MAT257 RSG 12 Sketch Solutions

Rishibh Prakash

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1. (a) Push
   (b) Push
   (c) Pull
   (d) Pull, pull, push, neither, neither
   (e) Push
   (f) Neither
   (g) Both
   (h) Push
   (i) Push
   (j) Pull
2. Consider the $n = 2k$ and take $T = I$.
3. Check that the dimensions only add up if $k = 2$
4.

$$L^*(\lambda \wedge \eta)(v_1, \ldots, v_{k+n} = (\lambda \wedge \eta)(Lv_1, \ldots Lv_{k+n})$$

and

$$\begin{aligned}(L^*(\lambda) \wedge L^*(\eta))(v_1, \ldots, v_{k+n}) &= \frac{1}{k!n!} \sum_{\sigma \in S_{k+n}} (-1)^\sigma L^*(\lambda)(v_{\sigma(1)}, \ldots, v_{\sigma(k)} L^*(\eta)(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+n)}) \\ &= \frac{1}{k!n!} \sum_{\sigma \in S_{k+n}} (-1)^\sigma \lambda(Lv_{\sigma(1)}, \ldots, Lv_{\sigma(k)} \eta(Lv_{\sigma(k+1)}, \ldots, Lv_{\sigma(k+n)}) \\ &= (\lambda \wedge \eta)(Lv_1, \ldots, Lv_{k+n})\end{aligned}$$

5.

$$\begin{aligned}(S \circ T)^*(\lambda)(w_1, \ldots, w_k) &= \lambda(S(Tw_1), \ldots, S(Tw_k)) \\ &= S^*(\lambda)(Tw_1, \ldots, Tw_k) \\ &= (T^* \circ S^*)(\lambda)(w_1, \ldots, w_k)\end{aligned}$$

6. We know that

$$S^*(\omega) = \det(S)\omega$$

Then

$$\det(AB)\omega = (AB)^*(\omega) = B^*(A^*(\omega)) = \det(B)(A^*(\omega)) = \det(B)\det(A)\omega$$

7. The definition should really be

$$(\lambda \odot \eta) = \sum_{\sigma \in S_{n+k}} \lambda(v_{\sigma(1)}, \dots, v_{\sigma(k)})\eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+n)})$$

where we probably need a scalar to 'normalise' things. Not entirely sure of the properties but I would be shocked if this wasn't associative and commutative.

8. The norms are $1, 3, 3, 1, 4, 1$. For orthogonally diagonalisable matrices, the norm is the largest absolute value of its eigenvalues. It's exactly symmetric matrices that are orthogonally diagonalisable (spectral theorem).

9. (a) Nope, not too hard to find a counterexample

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

   (b) Nope, consider a diagonal matrix with positive and negative entries

   (c) Really feel like this one should be true, but have no proper proof yet.