MAT257 RSG 12 Sketch Solutions

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1. Dimension is n^2. If $e_1, \dots, e_n$ is a basis for V then $e_i \otimes e_j$ for $1 \leq i, j \leq n$ is a basis for $V \otimes V$.

2. The most natural map one can think of, I hope, is $\varphi(v, w) = v \otimes w$, defined to be bilinear.

3. Define $\widehat{T}(v \otimes w) = T(v, w)$ for reduced tensors (I think that's what they're called?). Extend by linearity. Can check that it works

4. The collection of bilinear functionals on V then corresponds exactly with functionals on $V \otimes V$. The latter is denoted $(V \otimes V)^*$. Without too much difficulty you can see this is isomorphic to $V^* \otimes V^*$ which one would like to denote as $\mathcal{T}^2(V^*)$.

5. Take $U = V \otimes V$ and $T = \varphi$ from before. Show that $\overline{T}$ and $\widehat{\psi}$ are isomorphisms and indeed are inverses of one another.

6. There are many different ways of seeing this fact. I present one way below

$$v \otimes 0 + a \otimes b = v \otimes 0 \cdot b + a \otimes b = (0 \cdot v + a) \otimes b = a \otimes b$$

7. As I have mentioned before, this is more or less an exercise in unpacking definitions. Let $v \in V$. Then

$$T^{**}(\iota(v)) = \iota(v) \circ T^*$$

Let $\omega \in W^*$. Then

$$(\iota(v) \circ T^*)(\omega) = \iota(v)(\omega \circ T) = \omega(Tv) = \iota(Tv)(\omega)$$

Therefore $\iota(v) \circ T^* = \iota(Tv)$ and hence $T^{**} \circ \iota = \iota \circ T$. The important this about this question is how ι not only gives you a correspondence between vector spaces and their double duals but also a correspondence from maps between vector spaces to maps between the double duals of the vector spaces.