MAT257 RSG 13

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February 2022

1. Decide whether the following pull back, push forward, both or neither:
   (a) Sequences (convergent sequences?)
   (b) Polynomials
   (c) Partitions of unity
   (d) Open sets, closed sets, compact sets, dense sets, open and dense sets
   (e) Fundamental groups
   (f) Orthonormal bases
   (g) Linear subspaces
   (h) Vector spaces
   (i) Matrices
   (j) Vector fields

2. Suppose $T : \Lambda^k(V) \to \Lambda^{n-k}(V)$ is a linear map such that
$$\lambda \wedge T\lambda = 0$$
for every $\lambda \in \Lambda^k(V)$. Is it the case that $T = 0$?

3. Show that $S_k(V) \oplus \Lambda^k(V) = \mathcal{T}^k(V)$ if and only $k = 2$.

4. If $L : V \to W$, show that $L^*(\lambda \wedge \eta) = L^*(\lambda) \wedge L^*(\eta)$.

5. If $T : U \to V$ and $S : V \to W$, show that $(S \circ T)^* = T^* \circ S^*$ where by *, we mean the induced map on the k alternating tensors of the appropriate spaces.

6. Prove that the determinant is multiplicative, i.e. show that $\det(AB) = \det(A)\det(B)$.

7. Define a product for symmetric tensors, analogously to the wedge product, which we will call the symmetric product. What is the symmetric product of two linear functionals? Two inner products? What properties does the symmetric product have (associativity? commutativity?)? What properties uniquely characterise the symmetric product?

8. Find the operator norms of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   Find the eigenvalues of the above matrices and see if you can spot a pattern that *mostly* holds.

9. Given arbitrary matrices $A, B \in \mathbb{R}^{n \times n}$, which of the following properties holds for the operator norm. If the statements do not hold true in general, consider under what conditions they may be true (I'll be honest, I don't know all the answers).

   (a) $\|AB\| = \|BA\|$

   (b) $\left\|e^A\right\| = e^{\|A\|}$

   (c) $\|A\| = \left\|T^{-1}AT\right\|$ for every invertible $T \in \mathbb{R}^{n \times n}$