MAT257 RSG 1

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26 September, 2021

1. Recall from last year (and the homework) that there seems to be a certain equivalence between inner products and norms. We will show that this equivalence is actually a bit weaker than it appears.

   (a) For $p \geq 1$, define $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}^+$ as

   $$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

   Show that $\|\cdot\|_p$ is a norm.*
   Note: $\|\cdot\|_p$ are known as the ℓ^p norm.

   (b) Show that this does not hold when $0 < p < 1$.

   (c) Let $\|\cdot\|_\infty$ denote $\lim_{p\to\infty} \|\cdot\|_p$. Show that this is equal to the sup norm (i.e. when $|x| = \sup\{x_i\}$).

   (d) (Axler 6A.18) Prove that $\|\cdot\|_p$ has an associated inner product if and only if $p = 2$ (Hint: there is a certain equality that holds for norms from inner products that does not hold for norms in general).

2. Using the fact that the longest side of a right-angled triangle is the hypoteneuse, prove Cauchy-Schwarz (Hint: Recall, in our proof of Cauchy-Schwarz we considered $|y|^2 x - \langle x, y \rangle\, y$).

3. Let $\mathcal{L}(\mathbb{R}^n)$ denote the space of linear operators on $\mathbb{R}^n$.

   (a) Prove that $S = \{|T(h)| : h \in \mathbb{R}^n, |h| \leq 1\}$ is bounded for all $T \in \mathcal{L}(\mathbb{R}^n)$.

   (b) Define $\|T\|$ to be $\sup S$ (as defined above) for $T \in \mathcal{L}(\mathbb{R}^n)$ which is called the operator norm. Prove that this is indeed a norm on the space of linear operators on $\mathbb{R}^n$.

4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{R}^2$. We say that two norms are equivalent if there exist positive A and B such that $A\|v\|_1 \leq \|v\|_2 \leq B\|v\|_1$ for all $v \in \mathbb{R}^2$. Show that all norms on $\mathbb{R}^2$ are equivalent.*

5. Here we prove that every subset of a compact set is compact.
"Let $T \subset \mathbb{R}^n$ be compact and $S \subset T$ be arbitrary. Let $\{U_\alpha\}$ be an open cover of S. We extend this to an open cover of T by adding in an open set that contains T. By compactness of T, we know there is a finite subcover of T and since S is a subset of T, we also get a finite subcover of S, proving that S is compact."
What is wrong with the above proof?

6. In tutorial, we proved that $A \subset \mathbb{R}^n$ being compact implies that A is closed and bounded. However, we were given what I feel is a rather unsatisfactory proof for the closedness as we used an unfamiliar definition for closedness. So let us do it ourselves:

   Show that A being compact implies that A^c is open. (*Hint:* To each $x \in A^c$ associate a family of open sets that cover A and then use compactness of A to show there is an open set containing x that is contained within A^c.)

7. Show that $A \subset \mathbb{R}^n$ is closed if and only if $\text{Bd}A \subset A$. Do you think the opposite holds for open sets? That is, is it true that $A \subset \mathbb{R}^n$ is open if and only if $\text{Bd}A \subset A^c$?

8. Let $\Delta = \{(x, y) \in \mathbb{R}^2 | x - y = 0\}$. Find the interior, exterior and boundary of Δ.

9. Suppose $A \subset \mathbb{R}^n$ is such that $A = \text{Bd}A$. What can we say about A?

10. Suppose $A \subset \mathbb{R}^n$ is such that $\text{Bd}A$ is open. What can we say about A?

* indicates questions I either have no solution or partial solutions for