MAT257 RSG 12

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Assume any vector space is finite-dimensional and real.

(*Tensor Product*)

1. Let V be a vector space. Let $V \otimes V$ be the tensor product of V with itself (in other, it is the space spanned by elements of the form $v \otimes w$ for $v, w \in V$ where we assert that $\otimes$ is bilinear). What is the dimension of $V \otimes V$? (*Hint:* Choose a basis for V.)

2. Show there exists a 'natural' bilinear map φ from $V \times V$ to $V \otimes V$. (I admit this is a vague question, but I think you will understand what I mean).

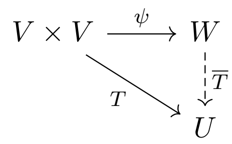
3. Suppose $T : V \times V \to \mathbb{R}$ is a bilinear map. Show there exists a unique linear map $\widehat{T} : V \otimes V \to \mathbb{R}$ such that $T = \widehat{T} \circ \varphi$. (*Hint:* This is easier than it looks! It's just a matter of parsing through the symbols)
P.S. This property is what makes φ natural.

4. Use the above to explain why Dror prefers $\mathcal{T}^2(V^*)$ to denote the set of all bilinear functionals on V.

(*Bonus*)

5. Suppose there exists a pair (W, ψ) where W is a vector is a vector space $\psi : V \times V \to W$ is a bilinear map. Suppose additionally that the pair satisfies the following property: given any vector space U and any bilinear map $T : V \times V \to U$ there exists a unique linear map $\overline{T} : W \to U$ such that $T = \overline{T} \circ \psi$. Show that W is isomorphic to

$V \otimes V$.



(*Hint*: Use Q3)

6. If the tensor product is really a ‘product’, then we should expect $v \otimes 0 = 0$ for all v. Show that this is true.

(*Dual spaces*)

7. Let V, W be vector spaces and $T : V \to W$ a linear map. Then we know there exists a linear map $T^* : W^* \to V^*$ where $T^*(\omega) = \omega \circ T$. Similarly we can have $T^{**} : V^{**} \to W^{**}$. Recall that we have maps $V \to V^{**}$ and $W \to W^{**}$ which we call ι (this is a slight abuse of notation since technically these maps are different as they have different domain and codomain. However, the definition of the map is the exact same in both cases). Show that $\iota \circ T = T^{**} \circ \iota$. In other words show that the following diagram commutes.