1. (a) Not sure how to show subadditivity, pretty sure I'm just missing something minor.
   (b) Subadditivity fails, e.g. $|(1,0)| = |(0,1)| = 1$ but $|(1,1)| = (1+1)^2 = 4$
   (c) MAT157 HW16 Q3
   (d) It's the parallelogram equality that fails.

$$\begin{aligned}
\|(1,0)+(0,1)\|^2 + \|(1,0)-(0,1)\|^2 &= 2(\|(1,0)\|^2 + \|(0,1)\|^2) \\
\|(1,1)\|^2 + \|(1,-1)\|^2 &= 2(\|(1,0)\|^2 + \|(0,1)\|^2) \\
(1^p+1^p)^{\frac{2}{p}} + (1^p+(-1)^p)^{\frac{2}{p}} &= 2((1^p+0^p)^{\frac{2}{p}} + (0^p+1^p)^{\frac{2}{p}}) \\
2 \cdot 4^{\frac{1}{p}} &= 2 \cdot 2 \\
4^{\frac{1}{p}} &= 2
\end{aligned}$$

   Thus $p = 2$.

2. It's sufficient to show this holds when x, y are of unit length. This leaves us with $x - \langle x, y\rangle\, y$ gives us an orthogonal decomposition for x. We have a right angle triangle with $\langle x, y\rangle\, y$ as one the sides and x at the hypotenuse. Note $\|\langle x, y\rangle\, y\| = |\langle x, y\rangle|$ which has to smaller than the length of the hypotenuse which is 1.

3. (a) We know from homework that $|T(h)| \leq M|h|$ for some sufficiently large M. Assuming $|h| \leq 1$, we directly get that $|T(h)| \leq M$ as desired.
   (b) MAT247 A4 Q3, simply note that $|(S+T)(x)| \leq |S(x)| + |T(x)|$ and then taking supremum over both sides for $|x| \leq 1$

4. I only have a partial solution for this one so far. I think the easiest way to go about this is to show that every norm is equivalent to the supremum norm. It is clear that the right hand side of the inequality holds.

$$\begin{aligned}
\|v\| &= \left\| \sum_{i=1}^{n} v_i e_i \right\| \\
&= |v_i| \sum_{i=1}^{n} \|e_i\| \\
&\leq \|v\|_\infty \sum_{i=1}^{n} \|e_i\|
\end{aligned}$$

So we just take $B = \sum_{i=1}^{n} \|e_i\|$. Not sure how to do the other direction.

5. This, I would argue, is one of the two good questions in this worksheet (the other one being Q9). The key is that the finite subcover we have at the end might contain the additional open set we threw in to cover T. Hence it tells us nothing about the open cover we started with to cover S and whether finitely many of those cover S.

6. Let $x \in A^c$. Consider the collection $\{\text{ext}(B_{\frac{1}{n}}(x))\}_{n \in \mathbb{N}}$ which covers all of $\mathbb{R}^n \backslash \{x\}$ so in particular covers A. Using compactness we only need finitely many $\text{ext}(B_{\frac{1}{n}}(x))$ to cover A. Since each ball exterior is contained within the next one, we can find a smallest ball around x who exterior contains A. Taking complements tells us that the closure of this ball is contained within A^c. We then just take the interior of the closure (which is of course the open ball itself) as the neighbourhood of x that is contained within A^c.

7. Suppose A is closed and let $x \in \text{Bd}A$. Suppose $x \in A^c$. But A^c being open contradicts x being in the boundary since there is an open set that lies entirely within A^c.

   Now suppose $\text{Bd}A \subset A$. Then every $x \in A^c$ has an open neighbourhood that never intersects A, hence lies entirely within A^c.

   Yes, the opposite does in fact hold. Suppose A is open. Then certainly it cannot contain it's boundary by definition of boundary and open. Then suppose $\text{Bd}A \subset A^c$. Then as before we argue that for every $x \in A$, we can find an open neighbourhood that never intersects A^c hence is entirely within A.

8. Interior is empty, the boundary is itself and exterior is the complement of Δ.

9. The most I have been able to say is that A is closed (see Q6) and that $\text{int}(A) = \phi$. The latter follows from recalling that $\mathbb{R}^n = \text{int}(A) \sqcup \text{Bd}(A) \sqcup \text{ext}(A)$ for any $A \subset \mathbb{R}^n$. ($\sqcup$ is disjoin union). If $A = \text{Bd}(A)$ then $A^c = \text{int}(A) \sqcup \text{ext}(A)$. Clearly A^c cannot contain $\text{int}(A)$ so it must be empty.

10. This I would argue is the only other good question in this list as there's at least some work to be done and it takes to some rather interesting places. We will show that A is such that every open set in $\mathbb{R}^n$ intersects A or A is empty.

First we note that the boundary of a set is always closed since its complement is the union of the interior and exterior which are both open. Thus $\mathrm{Bd}(A)$ is clopen implying that $\mathrm{Bd}A = \mathbb{R}^n$ or $\mathrm{Bd}A = \phi$.

I claim that the only sets with empty boundary are $\mathbb{R}^n$ and ϕ. This is because this implies that $\mathbb{R}^n = \mathrm{int}(A) \sqcup \mathrm{ext}(A)$. This means that $\mathrm{int}(A)^c = \mathrm{ext}(A)$ but $\mathrm{int}(A)$ and $\mathrm{ext}(A)$ are open by definition. Thus we once again have clopen sets. If $\mathrm{int}(A) = \mathbb{R}^n$ then clearly $A = \mathbb{R}^n$ and if $\mathrm{ext}(A) = \mathbb{R}^n$ then $A^c = \mathbb{R}^n$ so $A = \phi$.

Now suppose that $\mathrm{Bd}(A) = \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be any open set. For any $x \in U$, we know that $x \in \mathrm{Bd}(A)$ so we cannot have $U \subset A$ or $U \subset A^c$. Thus U intersects A (and A^c).