MAT257 RSG 7

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(*K-Theory*)

1. Let J be an abelian semi-group. That is J is a set with a commutative and associative operation, often denoted $+$. We define an equivalence relation on $J \oplus J$ where

$$(\xi, \eta) \sim (\xi', \eta') \Leftrightarrow \exists \theta \in J \text{ with } \xi + \eta' + \theta = \xi' + \eta + \theta$$

   Show that $\widehat{J} := J \oplus J / \sim$ is an abelian group. What is $\widehat{\mathbb{N}}$?

2. Let α be an $n \times n$ idempotent matrix, that is $\alpha^2 = \alpha$. Let $[0]_r$ denote the $r \times r$ zero matrix. Verify that $\alpha \oplus [0]_r$ given by

$$\alpha \oplus [0]_r = \begin{pmatrix} \alpha & 0 \\ 0 & 0_r \end{pmatrix}$$

   is also idempotent.

3. Let $\mathbb{F}$ be a field. Let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$. Now let $P_n(\mathbb{F})$ be the set of idempotent $n \times n$ matrices. That is

$$P_n(\mathbb{F}) := \{\alpha \in M_n(\mathbb{F}) | \alpha^2 = \alpha\}$$

   We then take

$$P(\mathbb{F}) = P_1(\mathbb{F}) \cup P_2(\mathbb{F}) \cup \dots$$

   using the inclusion given by the previous question.

   We will say that $\alpha \in P_n(\mathbb{F})$ and $\beta \in P_m(\mathbb{F})$ are *stably similar* if there exists some natural number r and invertible $\gamma \in M_r(\mathbb{F})$ such that $\beta = \gamma^{-1}\alpha\gamma$ (where α and β are extended to be $r \times r$ matrices as given by the previous question). Verify that stable similarity is an equivalence relation.

4. Let $J(\mathbb{F}) = P(\mathbb{F})/(\text{stable similarity})$. Given some $\alpha \in P_n(\mathbb{F})$ and $\beta \in P_m(\mathbb{F})$, we define an addition as follows:

$$[\alpha] + [\beta] = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Show that this addition is independent of the choice of representative. Show that $J(\mathbb{F})$ along with the addition defined above forms an *abelian* semi-group.

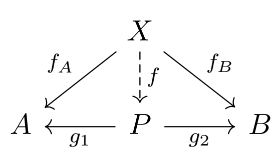
5. What is $\widehat{J(\mathbb{F})}$? This is often denoted $K_0\mathbb{F}$.

(*Universal Properties*)

1. Suppose A is a set such that given any set X, there is exactly one map f from X to A. Show that A is unique up to (some unique) bijection. (*Hint:* What can we say about the cardinality of A?).

2. Suppose A is a set such that given any set X there is exactly one map f from A to X. What is A?

3. Let A and B be sets. Let P be another set with $g_1 : P \to A$ and $g_2 : P \to B$ maps such that given any set X and any pair of maps $f_A : X \to A$ and $f_B : X \to B$, there exists a unique map $f : X \to P$ such that $f_A = g_1 \circ f$ and $f_B = g_2 \circ f$. We can summarise this information in the following diagram:



What is P? (*Hint*: Consider what happens if we take X to be a singleton set)

(*Nets and convergence*)

1. Suppose X is a metric space. Show that the topology on X is completely characterised by the family of convergent sequences in X. That is, if $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X arising from metrics d and d' such that a sequence converges in $\mathcal{T}$ if and only if it converges in $\mathcal{T}'$ then $\mathcal{T} = \mathcal{T}'$. (*Hint*: Recall the Sequence Lemma)

2. Show that the above does not hold if X is not a metric space (*Hint*: Consider the convergent sequences of $\mathbb{R}$ endowed with the countable complement topology).

3. We say that a set Λ is upward directed if it has a preorder, $\leq$ (i.e. a binary relation that is reflexive and transitive) and if for every $\lambda, \mu \in \Lambda$, there exists some $\theta \in \Lambda$ such that $\lambda \leq \theta$ and $\mu \leq \theta$.

   Let X be a topological space. Choose some $x \in X$ and let $\mathcal{O}(x) := \{U \in \mathcal{T} | x \in U\}$. Show that $\mathcal{O}(x)$ with $\subseteq$ is an upward directed set. Show that the same is true, if we consider the relation $\supseteq$.

4. A net in a topological space X is a pair (Λ, i) where Λ is an upward directed set and i is a map from Λ to i. These are often denoted $(x_\lambda)_{\lambda \in \Lambda}$ where $x_\lambda = i(\lambda)$.

   We say that a net *converges* to some point $x \in X$ if for every open neighbourhood U of x, there is some $\lambda' \in \Lambda$ such that for every $\lambda \geq \lambda'$ we have $x_\lambda \in U$. Show that if A is any subset of X then $x \in \overline{A}$ if and only if there exists a net in A converging to x. (*Hint*: Consider $\Lambda = \mathcal{O}(x)$). Conclude that a topological space is completely characterised by the family of convergent nets in it