MAT257 RSG 2

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1. We say that a collection of subsets $\{A_\alpha\}$ has the *finite intersection property* if any finite intersection of element in the collection is non-empty.

   (a) Give an example of a collection of closed subsets in $X = (0, 1)$ with the finite intersection property such that their intersection is empty (*Note:* A set F is closed in a subset X if $F = X \cap E$ where E is closed in $\mathbb{R}^n$).

   (b) Show that a space X is compact if and only if for any collection $\{F_\alpha\}$ of closed subsets of X with the finite intersection property, we have that

   $$\bigcap_\alpha U_\alpha \neq \phi$$

2. The following is one my favourite statements from linear algebra. We partially answered it in the past two 257 homeworks but we can certainly generalise things.

   Let V be a (possibly infinite dimensional) inner product space and let $T : V \to V$ be a linear operator. Show that the following are all equivalent:

   (a) T is bounded (i.e there exists some $M > 0$ such that $|T(h)| \leq M|h|$)

   (b) T is continuous at 0

   (c) T is continuous at a point

   (d) T is continuous everywhere

3. Professor Dror mentioned that in order to find derivatives of limit points in a non-necessarily-open set you find an extension of the function where you *can* take the derivative and take this to be the differential. This of course requires that all eligible extensions agree on what the derivative should be. Let us formalise this and prove it.

   Let A be a not-necessarily-open subset of $\mathbb{R}^n$ and let $a \in A$ be a limit point. Suppose we are give a map $f : A \to \mathbb{R}^m$. Let U, V be open sets containing a and $g : U \to \mathbb{R}^m$ and $h : V \to \mathbb{R}^m$ such that $f(x) = g(x) = h(x)$ for all $x \in U \cap V \cap A$. Suppose additionally that g and h are differentiable at a. Show that $g'(a) = h'(a)$. Show that it may not be true that $g(x) = h(x)$ for all $x \in U \cap V$.

4. Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then its graph Γ_f is closed in $\mathbb{R}^{2n}$. Does the converse hold?

5. Now that we have continuity, we should show that some of our basic functions are continuous. Show that the projection maps $\pi_i : \mathbb{R}^n \to \mathbb{R}$, given by $\pi_i(x_1, \ldots, x_n) = x_i$ are continuous.

   We also have natural maps going the other way, namely the inclusion maps. Define $\iota : \mathbb{R} \to \mathbb{R}^n$, given by $\iota(x) = (x, 0, \ldots, 0)$. Is ι continuous?

6. (Pederson, *Analysis Now*, E 1.4.6) Given $f : \mathbb{R}^2 \to \mathbb{R}$, we say that f is *separately continuous in each variable* if for every $x', y' \in \mathbb{R}$, the maps $f_{x'}(y) = f(x', y)$ and $f_{y'}(x) = f(x, y')$ are continuous.

   (a) Show if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then f is separately continuous in each variable

   (b) Show that $f : \mathbb{R}^2 \to \mathbb{R}$ being separately continuous on each variable does *not* imply that f is continuous. (*Hint:* Consider `www.math3d.org/OR9Feg3Y`)

7. (Our lord and saviour Saied) Suppose we have a map $f : \mathbb{R} \to \mathbb{R}^n$.

   (a) Show that there exist $f_1 \dots, f_n : \mathbb{R} \to \mathbb{R}^n$ such that

   $$f(x) = (f_1(x), \dots, f_n(x))$$

   (b) Show that f is continuous if and only if all the $f_1, \dots, f_n$ are continuous.

8. Show that there is no continuous map from $\mathbb{R}$ onto S^1 (i.e. surjective map), where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (*Hint:* There's a one line answer!).

9. Show that there is a continuous, bijective function from $\mathbb{R}$ to $S^1 \backslash \{(1, 0)\}$.