$

where $A, B, C$ are arbitrary functions of $t$. And so,

$$u_t = A'(t)x^2 + B'(t)x + C'(t),$$

$$u_{xx} = 2A(t).$$
Since $u$ satisfies the diffusion equation $u_t = u_{xx}$, we get
\[ A'(t)x^2 + B'(t)x + C'(t) = 2A(t). \]

Equating the coefficients gives $C'(t) = 2A(t)$, $B'(t) = 0$, $A'(t) = 0$. And so $A(t) = c_1$, $B(t) = c_2$, and $C(t) = 2c_1 + c_3$, where $c_1, c_2, c_3$ are constants. Now put in the initial conditions,
\[ x^2 = u(x, 0) = c_1x^2 + c_2x + 2c_1t + c_3. \]

Equate coefficients again to get $c_1 = 1$ and $c_2 = 0 = c_3$. So $u = x^2 + 2xt$

7. (Chapter 2.4)
Solve the equation
\[ u_t = ku_{xx} - bu, \quad x \in \mathbb{R}, \ t > 0 \text{ with } u(x,0) = \phi(x), \]
where $b > 0$ is a constant. This is called the heat equation with constant dissipation at rate $b$. (Hint: define $v = v(x,t)$ by $u(x,t) = e^{-bt}v(x,t)$ and check what PDE $v$ satisfies. It should be something you are familiar with.)

**Solution:**
Let $v(x,t) = e^{bt}u(x,t)$. Then
\[
v_t = be^{bt}u + e^{bt}u_t = be^{bt}u + e^{bt}(kux_x - bu) \quad \text{by the diffusion equality for } u = \kappa e^{bt}u_{xx}.
\]

Since $v_{xx} = e^{bt}u_{xx}$, we get that $v$ satisfies the usual diffusion equation. Moreover, $v(x,0) = u(x,0) = \phi(x)$. And so
\[
v(x,t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy
\implies u(x,t) = e^{-bt}v(x,t) = e^{-bt}\int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy.
\]