MAT257 RSG 6

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(*Differentiability*)

1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at some $a \in \mathbb{R}^n$. Show that all partial derivatives at a exist.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that all partial derivatives exist at some point $a \in \mathbb{R}^n$ and moreover these partial derivatives are continuous on some neighbourhood of $a \in \mathbb{R}^n$. Show that f is differentiable at a.

3. Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}^m$ a differentiable function. Suppose there is some $M > 0$ such that $|D_i f_j(x)| \leq M$ for all $x \in \text{int}(R)$ and all i, j. Then show that $|f(x) - f(y)| \leq nmM|x - y|$ for all $x, y \in R$.

(*Inverse Function Theorem*)

4. When proving the Inverse Function Theorem, why is it sufficient to only consider the case when $f'(a) = I$?

5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and such that $f'(a) = I$ for some $a \in \mathbb{R}^n$. Show that given any $M > 0$ one can find a neighbourhood U of a where $|(f(x_1) - f(x_2)) - (x_1 - x_2)|$ is bounded by $M|x_1 - x_2|$ for all $x_1, x_2 \in U$.

6. Show that there exists some open set $W \subset \mathbb{R}^n$ such that for all $y \in W$ there exists a sequence in the image of f that converges to y. Show moreover that this sequence is Cauchy to conclude that there is some $x \in U$ such that $f(x) = y$.

7. Let g be the map defined by the above procedure. That is given some $y \in W$, $g(y)$ is the element in U such that $f(g(y)) = y$. Show that g is 1–1.

8. Show that there is some $N > 0$ such that $|g(y_1) - g(y_2)| \leq N|y_1 - y_2|$ for all $y_1, y_2 \in W$ and conclude that g is continuous.

9. Show that g is differentiable at $f(a)$ (recall we already have a candidate for the differential). Show moreover that g is continuously differentiable at $f(a)$.

10. Use the Inverse Function Theorem to prove the Implicit Function Theorem.

(*Integrability*)

11. Let $f : R \to \mathbb{R}$ be a bounded function, where $R \subset \mathbb{R}^n$ is a rectangle. Show $L(f) \leq U(f)$.

12. Show that $O(f, a)$ (i.e. the oscillation of f at a) is 0 if and only if f is continuous at a.

13. Show that continuous functions are integrable.