MAT257 RSG 5

Rishibh Prakash

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3. (Our Lord and saviour Saied) Consider the following system of equations:

$$wxyz + x^2 e^y + y\cos(wz) = 0$$
$$4\sin(w) + 3\cos(x) - 2e^y - z = 1$$

   (a) Show that there exists a neighbourhood B of $(0, 0)$ with a unique function $g : B \to \mathbb{R}^2$ of class C^∞ such that:

   - $g(0, 0) = (0, 0)$ and
   - if $(w, x) \in B$ and $(y, z) = g(w, x)$ then (w, x, y, z) satisfies the system of equations above.

   (b) Determine $Dg(0, 0)$.

3. Show that the Implicit Function Theorem implies the Inverse Function Theorem.

3. (Our Lord and saviour Saied) Let $n > 1$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ be of class C^∞. Define

$$C = \{x \in \mathbb{R}^n : g(x) = 0\}$$

   We say $x \in \mathbb{R}^n$ is a *constrained local extremum of* f if x is a local extremum of $f|_C$. Let c be such a point.

   (a) Suppose $\frac{\partial g}{\partial x_n}(c) \neq 0$. If $c = (a, b)$ with $a \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$, show that there exists $B \subset \mathbb{R}^{n-1}$ open containing a and $h : B \to \mathbb{R}$ of class C^∞ such that

   - $h(a) = b$ and
   - $g(x, h(x)) = 0$ for all $x \in B$

   (b) Define $F : B \to \mathbb{R}, F(x) = f(x, h(x))$. Use F to show that there exists some $\lambda \in \mathbb{R}$ such that $Df(c) = \lambda Dg(c)$.

For the nerds

3. Let $\mathcal{H}$ be a (possibly infinite dimensional) real inner product space, such that that a sequence converges in $\mathcal{H}$ if and only if it is Cauchy.

   Let K be a closed linear subspace of $\mathcal{H}$. Let $v \in \mathcal{H}$ be arbitrary.

   (a) Let $(k_n)_{n\in\mathbb{N}}$ be a sequence in K such that $\|k_n\|$ converges to some $d \in \mathbb{R}$. Show that $(k_n)_{n\in\mathbb{N}}$ is Cauchy. (*Hint*: Consider $\left\|\frac{k_n - k_m}{2}\right\|$ and use the Parallelogram equality)

   (b) Show there exists some $u \in K$ such that

   $$\|v - u\| = \inf\{\|v - w\| : w \in K\}$$

   (c) Let $w \in K$ be arbitrary. We know from above that $\|v - u\| \leq \|v - u - tw\|$ for all t. Use this to show that $\langle v - u, w\rangle = 0$.

   (d) Conclude that $\mathcal{H} = K \oplus K^{\perp}$.