MAT257 RSG 8

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1. Show that if U is a non-empty open set, then it does not have measure zero.
2. The above statement can be extended to conclude that if $A \subset \mathbb{R}^n$ contains any open set, then it is not of measure zero. Is the converse true? That is, if A is not of measure zero, does it follow that A necessarily contains some open set(s)? (*Hint*: Consider the contrapositive)
3. Is the (continuous) image of a measure zero set measure zero? The preimage? What if the domain and codomain are of the same dimension?
4. There is at least something we can say about measure zero sets: suppose $A \subset \mathbb{R}^n$ is of measure zero. Show that A^c is dense in $\mathbb{R}^n$ (this means that every open set in $\mathbb{R}^n$ intersects A^c). For completeness' sake show that the converse is not true.
5. Let $A \subset \mathbb{R}^n$ be arbitrary and let $f, g : A \to \mathbb{R}^m$ be continuous functions such that $f \neq g$. Show that the set of points on which they differ cannot have measure zero (that is you cannot have two continuous functions that are the same 'almost everywhere').
6. In the very first RSG, we made some comments about sets $A \subset \mathbb{R}^n$ where $A = \text{Bd}(A)$. Thinking through a few examples, we may conjecture that such sets have measure zero. Even this turns out to be false.

   Consider the following construction of the so-called 'fat Cantor set': Take the unit interval and remove the middle $1/4$. From each of the two remaining intervals remove the middle $1/16$. Continue this process where in step n you remove the middle $1/4^n$ from each of the 2^{n-1} intervals. Note the interior of this set is empty (why?) hence it must be equal to its boundary. Find the total length of the intervals removed to conclude that this set cannot have measure zero.