1. (a) Let $A = [0, 1) \cup [2, 3]$ and

$$f(x) = \begin{cases} x, & x \in [0,1) \\ x - 1 & x \in [2,3] \end{cases}$$

   (b) We just have to show that preimages of closed sets under f^{-1} are closed. Suppose $F \subset A$ is closed. Since F is a closed subset of a compact set, it must be compact. Then $(f^{-1})^{-1}(F) = f(F)$ which is compact since f is continuous. Thus in particular it is closed.

2. Use Cauchy-Schwarz for each component

3. (a) When calculating $D_i D_j f(a)$ only two coordinates are not being held constant so we only need the consider the case for $\mathbb{R}^2$.

   Define $\phi : [a, a+h] \to \mathbb{R}$, given by $\phi(t) = f(t, b)$. Then by MVT there exists some $q_1 \in (a, a+h)$ such that $h \cdot D_1 f(q_1, b) = f(a + h, b) - f(a, b)$. Then define $\psi : [b, b+k] \to \mathbb{R}, \psi(t) = D_1 f(q_1, t)$. We note that $h\psi(b+k) - \psi(b) = \lambda(h, k)$. We use MVT once again to find $q_2 \in (b, b+k)$ such that $D_2 D_1 f(q_1, q_2) \cdot hk = \lambda(h, k)$. By symmetry we can find (p_1, p_2) such that $D_1 D_2 f(p_1, p_2) = \lambda(h, k)$.

   (b) We consider $R = [a, a+t] \times [b, b+t]$. By above, we can find points p, q in the interior of this rectangle such that $D_1 D_2 f(p) = D_2 D_1 f(q)$. Once we take the limit at t goes to 0, p, q must coincide by continuity of the partial derivatives.

4. (a) For $v \in \mathcal{H}$ we define

$$f_v(u) = \langle u, v \rangle$$

   which is continuous (it is linear and we know it is bounded by Cauchy-Schwarz).

   Then

$$S^\perp = \bigcap_{s \in S} f_s^{-1}(\{0\})$$

   where each $f_s^{-1}(\{0\})$ is closed by continuity of f_s. So $S^\perp$ is closed. The fact that $S^\perp$ is a linear subspace is clear and easily verified.

   (b) We see from above that $(S^\perp)^\perp$ is a closed set that contains S. Thus by definition of closure, we know that $\overline{S} \subset (S^\perp)^\perp$. All that remains to show is the reverse inclusion. We will need to assume part c for this.

First we claim that $\overline{S}^{\perp} = S^{\perp}$. It is clear that $\overline{S}^{\perp} \subset S^{\perp}$. So let us show the reverse and assume $u \in S^{\perp}$. Then for all $v \in \overline{S}$. We know there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in S that converges to v. Then

$$\langle v, u\rangle = \left\langle \lim_{n\to\infty} v_n, u \right\rangle = \lim_{n\to\infty} \langle v_n, u\rangle = 0$$

As this holds for all $v \in \overline{S}$, we conclude that $u \in \overline{S}^{\perp}$.

Now assuming c, we can write $\mathcal{H} = \overline{S} \oplus \overline{S}^{\perp}$. By above, we get $\mathcal{H} = \overline{S} \oplus S^{\perp}$. Let $v \in (S^{\perp})^{\perp}$. By assumption there exists $u \in \overline{S}$ and $W \in S^{\perp}$ such that $v = u + w$. Then $v - u \in S^{\perp}$. On the other hand we know that $u \in (S^{\perp})^{\perp}$ so $v - u \in (S^{\perp})^{\perp}$. This means that $v - u \in (S^{\perp})^{\perp} \cap S^{\perp}$ thus $v - u$ must be orthogonal to itself. This implies that $v = u$ hence $v \in \overline{S}$ as desired.

(c) I don't even know man