MAT454: Complex Analysis II

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Contents

0 Preface

These notes are based on a series of lecture given by Professor Edward Bierstone at the University of Toronto in Winter 2023 for MAT454: Complex Analysis II[1].

1 Introduction/Review

The details and proofs of most things in this section can be found in my MAT354 notes here.

1.1 Holomorphic Functions

If $f(z)$ is a complex-valued function on the complex numbers, we say f is *holomorphic* if

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists. Suppose we call the limit c. Then the above condition is equivalent to saying there exists some $\phi(h)$ such that

$$f(z+h) = f(z) + ch + \phi(h)h$$

where $\lim_{h\to 0} \phi(h) = 0$.

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider what conditions it places on the derivatives (in the real sense). So suppose we have $c = (a, b)$ and $h = (\xi, \eta)$. Then the map $h \mapsto ch$ becomes $\xi + i\eta \mapsto (a+ib)(\xi+i\eta) = a\xi - b\eta + i(b\xi + a\eta)$. In terms of the real variables, we get

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

The matrix is the differential of f. Thus looking at the columns we see the following equation is satisfied

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

This is known as the Cauchy-Riemann equation(s). If we write $f = u + iv$, we can again use the matrix to get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are another way of formulating the Cauchy-Riemann equations.

Recall that if f is differentiable, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

[1] Archived link

This means that for the identity map $z = x + iy$ and conjugation map $\overline{z} = x - iy$ we have

$$dz = dx + idy$$
$$d\overline{z} = dx - idy$$

We can then solves for dx and dy to get

$$dx = \frac{1}{2}(dz + d\overline{z})$$
$$dy = \frac{1}{2i}(dz - d\overline{z})$$

Thus we get

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\overline{z}$$

This motivates us to define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

One thinks of these as the duals to dz and $d\overline{z}$. With this notation we can once again rewrite the Cauchy-Riemann equations to

$$\frac{\partial f}{\partial \overline{z}} = 0$$

Roughly speaking, this says that a holomorphic function should only depend on z and not $\overline{z}$.

1.2 Harmonic functions

Apart from holomorphic functions, another important class of functions are the harmonic functions.

Definition 1.1 (Harmonic functions)**.** A real- or complex-valued function $f(x, y)$ is said to be *harmonic* if it is C^2 and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

The operator

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is often called the Laplacian. It is easy to verify from the definition that if a complex-valued function is harmonic then so are its real and imaginary parts. Using the definitions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ we get

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}$$

This immediately implies that holomorphic functions are harmonic since they satisfy $\frac{\partial f}{\partial \overline{z}}$ is 0, which also means that the real and imaginary parts of a holomorphic function are harmonic.

In fact the relationship between harmonic and holomorphic functions runs deeper than that. Any real-valued harmonic function is locally the real part of a holomorphic function, which is uniquely determined up to the addition of a constant (shown in Subsection 1.6). The holomorphic function need not be define globally. An example of a harmonic function whose associated holomorphic function is only locally defined is $\log |z|$. This is harmonic on $\mathbb{C}\setminus\{0\}$ but the corresponding holomorphic function would have to be $\log(z)$ which does not have a holomorphic branch on the entire punctured plane.

1.3 Geometric Models

We are often interested in the behaviour of holomorphic functions at ∞ to the extent that we often include it in our domain. If we add the point at infinity to $\mathbb{C}$ we get $\mathbb{C} \cup \{\infty\}$ the extended complex plane. Two primary ways of modelling this space are the Riemann sphere and one-dimensional complex projective plane (which we will see are equivalent).

Recall that the plane is homeomorphic to the sphere without a point. Thus if we add a point to the plane (namely the point at infinity) we get exactly a sphere. We can cover the sphere using two charts via stereographic projection. For example we have the homeomorphism $S^2 \setminus \{N\}$

$$\begin{aligned} S^2 \setminus \{N\} &\to \mathbb{C} \\ (x, y, t) &\mapsto z := \frac{x + iy}{1 - t} \end{aligned}$$

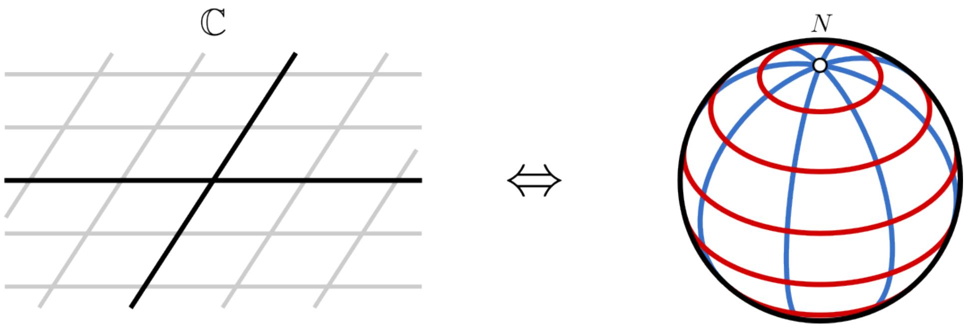


Figure 1: Points in $\mathbb{C}$ can be identified with points in $S^2 \setminus \{N\}$

The point at infinity then is the north pole. We can cover this point by stereographic projection from the south pole where the chart is given by

$$\begin{aligned} S^2 \setminus \{S\} &\to \mathbb{C} \\ (x, y, t) &\mapsto z' := \frac{x - iy}{1 + t} \end{aligned}$$

If we just wanted a chart, we would use $\frac{x+iy}{1+t}$ which is the usual projection from $S^2 \setminus \{S\}$. But since we want to impose a complex structure on S^2 we want the transition maps to be holomorphic so we take the complex conjugate instead. Note that with the above choice we have

$$zz' = \frac{x + iy}{1 - t} \cdot \frac{x - iy}{1 + t} = \frac{x^2 + y^2}{1 - t^2} = 1$$

This means that $z' = \frac{1}{z}$ which allows us to translate to coordinates at infinity. For example, given a map f which is defined on the complement of a (large) disk, we say f is holomorphic at ∞ if $f(\frac{1}{z})$ is holomorphic at 0.

A seemingly different but ultimately equivalent geometric model is the one-dimensional complex projective space. Define $P^1(\mathbb{C}) := \mathbb{C}^2 \setminus \{0\}/\sim$ where $(x_0, x_1) \sim (y_0, y_1)$ if and only if there is some non-zero complex number λ such that $(x_0, x_1) = \lambda(y_0, y_1)$. Let $[x_0, x_1]$ denote the equivalence class of (x_0, x_1).

Once again we can cover this space with two coordinate charts. For $i = 0$ and $i = 1$ we define $U_i := \{[x_0, x_1] \in P^1(\mathbb{C}) : x_i \neq 0\}$. Then we can define charts

$$U_0 \to \mathbb{C}$$
$$[x_0, x_1] \mapsto z := \frac{x_1}{x_0}$$

and

$$U_1 \to \mathbb{C}$$
$$[x_0, x_1] \mapsto z' := \frac{x_0}{x_1}$$

These are well-defined due the equivalence placed upon the point. These are both homeomorphisms onto $\mathbb{C}$. Once again we see that $zz' = 1$. This means that $P^1(\mathbb{C})$ is obtained by gluing two copies of $\mathbb{C}$ along the complement of the origin by the formula $z' = \frac{1}{z}$, exactly like we had with the sphere.

1.4 Cauchy's Theorem

Much of complex analysis is about studying the properties of holomorphic functions and one of the fundamental results in this area is Cauchy's Theorem. Before we get to the theorem, we should maybe establish some basic facts about (differential) forms.

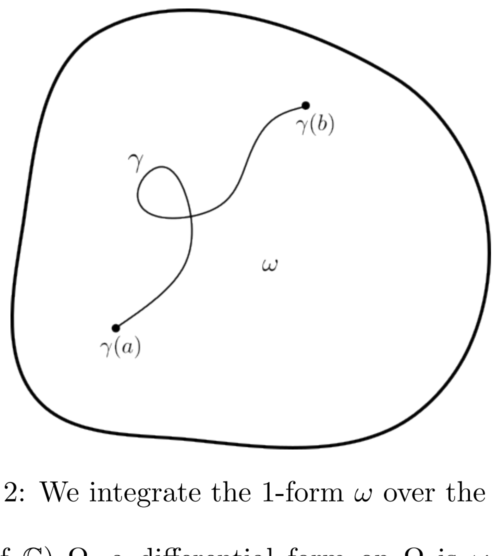


Figure 2: We integrate the 1-form ω over the curve γ

Given an open set (of $\mathbb{C}$) Ω, a differential form on Ω is $\omega = Pdx + Qdy$ with P, Q continuous functions (taking values in $\mathbb{C}$) on Ω. We can integrate a form along a piecewise C^1 curve $\gamma : [a, b] \to \Omega$ by the formula

$$\int_\gamma \omega = \int_a^b f(t)dt$$

where

$$f(t) := P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)$$

We see this by computing the pullback of ω by γ. The reason we integrate forms and not functions is because then the integral is independent of how we parameterise the curve (in order to verify this we simply use integration by substitution).

Now suppose we are given a form ω and suppose there exists a function F so that

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Then we call F a *primitive* of ω. If ω has a primitive then

$$\begin{aligned}
\int_\gamma \omega &= \int_\gamma \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \\
&= \int_a^b \frac{\partial F}{\partial x}d(x(t)) + \frac{\partial F}{\partial y}d(y(t))dt \\
&= \int_a^b (F \circ \gamma)'(t)dt \\
&= F(\gamma(b)) - F(\gamma(a))
\end{aligned}$$

It is clear then, that if γ is any closed curve (i.e. $\gamma(a) = \gamma(b)$) then $\int_\gamma \omega = 0$ provided that ω has a primitive. In fact the converse is also true.

Theorem 1.2 *A form ω on a connected open set Ω has a primitive if and only if the integral of ω over any closed curve is 0.*

Proof. We choose a basepoint $(x_0, y_0) \in \Omega$. Then we define the primitive $F(x, y)$ by integrating along a path from (x_0, y_0) to (x, y). This is well-defined due to the fact that the integral over closed curves is 0 and one can verify that indeed defined this way, we have $dF = \omega$. The details can be found in Proposition 9.2 in my MAT354 notes here. □

If we have a disk, then it is much easier to check whether or not a form has a primitive. Namely, a form has a primitive if and only if the integral over the boundary of any rectangle (by which we mean a rectangle whose sides are parallel to the axes) is 0. This is simply because in a disk, we can connect any point to the center via paths that run parallel to the axes (for example we first travel only in the x-direction and then only in the y-direction). Thus although the existence of a primitive might be difficult to check globally, it is fairly straightforward to do locally. This inspires the following definition.

Definition 1.3 (Closed forms)**.** We say a form ω is closed if the integral over the boundary of any (small) rectangle is 0.

By the above discusssion, this is equivalent to saying that ω has a local primitives. Moreover, if the integral over the boundary of small rectangles is 0, then the integral over the boundary of any rectangle is 0 since any rectangle can be cut into smaller rectangles.

Importantly, however, closed forms need not have a global primitive. The classic example of a form which has local primitives but not global ones is

$$\omega = \frac{dz}{z}$$

defined on $\mathbb{C} \setminus \{0\}$. Local primitives are easy to find since these are just branches of $\log(z)$. However ω cannot have a global primitive since the integral over the unit circle is not zero. If we define $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$ then

$$\int_\gamma \omega = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = 2\pi i$$

Now we can state and prove Cauchy's theorem.

Theorem 1.4 (Cauchy's Theorem) *If $f(z)$ is holomorphic then the differential form $f(z)dz$ is closed.*

Proof. Once again, the proof can be found in my MAT354 notes under Theorem 9.5. □

Remark 1.5. In Cauchy's Theorem, it is enough to assume that f is continuous on Ω and holomorphic everywhere except possibly on a line.

Corollary 1.6 *Holomorphic functions $f(z)$ locally have a holomorphic primitive.*

Proof. We have seen above that $f(z)dz$ has a primitive. Now we show that the primitive is in fact holomorphic. Suppose the primitive is given by F. Then

$$f(z)dz = dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z}$$

Since $dz, d\bar{z}$ form a basis they are in particular linearly independent which means that $\frac{\partial F}{\partial \bar{z}}$ must be 0. □

We have the following important theorem about closed forms on the plane (or subsets thereof).

Theorem 1.7 *Let ω be a closed differential form in open $\Omega \subset \mathbb{R}^2$. If $\gamma_0, \gamma_1 : [0, 1] \to \Omega$ homotopic curves (either with fixed endpoints or as closed curves) then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

From this, Theorem 1.8 follows as a corollary since it immediately implies that the integral over any closed curve is 0.

Theorem 1.8 *A closed differential form ω in a simply connected open set $\Omega \subset \mathbb{R}^2$ has a global primitive.*

1.5 Cauchy's Integral Formula

We are very close to being able to state and prove Cauchy's Integral Formula. The one thing that is left is the winding number.

Definition 1.9. The winding number of a closed curve γ with respect to a point a (not on γ) is given by

$$w(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} dz$$

It is clear that $w(\gamma, a)$ is an integer because the integral is the difference between 2 branches of log. If γ is the boundary of a circle then

$$w(\gamma, a) = \begin{cases} 1 & a \text{ inside circle} \\ 0 & a \text{ outside circle} \end{cases}$$

From Theorem 1.7, it follows that $w(\gamma, a)$ is invariant under homotopy of γ that does not pass through a. Since moving γ a little bit is the same as moving a a little bit, it also follows that $w(\gamma, \cdot)$ is constant on the connected components of the complement of γ.

Theorem 1.10 *Suppose $f(z)$ is holomorphic in an open $\Omega \subset \mathbb{C}$ and a is a point in Ω. Let γ be a nullhomotopic closed curve in Ω. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = f(a) w(\gamma, a)$$

Proof. We define

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

We see that g is continuous in Ω and holomorphic on $\Omega \setminus \{a\}$. Therefore $g(z)dz$ is closed by Cauchy's Theorem. Then the nullhomotopy of γ implies that

$$\int_\gamma \frac{f(z) - f(a)}{z-a} dz = 0$$

Splitting the sum gets us the desired result

$$\int_\gamma \frac{f(z)}{z-a} dz = \int_\gamma \frac{f(a)}{z-a} dz = 2\pi i f(a) w(\gamma, a)$$

□

A very nice and important consequence of Cauchy's integral formula is that holomorphic functions are infinitely differentiable. Suppose f is a holomorphic function in a neighbourhood of a closed disk $|z| \leq r$. For $|z| < r$, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

Then we can differentiate both sides (by differentiating under the integral sign) to get

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

and more generally

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

We can summarise all this information about holomorphic function as follows.

Theorem 1.11 *Suppose $f(z)$ is a continuous function on an open set Ω. Then the following are equivalent:*

1) $f(z)$ is holomorphic

2) $f(z)dz$ is closed

3) Given γ the boundary of a circle of radius r and $|z| < r$, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

Proof. We know that 1) $\Rightarrow$ 2) is Cauchy's theorem. We have shown 1) $\Rightarrow$ 3) above. 3) $\Rightarrow$ 1) is easy to see since we can differentiate under the integral sign. This only leaves 2) $\Rightarrow$ 1) which is known as Morera's theorem.

If $f(z)dz$ is closed then $f(z)dz$ locally has a primitive $g(z)$. Then

$$f(z)dz = dg = \frac{\partial g}{\partial z} dz + \frac{\partial g}{\partial \bar{z}} d\bar{z}$$

Since dz and $d\bar{z}$ are linearly independent, we must have $\frac{\partial g}{\partial \bar{z}} = 0$. This also tells us that $f(z) = \frac{\partial g}{\partial z}$ and hence f is holomorphic since the derivative of holomorphic functions is holomorphic. □

Not only are holomorphic functions infinitely differentiable, they are also analytic, which is to say every holomorphic function has a convergent power series expansion (defined locally of course) that represents the function.

By Cauchy's Integral formula we know that if $f(z)$ is a holomorphic function in a neighbourhood of $|z| \leq r$, then

$$f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

where γ is the boundary of the closed disk, $|z| = r$. We can then write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^n}$$

We can substitute this into the integral formula to get

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta^{n+1}} d\zeta \right) z^n$$

The coefficient of z^n agrees with what we would expect since we know that $a_n = \frac{f^{(n)}(0)}{n!}$. This series converges whenever $|z| < r$.

We can also bound the Taylor coefficients. For example by substituting $z = re^{i\theta}$ we get

$$f(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta}$$

We can multiply both sides by $e^{-in\theta}$ and integrate from $\theta = 0$ to $\theta = 2\pi$. This will cancel out all but the $a_n r^n e^{in\theta}$ term in the series. Hence we get

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$$

If we have $M(r) := \sup_{|z|=r} |f(z)|$ then

$$|a_n| \leq \frac{M(r)}{r^n}$$

These are known as *Cauchy's inequalities.*

This immediately gives us Liouville's Theorem (a corollary of which is the Fundamental Theorem of Algebra).

Theorem 1.12 (Liouville's Theorem) *If $f(z)$ is holomorphic on $\mathbb{C}$ and bounded then f is constant.*

Proof. There is some M such that $M(r) \leq M$ for all r. Therefore

$$|a_n| \leq \frac{M}{r^n}$$

Therefore for $n \geq 1$, we can send $r \to \infty$ to conclude that $a_n = 0$. Therefore $f(z) = a_0$ is constant. □

A direct consequence of the integral formula is that holomorphic functions satisfy the Mean Value Property which is to say that given a closed disk D of radius r, centered at a point a, the value of $f(a)$ is given by the mean value along the boundary of the disk. In formulae, we write

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$$

Continuous functions that have the mean value property also satisfy the very important maximum modulus principle.

Theorem 1.13 (Maximum Modulus Principle) *Suppose $f(z)$ is a continuous complex-valued function defined on an open set Ω and f satisfies the Mean Value Property. If $|f|$ has a local max at $z_0 \in \Omega$, then f is constant in a neighbourhood of z_0.*

1.6 Harmonic Functions, revisited

Earlier it was claimed that all real-valued harmonic functions are (locally) the real part of a holomorphic function and moreover this holomorphic function is unique up to the addition of a constant. Let us verify this claim. Let g be a real-valued harmonic function. Then

$$\frac{\partial^2 g}{\partial z \partial \overline{z}} = 0$$

This means that $\frac{\partial g}{\partial z}$ is holomorphic and therefore $\frac{\partial g}{\partial z}dz$ locally has a holomorphic primitive. Let us call this primitive f. Then

$$df = \frac{\partial g}{\partial z}dz$$

Conjugating both sides we get

$$d\overline{f} = \frac{\partial g}{\partial \overline{z}}d\overline{z}$$

where g is not conjugated since it is real valued. Therefore we have

$$df + d\overline{f} = d(f + \overline{f}) = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \overline{z}}d\overline{z} = dg$$

Therefore $g = f + \overline{f}$ up to the addition of a constant.

It is clear that if $f(z)$ is a complex-valued functions with the Mean Value Property then the real and imaginary parts of f have this property (we can simply equate the real and imaginary parts on both sides). Since harmonic functions are locally the real part of a holomorphic function, it follows that harmonic functions also satisfy the Mean Value Property. In fact any continuous function satisfying the Mean Value Property is harmonic, which we will soon prove.

The natural question that arises now is whether given a real-valued harmonic function we can figure out what the corresponding holomorphic functions should be. Suppose the harmonic function is $g(z)$ and the holomorphic function is $f(z)$. We know there for some R and any $|z| < R$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Moreover since f is unique up to addition of a constant, we can assume that $f(0) = a_0$ is real. Then substituting $z = re^{i\theta}$ and equating the real part we get

$$g(r\cos\theta, r\sin\theta) = a_0 + \frac{1}{2}\sum_{n=0}^{\infty}(a_n r^n e^{in\theta} + \overline{a_n} r^n e^{-in\theta})$$

Therefore we get that

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)d\theta$$

For the remaining coefficients, we can use our usual trick of multiplying by $e^{-in\theta}$ to conclude

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\frac{1}{r^n e^{in\theta}}d\theta$$

Substituting these back into the expansion of f we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \left[1 + 2\sum_{n=0}^{\infty} \left(\frac{z}{re^{i\theta}} \right)^n \right] d\theta$$

We can evaluate the series and simplify things to get

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta$$

Equating real parts again we get

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta$$

Remark 1.14. We call the function

$$\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}$$

the Poisson kernel.

A classic problem with harmonic functions is the Dirichlet problem. In this case we work on the disk.

Theorem 1.15 *Given a continuous function $f(\theta)$ which is periodic and has period 2π and given some $r > 0$, there exists a continuous function $F(z)$ on the closed disk $|z| \leq r$ which is harmonic on the open disk (of radius r) with $F(re^{i\theta}) = f(\theta)$. Moreover F is unique.*

Proof. We can assume that f is real-valued (otherwise we work with the real and imaginary parts of f separately).

The uniqueness of F follows from the maximum modulus principle. For existence we can define

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} d\theta$$

which is harmonic because it is the real part of the holomorphic function

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{re^{i\theta} + z}{re^{i\theta} - z} d\theta$$

All that remains to check is that $\lim_{z \to re^{i\theta_0}} F(z) = f(\theta_0)$ which is a direct computation. As usual, more details can be found in my MAT354 notes. □

Corollary 1.16 *A continuous function $f(z)$ defined on an open $\Omega \subset \mathbb{R}^2$ with the Mean Value Property is harmonic.*

Proof. It suffices to check things locally. So let z_0 be some point in Ω and D some disk in Ω that contains z_0. Then there exists a function F which is continuous on $\overline{D}$ and harmonic on D and which agrees with f on the boundary of D. Since F and f both have the Mean Value Property so does $F - f$. Note this function is 0 on the boundary of D and therefore is identically 0 on D by the maximum modulus principle. □

1.7 Zeros, poles and singularities

Suppose $f(z)$ is a holomorphic function such that $f(z_0) = 0$. Then near z_0 we can use the power series of f to write

$$f(z) = (z - z_0)^k f_1(z)$$

where f_1 is holomorphic and non-zero at z_0. The integer k is known as the order or multiplicity of z_0.

> **Definition 1.17** (Meromorphic functions)**.** A *meromorphic function* on an open set Ω is a function that is holomorphic on the complement of a discrete subset of Ω and expressible in a neighbourhood of any point of Ω as the quotient of holomorphic functions $\frac{f(z)}{g(z)}$ (where of course g is not identically 0).

If $f(z)$ and $g(z)$ are holomorphic functions, we can write

$$f(z) = (z - z_0)^k f_1(z)$$
$$g(z) = (z - z_0)^l g_1(z)$$

where f_1 and g_1 are both non-zero at z_0. Then

$$\left(\frac{f}{g}\right)(z) = (z - z_0)^{k-l} \left(\frac{f_1}{g_1}\right)(z)$$

If $k \geq l$, then f/g extends holomorphically at z_0. Otherwise we have $\lim_{z \to z_0} (\frac{f}{g})(z) = \infty$, so z_0 is a pole of order $l - k$. The limit can be thought of as a convergence to the point at infinity in the Riemann sphere. Therefore we can also consider meromorphic functions as functions with values in S^2. With this we see that meromorphic functions are simply holomorphic functions with values in S^2.

Given a holomorphic function $f(z)$ defined on an annulus $0 \leq R_2 < |z| < R_1 \leq \infty$, we can always find its Laurent series. This means that there exists coefficients a_n for $n \in \mathbb{Z}$ so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

for z in the annulus. Such a series converges if series with negative indices and non-negative indices converge separately.

Let γ_1 and γ_2 be the boundary of a disk of radius r_1 and r_2 respectively where $R_2 < r_2 < r_1 < R_1$. By Cauchy's Integral formula we get

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

We have already seen above how to express the first integral as a series. For the second integral we can write $(\zeta - z)^{-1} = -z^{-1}(1 - \frac{\zeta}{z})^{-1}$ and expand this using the geometric series. In this case we get that the a_n are the exact same except we are integrating over γ_2 instead. In summary, we can write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$$

where $i = 1$ if $n \geq 0$ and $i = 2$ if $n < 0$. This series converges uniformly and absolutely in $r_2 \leq |z| \leq r_1$. The portion of the Laurent series with the negative indices is called its principal part. We can use the Laurent series to prove some very nice statements.

Theorem 1.18 *A meromorphic function $f(z)$ on S^2 is rational.*

Proof. Since S^2 is compact, there can only be finitely many poles say $b_1, \ldots, b_k$ and possibly ∞. The corresponding principal parts are $P_j(\frac{1}{z-b_j})$ for each of the b_j and $P_\infty(\frac{1}{\zeta})$ where ζ is the coordinate at ∞. Since $\zeta = 1/z$, P_∞ is actually a polynomial in z. Then we see that

$$f(z) - P_\infty(z) - \sum_{j=1}^{k} P_j\left(\frac{1}{z-b_j}\right)$$

is a holomorphic function on S^2. Moreover since S^2 is compact it must also be bounded. But then Liouville's Theorem allows us to conclude that f is actually constant. Therefore

$$f(z) = c + P_\infty(z) + \sum_{j=1}^{k} P_j\left(\frac{1}{z-b_j}\right)$$

is rational. In fact, this even gives us the partial fraction decomposition of the rational function. □

If we a holomorphic function defined on a punctured neighbourhood of 0 (i.e. on $0 < |z| < R$) then 0 is said to be an isolated singularity. If f extends holomorphically to 0, then 0 is said to be a *removable* singularity. We have such an extension if and only if f is bounded in a (punctured) neighbourhood of 0 (this follows from Cauchy's inequalities which still hold for coefficients in the Laurent expansion and allows us to show that all the negative index coefficients must be 0).

If f does not extend holomorphically to 0 then there are essentially 2 different behaviours of f (at 0), which are determined by the Laurent series. If the Laurent series has only finitely many terms with negative indices then we have a *pole* at 0. Otherwise there are an infinite number of terms with negative indices and we have an *essential singularity* at 0.

Theorem 1.19 (Weierstrass' Theorem) *If 0 is an essential singularity, then for any $\epsilon > 0$, we have $f(0 < |z| < \epsilon)$ is dense in $\mathbb{C}$.*

1.8 Residue Theorem

Suppose $f(z)$ is holomorphic. The *residue* of $f(z)dz$ at a point a is defined to be

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz$$

where γ is a curve of winding number 1 (most typically a circle) around a. If we write $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ then we can immediately compute that the residue of $f(z)dz$ at 0 is a_{-1} (when integrating the other terms disappear since they have a primitive).

The residue at ∞ is defined in the exact same way. Let γ be a small circle around $z = \infty$. In coordinates at infinity, we can write $\zeta = 1/z$. This means

$$f(z)dz = f(\zeta) \cdot -\frac{1}{\zeta^2} d\zeta$$

If γ is a small circle around ∞ in the z-plane then its image in the ζ-plane is a large circle around 0, with the opposite orientation. Therefore

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = -\frac{1}{2\pi i}\int_{\gamma'} \frac{1}{\zeta^2} f\left(\frac{1}{\zeta}\right) d\zeta$$

Therefore if we have $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ then the residue at ∞ is $-a_{-1}$ (in fact one might think the residue would be given by $-a_1$ but the multiplication with $1/\zeta^2$ forces you to shift the index by 2 bringing us back to $-a_{-1}$).

Theorem 1.20 (Residue Theorem) *Let Ω be an open subset of S^2 and $f(z)$ a holomorphic function in Ω except for isolated singularities which may occur at ∞. Let K be a compact subset of Ω with piecewise C^1 boundary Γ. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{z_k\in K} Res(f, z_k)$$

where S is the set of singular points of f in K.

2 Topology of space of Holomorphic Functions

Let Ω be an open neighbourhood of $\mathbb{C}$ (or possibly even S^2). Then we use $\mathcal{C}(\Omega)$ to denote the ring of continuous complex-valued functions on Ω and $\mathcal{H}(\Omega)$ for the subring of holomorphic functions.

There is a natural topology on $\mathcal{C}(\Omega)$ (and therefore on $\mathcal{H}(\Omega)$ via the subspace topology). In the study of functions one often defines a topology but defining how a sequence of functions should converge (recall that a metric space is determined completely by its set of convergent sequences). In this case, we will say that a sequence of continuous functions $\{f_n\}$ converges if we have uniform convergence on every compact subset of Ω. More symbolically, we say $\{f_n\}$ converges if, given any compact set $K \subset \Omega$, we have $\{f_n|_K\}$ converges uniformly.

We can in fact describe the open sets in this topology quite explicitly. Given compact set $K \subset \Omega$ and $\epsilon > 0$, the set

$$V(K, \epsilon) := \{f \in \mathcal{C}(\Omega) : |f(z)|_{z\in K} < \epsilon\}$$

is an open neighbourhood of 0. Then the open neighbourhoods of any g can be found by simply translating these. In other words,

$$V(g, K, \epsilon) := \{f \in \mathcal{C}(\Omega) : |f(z) - g(z)|_{z\in K} < \epsilon\}$$

is an open neighbourhood of g.

If our claim is that this topology is determined by convergent sequences, then we should be able to specify what the metric is. Suppose we cover Ω by countably many closed disks D_i (for example we can take all the disks of rational radii with centres at rational coordinates). Then we can define

$$|f| = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, M_i(f)\}$$

where $M_i(f) := \max\{|f(z)| : z \in D_i\}$. This defines a metric that is translation invariant and in fact induces the above topology. An important remark is that $\mathcal{C}(\Omega)$ is complete with respect to this topology.

In order to see this, suppose $\{f_n\}$ form a Cauchy sequence with respect to the above metric. In particular this means that, for any $z \in \Omega$, the sequence $\{f_n(z)\}$ is a Cauchy sequence of complex numbers therefore converges to some value we call $f(z)$. We want to show this pointwise limit is continuous so fix some $z_0 \in \Omega$. There is a compact set K of Ω containing z_0. This compact set is covered by the interiors of finitely many of the D_i. This places a bound on $|f(z) - f_n(z)|$ for $z \in K$ [WHAT IS IT] which means that the f_n converge to f uniformly on K. Since we know the uniform limit of a sequence of continuous functions is continuous, we know f is continuous at z_0.

Now that we understand the topology on $\mathcal{C}(\Omega)$ a bit better, we want to try looking at $\mathcal{H}(\Omega)$ as well. In particular, we want to say that $\mathcal{H}(\Omega)$ is a closed subset of $\mathcal{C}(\Omega)$ and that the differentiation operator is continuous. We translate these into statements about sequences.

Theorem 2.1 (Weierstrass) *If $\{f_n\} \subset \mathcal{H}(\Omega)$ is a sequence of holomorphic functions that converges uniformly on compact sets, then $f = \lim_{n\to\infty} f_n$ is holomorphic. Moreover, $\{f_n'\}$ converges uniformly to f' on compact sets.*

Proof. In order to prove the first statement it suffices to show that $f(z)dz$ is closed (see Theorem 1.11). Let D be an open disk in Ω and γ a closed curve in D. Then

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0$$

where we can swap the limit and integral by uniform convergence. Therefore $f(z)dz$ is closed and by Morera's theorem we know $f(z)$ is holomorphic.

In order to see that the derivatives converge, let D be a closed disk in Ω. If suffices to show that f_n' converge uniformly to f' on D [TODO: Why?]. Let γ be the boundary of a larger circle (than D) in Ω. Then for any $z \in D$ we have

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \lim_{n\to\infty} \frac{f_n(\zeta)}{(\zeta - z)^2} \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2} \\ &= \lim_{n\to\infty} f_n'(z) \end{aligned}$$

□

When speaking of sequences, one must also mention series, i.e. infinite sums. The above statement can also be translated to work with these as well.

Corollary 2.2 *If a series of holomorphic functions* $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on compact subsets of* Ω *to* $f(z)$ *then* f *is holomorphic and we can differentiate term by term.*

Proof. Recall a series converges if and only if the partials sum converge. The partial sums are all holomorphic as well therefore if the series converges the limit must be holomorphic, by the above theorem. The second part of the above theorem tells us that the derivatives of the partial sums converge to f' which means exactly that we can compute $f'(z)$ by differentiating the series term by term. □

Proposition 2.3 (Hurwitz) *Suppose* Ω *is a domain and* $\{f_n\} \subset \mathcal{H}(\Omega)$ *where* f_n *are all nowhere zero on* Ω *and converge uniformly on compact sets. Then either the limit function* f *is also nowhere zero or it is identically zero on* Ω*.*

Remark 2.4. A domain is an open, connected subset of $\mathbb{C}$.

Proof. Suppose f is not identically 0. Then since f is holomorphic, its zeroes are isolated. Let $z_0 \in \Omega$ be arbitrary. If it is a zero, we can compute its multiplicity via the Argument Principle

$$\int_\gamma \frac{f'(z)}{f(z)} dz$$

where γ is a small circle around z_0. But then

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_\gamma \lim_{n\to\infty} \frac{f_n'(z)}{f_n(z)} dz = \lim_{n\to\infty} \int_\gamma \frac{f_n'(z)}{f_n(z)} dz$$

where the limit must be 0 since that is multiplicity of z_0 as a zero of the f_n for every n. Therefore z_0 is not a zero of f. □

Corollary 2.5 *If* Ω *is a domain and* $\{f_n\} \subset \mathcal{H}(\Omega)$ *where* f_n *are injective and converge uniformly on compact sets to* f*, then* f *is either constant or also injective.*

Proof. Suppose f is not constant and not 1-1. Then there are distinct points $z_1, z_2 \in \Omega$ so that $f(z_1) = f(z_2) =: a$. Let V_1, V_2 be disjoint open neighbourhoods of z_1 and z_2 respectively. Since $f(z) - a$ vanishes on V_1 we know that f_n must have a zero on V_1 as well by the previous proposition. But this holds for V_2 as well which contradicts injectivity of the f_n. □

2.1 Series of Meromorphic Functions

Quite often we will be interested not (only) in convergence of a series of holomorphic functions but rather in the series of meromorphic functions. We say that a series of meromorphic functions

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly (or absolutely and uniformly) on open $\Omega \subset \mathbb{C}$ if we have uniform convergence (or absolute and uniform convergence) on compact subsets of Ω after discarding finitely many terms. In other words, if K is any compact subset of Ω then only finitely many of the f_n should have poles in K. If we ignore these, then we have a series of holomorphic functions on K and we know what it means for a series of holomorphic functions to converge. Therefore if we have convergence in this manner for every compact set K (where perhaps we need to discard different f_n for different K) then we say that the series of meromorphic functions itself converges.

Corollary 2.6 *If a series of meromorphic functions $\sum f_n$ converges uniformly on compact subsets of Ω then $f = \sum f_n$ is meromorphic on Ω and $\sum f_n'$ converges uniformly to f'.*

2.1.1 Example 1

It is easiest to understand this via an example. Consider the series

$$f(z) := \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

We claim that this converges absolutely and uniformly on compact subsets of $\mathbb{C}$. In fact we can make the even stronger statement that we have absolute and uniform convergence not just on compact subsets but on any vertical strip $a_1 \leq \mathrm{Re}(z) \leq a_2$ (where a_1, a_2 are some fixed real numbers).

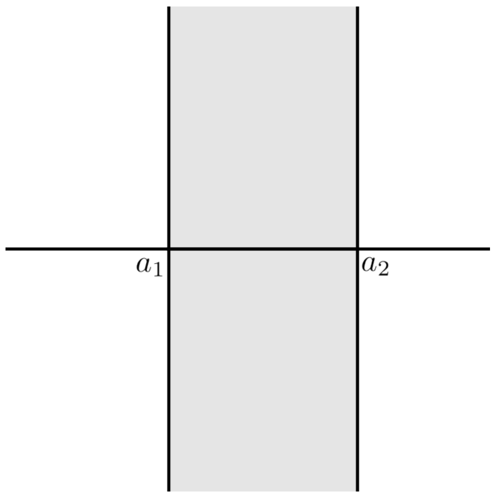


Figure 3: The series for $f(z)$ converges on every vertical strip in the complex plane

For $n < a_1$ and $n > a_2$, the functions $1/(z-n)^2$ are holomorphic on this strip, thus we can ignore all n that lie in (a_1, a_2). Moreover for $n < a_1$ we have $1/|z-n| < 1/|a_1 - n|$ for

z in the strip (see Figure 4). Therefore

$$\sum_{n=-\infty}^{a_1} |f_n(z)| < \sum_{n=-\infty}^{a_1} \frac{1}{(a_1-n)^2}$$

which converges as it is comparable to $\sum 1/n^2$. The analogous argument holds for $n > a_2$. Therefore the series converges to a meromorphic function.

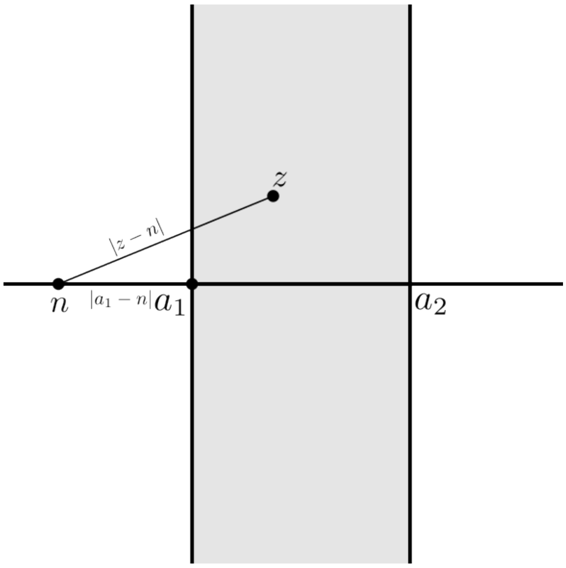


Figure 4: We have $1/|z-n| < 1/|a_1-n|$ for $n < a_1$

We want to find f more explicitly. Let us consider what properties $f(z)$ has. We know f is periodic with period 1 and it has double poles at the integers. There is another function that has these properties, namely

$$g(z) := \left(\frac{\pi}{\sin \pi z}\right)^2$$

We claim that these two functions are in fact equal.

Since f and g have the same principal parts, their difference $f-g$ is holomorphic on $\mathbb{C}$. Therefore, if we can show that $f-g$ is bounded we will be able to use Liouville's theorem to conclude that it is constant. We first show that $|f(z)| \to 0$ as $\mathrm{Im}(z) = y \to \infty$ uniformly with respect to x. This means that for any $\epsilon > 0$ we can find $b > 0$ such that for $|y| \geq b$ we have $|f(z)| < \epsilon$. By periodicity of f, it suffices to show this on a vertical strip of width 1.

Suppose z is in a strip of width 1 where its imaginary part y satisfies $|y| > b$ for some $b > 0$. Notice that the terms of the series are holomorphic on this subset and converge uniformly and absolutely on compact subsets. There is some large N such that for all z we have

$$\sum_{|n|>N} \frac{1}{|z-n|^2} < \frac{\epsilon}{2}$$

as this the tail of a convergent series.

The finitely many terms in the sum that remain also go to 0 uniformly with respect to x and can individually be bounded (in particular we choose b so that all the remaining terms are small). To be precise, for every $|n| \leq N$, we consider $1/(z-n)^2$ which goes to 0 uniformly as $|y| \to \infty$. In particular then there exists some b_n such that if $|y| > b_n$ then $\left|1/(z-n)^2\right| < \epsilon/4N$. Now we take b to be greater than all these b_n. Then notice that for $|y| > b$ we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{|z-n|^2} &= \sum_{n=-N}^{N} \frac{1}{|z-n|^2} + \sum_{|n|>N} \frac{1}{|z-n|^2} \\ &< \sum_{n=-N}^{N} \frac{\epsilon}{4N} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

Notice that g also has this property since

$$|\sin(\pi z)|^2 = \sin^2(\pi x) + \sinh^2(\pi y)$$

Now it is easy to see that $f - g$ is bounded. In particular on any strip, $f - g$ is bounded for $|y| \leq b$ by compactness and we know the difference goes to 0 for $|y| > b$. Therefore $f - g$ must be a constant and since the limit is 0 as $|y| \to \infty$, the constant must be 0.

2.1.2 Example 2

For a second example consider the series

$$f(z) := \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

The series does indeed converge on compact subsets of $\mathbb{C}$ (to a meromorphic function which we prematurely called f) since each term in the series is of the form $z/n(z-n)$ which is comparable to $1/n^2$ (on compact sets because we can bound z). Moreover if we differentiate the series for f term by term we get

$$\begin{aligned}f'(z) &= -\frac{1}{z^2} - \sum_{n \neq 0} \frac{1}{(z-n)^2} \\ &= -\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \\ &= -\left(\frac{\pi}{\sin(\pi z)} \right)^2 \\ &= \frac{d}{dz} \pi \cot(\pi z)\end{aligned}$$

Therefore $f(z) - \pi \cot(\pi z)$ is a constant (since their derivatives are equal) and this constant must be 0 since the functions are odd. Therefore

$$\frac{1}{z} + \sum_{n \neq 0} \frac{1}{z-n} + \frac{1}{n} = \pi \cot(\pi z)$$

3 Weierstrass $\wp$-function

We say a function $f(z)$ is doubly periodic if it is periodic with respect to a discrete subgroup Γ of $\mathbb{C}$ with 2 generators. This means that $f(z+\omega) = f(z)$ for every $\omega \in \Gamma$ for a subgroup Γ of the form

$$\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\}$$

where e_1, e_2 are complex numbers are linearly independent over $\mathbb{R}$ (see Figure 5). Equivalently, one can say that f has Γ as its group of periods.

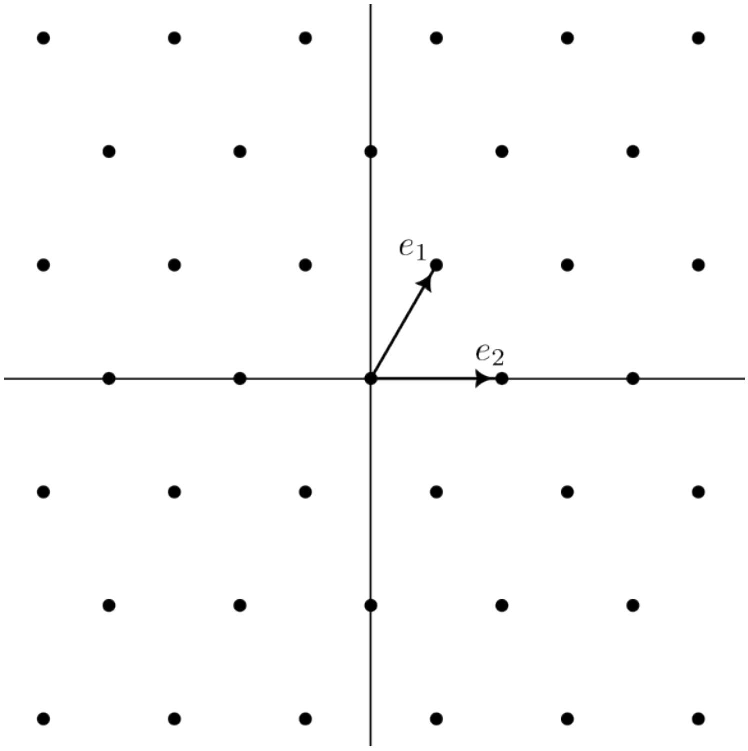


Figure 5: Γ is a discrete subgroup or lattice of $\mathbb{C}$

The Weierstrass $\wp$ function is defined with respect to a group of periods. So let Γ be a discrete subgroup of $\mathbb{C}$ as described above. Then

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

We claim that this is uniformly and absolutely convergent on compact subsets of $\mathbb{C}$. In order to see this, we first need the following lemma.

Lemma 3.1 *Given a discrete subgroup Γ, we have*

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{|\omega|^3} < \infty$$

Proof. In order to verify that the sum converges, we will sum over the points in a clever way, from the center outwards in a radial manner. Let $P_n = \{t_1 e_1 + t_2 e_2 : \max\{t_1, t_2\} = n\}$.

These are the $8n$ points that lie on the n-th parallelogram from the middle. Let k be the minimum distance between the origin and P_1. Then the distance between the origin and P_2 is $2k$ and in general the distance between the origin and P_n is nk (see Figure 6). Therefore

$$\begin{aligned}\sum_{\omega\in\Gamma\setminus\{0\}} \frac{1}{|\omega|^3} &= \sum_{n=1}^{\infty}\sum_{\omega\in P_n} \frac{1}{|\omega|^3} \\ &\leq \sum_{n=1}^{\infty} 8n \cdot \frac{1}{(nk)^3} \\ &= \sum_{n=1}^{\infty} \frac{8}{n^2k^3}\end{aligned}$$

and we know the final series converges. □

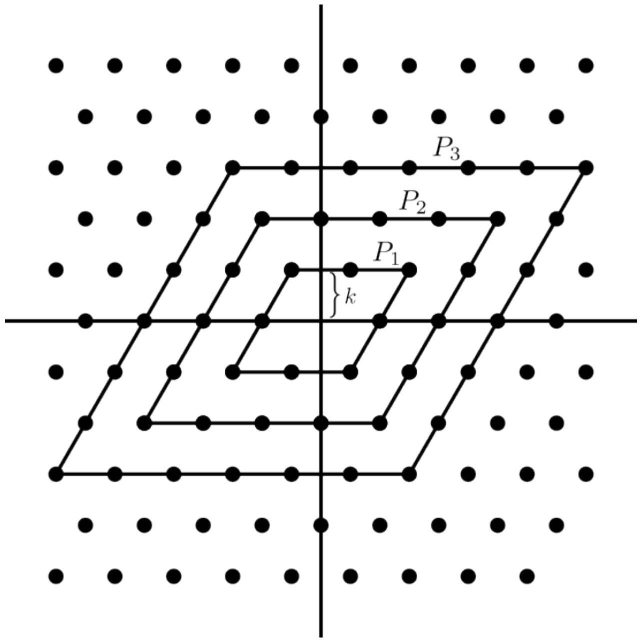


Figure 6: Sum 'radially' from the origin

Proposition 3.2 *Given a discrete subgroup* $\Gamma \subset \mathbb{C}$, *the series*

$$\frac{1}{z^2} + \sum_{\omega\in\Gamma\setminus\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Proof. It suffices to show that the series converges on closed disks $|z| \leq r$ for every r since any compact set is contained in such a disk. Then fix some $r > 0$. We see that for $|z| \leq r$

and $|\omega| \geq 2r$ we have

$$\begin{aligned}
\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| &= \left|\frac{\omega^2 - (z^2 - 2\omega z + \omega^2)}{\omega^2 (z-\omega)^2}\right| \\
&= \frac{\left|2\omega z - z^2\right|}{|\omega^2| \, |z-\omega|^2} \\
&= \frac{|\omega z| \, |2 - z/\omega|}{|\omega|^4 \, |1 - z/\omega|^2} \\
&= \frac{|z| \, |2 - z/\omega|}{|\omega|^3 \, |1 - z/\omega|^2} \\
&\leq \frac{|z| \, (2 + |z/\omega|)}{|\omega|^3 \, (1 - |z/\omega|)^2} \\
&\leq \frac{r \cdot 5/2}{|\omega|^3 \cdot 1/4}
\end{aligned}$$

We know the series $\sum 1/|\omega|^3$ converges by the previous lemma and hence the given series also converges by the Weierstrass M-test. □

The function given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

for a given subgroup Γ is a meromorphic function on $\mathbb{C}$. The poles of $\wp$ are exactly the points of Γ, which are in fact double poles. It is also easy to see that $\wp(z)$ is even (this requires the fact that Γ is a group so if $\omega \in \Gamma$ then $-\omega \in \Gamma$). What is less obvious is the fact that $\wp$ is doubly-period with group of periods Γ. In order to see this we will need to use the fact that $\wp'(z)$ is periodic. By differentiating term by term, we get that

$$\wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}$$

which *is* obviously periodic with respect to Γ (computing $\wp'(z+\omega)$ amounts to simply reordering the sum). Therefore

$$\wp(z + e_i) - \wp(z)$$

for $i = 1, 2$ is constant since the derivative is 0. Taking $z = -e_i/2$ and using the fact that $\wp$ is even, we get that the constant is

$$\wp\left(\frac{e_i}{2}\right) - \wp\left(-\frac{e_i}{2}\right) = \wp\left(\frac{e_i}{2}\right) - \wp\left(\frac{e_i}{2}\right) = 0$$

Hence $\wp(z) = \wp(z + e_i)$ (for $i = 1, 2$).

Consider the Laurent expansion of $\wp$ at 0. It looks like

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots$$

This is because $\wp$ is even and

$$\wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

is holomorphic around 0 and is 0 at 0. We know the right hand side is equal to $a_2 z^2 + a_4 z^4 + \cdots$. By differentiating the series the appropriate number of times we can work out a_2 and a_4 explicitly. For example,

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = a_2 z^2 + a_4 z^4 + \cdots$$

$$\sum_{\omega \in \Gamma \setminus \{0\}} -\frac{2}{(z-\omega)^3} = 2a_2 z + 4a_4 z^3 + \cdots$$

$$\sum_{\omega \in \Gamma \setminus \{0\}} \frac{6}{(z-\omega)^4} = 2a_2 + 12 a_4 z^2 + \cdots$$

Substituting $z = 0$, we get

$$a_2 = 3 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^4}$$

Similarly we get

$$a_4 = 5 \sum_{\omega \in \Gamma \setminus \{0\}} \frac{1}{\omega^6}$$

We want to relate $\wp(z)$ and $\wp'(z)$ to get a differential equation. First we see that

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \cdots$$

Therefore, in order to relate $\wp$ and $\wp'$ to get a holomorphic function we need to at least cube $\wp$ and square $\wp'$ so we can start cancelling out the principal parts. We see that

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + z^2(\cdots)$$

and

$$\wp(z)^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + z^2(\cdots)$$

Therefore

$$\wp'(z)^2 - 4\wp(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + z^2(\cdots)$$

Now we observe that $\frac{-20a_2}{z^2} = -20a_2\wp(z) + z^2(\cdots)$. Therefore by absorbing the remaining portion into the z^2 term we get

$$\wp'(z)^2 - 4\wp(z)^3 + 20a_2\wp(z) + 28a_4 = z^2(\dots)$$

Notice that this is holomorphic near 0, is 0 at 0 and is periodic. Therefore

$$\wp'(z)^2 - 4\wp(z)^3 + 20a_2\wp(z) + 28a_4$$

is a bounded entire function so must be constant by Liouville's Theorem and by evaluating at 0 we see that the constant must be 0. Consider a curve in $\mathbb{C}^2$ given by $y = \wp'(z)$ and $x = \wp(z)$. Then we know that this satisfies the equation

$$y^2 = 4x^3 - 20a_2x - 28a_4$$

In fact we will see that any curve satisfying such an equation (where recall a_2 and a_4 are dependent on a discrete subgroup of $\mathbb{C}$) is given by $(\wp(z), \wp'(z))$ for $\wp(z)$ with appropriate group of periods.

3.1 Doubly Periodic Functions

Although we will mostly apply them to the Weierstrass $\wp$ function, it is useful to keep some facts about general doubly-periodic functions in mind.

Proposition 3.3 *Suppose $f(z)$ is a non-constant meromorphic function with Γ as a group of periods. Then the number of zeroes of f in a period parallelogram is equal to the number of poles of f in the parallelogram when both are counted with multiplicity (provided that there are no poles or zeroes on the boundary)*

Proof. This is a consequence of the argument principle.

A period parallelogram is found by taking any $z_0 \in \mathbb{C}$ and considering the parallelogram given by the points $z_0, z_0 + e_1, z_0 + e_1 + e_2, z_0 + e_2$ where e_1, e_2 are the generators for Γ. Let γ be boundary of this parallelogram. By choosing z_0 appropriately, we can ensure that no poles lie on the γ. Then consider

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

On the one hand we know by the argument principle that this is equal to the number of zeroes minus the number of poles. On the other hand the periodicity of f (and therefore f') implies that the integral is 0 (for example the integral over the bottom edge is the same as the integral over the top edge but with a flipped sign). Therefore the number of poles and zeroes is equal. □

Similar to the above result we can also comment on the sum of the poles and zeroes.

Proposition 3.4 *Suppose $f(z)$ is a non-constant meromorphic function with Γ as a group of periods. Let $a \in \mathbb{C}$ be arbitrary. Let α_i be the roots of $f(z) - a$ and β_i the poles of $f(z)$ (both counted with multiplicity) in a period parallelogram. Then*

$$\sum \alpha_i \equiv \sum \beta_i \bmod \Gamma$$

In particular the sum of roots of $f(z) = a$ is independent of a.

Proof. A consequence of the argument principle is that the sum of zeroes minus the sum of poles (counted with multiplicity) in the period parallelogram is given by

$$\frac{1}{2\pi i}\int_{\gamma}\frac{zf'(z)}{f(z)-a}dz$$

where γ is the boundary of the parallelogram. Let γ_1 be the bottom edge and γ_3 the top edge. Notice that $\gamma_3(t)=\gamma_1(1-t)+e_2$ (see Figure 7).

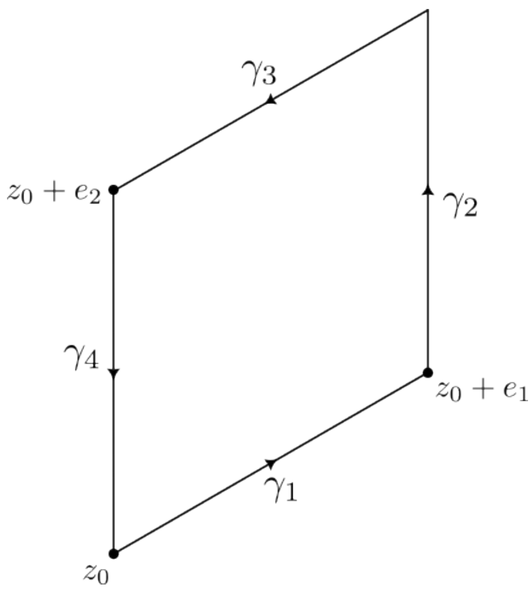


Figure 7: $\gamma_3(t)=\gamma_1(1-t)+e_2$

Therefore

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_3}\frac{zf'(z)}{f(z)-a}dz &= -\frac{1}{2\pi i}\int_{\gamma_1}\frac{(z+e_2)f'(z+e_2)}{f(z+e_2)-a}dz\\ &= -\frac{1}{2\pi i}\int_{\gamma_1}\frac{zf'(z)}{f(z)-a}dz - e_2\cdot\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)-a}dz\end{aligned}$$

Therefore in particular

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)-a}dz+\frac{1}{2\pi i}\int_{\gamma_3}\frac{f'(z)}{f(z)-a}=-e_2\cdot\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)-a}dz$$

Moreover the coefficient of e_2 is an integer since

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)-a}dz=\frac{1}{2\pi i}\int_{(f-a)\circ\gamma_1}\frac{1}{w}dw$$

is simply the winding number of $(f-a)\circ\gamma_1$ with respect to 0.

A similar thing happens with the left and right edges which we label γ_2 and γ_4 respectively. Therefore

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-a}dz &= \sum_{i=1}^{4}\frac{1}{2\pi i}\int_{\gamma_i}\frac{f'(z)}{f(z)-a}dz \\
&= e_1\cdot\underbrace{-\frac{1}{2\pi i}\int_{\gamma_2}\frac{f'(z)}{f(z)-a}dz}_{\in\mathbb{Z}} + e_2\cdot\underbrace{-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)-a}dz}_{\in\mathbb{Z}}
\end{aligned}$$

which is in Γ. □

We immediately apply the results above to the case of $\wp(z)$.

Theorem 3.5 *The right hand side of the equation satisfied by $(x,y)=(\wp(z),\wp'(z))$, namely*

$$y^2 = 4x^3 - 20a_2x - 28a_4 \tag{3.1}$$

has 3 distinct roots. Moreover for all (x,y) on this curve there exists a unique $z\in\mathbb{C}/\Gamma$ such that $(x,y)=(\wp(z),\wp'(z))$.

Proof. Given $a\in\mathbb{C}$, we know that $\wp(z)=a$ has 2 roots in the period parallelogram and $\wp'(z)=a$ has 3 roots. This follows from Proposition 3.3 and the fact that $\wp(z)$ and $\wp'(z)$ have a double and triple pole (respectively) at $\omega\in\Gamma$.

We want to consider points $z\in\mathbb{C}$ such that $2z\in\Gamma$ but $z\notin\Gamma$. These points are interesting because they are exactly the points satisfying $z\equiv -z \bmod \Gamma$. It is easy to see that the only such points modulo Γ are $e_1/2, e_2/2$ and $(e_1+e_2)/2$. This is because if

$$2z = n_1e_1 + n_2e_2$$

Then

$$z = \frac{n_1}{2}e_1 + \frac{n_2}{2}e_2$$

so the only solutions modulo Γ are when one or both of the coefficients of e_1, e_2 are $1/2$. If z is any of the three points above then $z\equiv -z$ so

$$\wp'(z) = \wp'(-z)$$

On the other hand $\wp'$ is odd so for any z at all we have

$$\wp'(-z) = -\wp'(z)$$

Therefore we conclude that $\wp'(z)=0$ at the three points above. This means that at these three points the left hand side of the equation (3.1) is 0 and thus $\wp(e_1/2), \wp(e_2/2), \wp((e_1+e_2)/2)$ are zeroes of the right hand side. All that remains to show is that these are distinct.

Let z_0 be one of the 3 special points. Then we know that $\wp(z)-\wp(z_0)$ is 0 at z_0 and its derivative $\wp'(z)$ is also 0 at z_0. Therefore $\wp(z)-\wp(z_0)$ has a double root at z_0. Since $\wp(z)-a$ has exactly 2 roots for any a we know that $\wp(z_0)$ cannot be achieved by any other point in the period parallelogram. Therefore the 3 zeroes to the equation are indeed distinct.

For the second part of the statement, we already know the case for $y=0$ (in particular since (x,y) is on the curve, if $y=0$ then x would be a root of the right hand side and we have seen in this case x is necessarily one of $\wp(e_1/2), \wp(e_2/2)$ and $\wp((e_1+e_2)/2)$). Suppose $y\neq 0$.

Then we know that $\wp(z) = x$ has 2 roots and since $\wp$ is even we know that $\wp(z) = \wp(-z)$ so the two roots are given by z and $-z$. Since y is not 0 we know that $2z \notin \Gamma$ so in particular z and $-z$ are distinct. Looking at the equation, we can see that for a fixed x, there are exactly two choices of y (provided $y \neq 0$) that only differ by a sign. Since $\wp'$ is odd, we know then that y must be given by $\wp'(z)$ or $\wp'(-z)$. Thus for every (x, y) on the curve, there exists a unique $z \in \mathbb{C}/\Gamma$ satisfying $(x, y) = (\wp(z), \wp'(z))$. □

3.2 Compactification of Elliptic Curve

Let $X \subset \mathbb{C}^2$ be the curve satisfying

$$y^2 = 4x^3 - 20a_2x - 28a_4$$

We know that the right hand side has 3 distinct roots and we would like to use the Implicit Function Theorem to conclude that X is smooth. Unfortunately, we don't yet have the theorem for several complex variables (indeed we don't even know yet what it means for a function in more than one variable to be holomorphic). But there is a weaker version of the statement that is enough for us.

Theorem 3.6 ((Weak) Implicit Function Theorem) *Suppose $f(x, y)$ is a C^1 function (when viewed as a function from $\mathbb{R}^4$ to $\mathbb{R}^2$) that is separately holomorphic in each variable. Then if $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then we can solve for y as a function of x (i.e. we get $y = y(x)$) with $y(x_0) = y_0$.*

Proof. Suppose we write $x = x_1 + ix_2, y = y_1 + iy_2$ and $z = f(x, y)$ so that $z = z_1 + iz_2$ or $f = f_1 + if_2$. For a fixed x we have

$$dz = \frac{\partial f}{\partial y} dy$$

and

$$d\overline{z} = \overline{\frac{\partial f}{\partial y}} d\overline{y}$$

Therefore

$$dz \wedge d\overline{z} = \left|\frac{\partial f}{\partial y}\right|^2 dy \wedge d\overline{y}$$

Since $dy = dy_1 + idy_2$ and $d\overline{y} = dy_1 - idy_2$ we compute that

$$dy \wedge d\overline{y} = -2idy_1 \wedge dy_2$$

Similarly of course we get

$$dz \wedge d\overline{z} = -2idz_1 \wedge dz_2$$

This means that

$$dz_1 \wedge dz_2 = \left|\frac{\partial f}{\partial y}\right|^2 dy_1 \wedge dy_2$$

In particular this means that $\frac{\partial(z_1,z_2)}{\partial(y_1,y_2)}$ is invertible at (x_0, y_0) and so by the Real Implicit Function Theorem we can write y as a function of x so that $f(x, y(x)) = 0$ for all x in some open neighbourhood of x_0. All that remains to do is show that y is holomorphic. Differentiating $f(x, y(x))$ with respect to x we get

$$0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial \overline{x}}d\overline{x}\right)$$

Note that there is no $\partial/\partial\overline{x}$ term on the left. Therefore by linear independence of $\partial/\partial x$ and $\partial/\partial\overline{x}$ we conclude that

$$\frac{\partial y}{\partial \overline{x}} = 0$$

Thus y is indeed holomorphic. □

Now that we have this result we can consider the curve again. Suppose we have

$$f(x, y) = y^2 - (4x^3 - 20a_2x - 28a_4)$$

We want to show that we have local coordinates for every point in the zero set of f. Notice that

$$\frac{\partial f}{\partial y} = 2y$$

so for $y \neq 0$ we know that $\frac{\partial f}{\partial y}$ is invertible so we can solve for y as a function of x. We know y is 0 for exactly 3 points, namely the roots of the cubic polynomial in x. However these roots must be simple (a cubic polynomial can have at most 3 distinct roots and we have exactly that) so in particular $\frac{\partial f}{\partial x}$ is non-zero at these points. Therefore in a neighbourhood of these points we can solve for x as a function of y.

Recall how it is very useful to adjoin a point at ∞ to $\mathbb{C}$. We want to try doing the same thing for the curve X by compactifying it. For this we will need to know about n-dimensional complex projective space.

The n-dimensional complex projective space is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. In other words

$$P^n(\mathbb{C}) := \mathbb{C}^{n+1} \setminus \{0\}/\sim$$

where

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \Leftrightarrow (x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$$

for some $\lambda \in \mathbb{C}$.

This is an n-dimensional complex manifold which we can cover with $n+1$ coordinate charts. For $i = 0, \dots, n$ we define

$$U_i := \{[x_0, \dots, x_n] \in P^n(\mathbb{C}) : x_i \neq 0\}$$

Then we can have

$$\begin{aligned} \phi_i : U_i &\to \mathbb{C}^n \\ [x_0, \dots, x_n] &\mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}\right) \end{aligned}$$

The inverse is given by

$$\begin{aligned} \phi_i^{-1} : \mathbb{C}^n &\to U_i \\ (z_1, \dots, z_n) &\mapsto [z_1, \dots, z_i, 1, z_{i+1}, \dots, z_n] \end{aligned}$$

It is then easy to see that the transition maps between these charts are given by rational functions (although we still haven't properly defined what it means for a multivariate complex function to be holomorphic certainly the theory should include rational functions).

Since we are looking at a curve in $\mathbb{C}^2$ we only really need to focus on $P^2(\mathbb{C})$. We can decompose it like so

$$P^2(\mathbb{C}) = \underbrace{\{[x,y,t] \in P^2(\mathbb{C}) : t \neq 0\}}_{\mathbb{C}^2} \sqcup \underbrace{\{[x,y,t] \in P^2(\mathbb{C}) : t = 0\}}_{P^1(\mathbb{C})}$$

For convenience we will call the first set above U_0. In $U_0 \subset P^2(\mathbb{C})$, the coordinates are given by $(x/t, y/t)$. Writing the equation of the curve in these coordinates we get

$$\left(\frac{y}{t}\right)^2 = 4\left(\frac{x}{t}\right)^3 - 20a_2\left(\frac{x}{t}\right) - 28a_4$$

Written like this, the equation is just begging to be homogenised. Doing so, gives us the closure of X in $P^2(\mathbb{C})$

$$X' : y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4 t^3$$

The points of X of course still lie in X' which are given when $t \neq 0$. Therefore the new points occur when $t = 0$. Notice when $t = 0$, we must have $x = 0$. Therefore taking the closure we only have one new point, $[0,1,0]$ which we call a (or in this case the) point at infinity. Around this point the coordinates of X' are given by $(x', t') := (x/y, t/y)$ so we dehomogenise with respect to y (i.e. we divide through by y^3) giving us

$$t' = 4x'^3 - 20a_2 x' t'^2 - 28a_4 t'^3$$

In fact we can solve for t' as a holomorphic function of x' and even write out the first few terms of its power series

$$t' = 4x'^3 - 320a_2 x'^7 + \dots$$

Note that since X is a curve in $\mathbb{C}^2$ we have a natural map onto $\mathbb{C}$ which is given by simply projecting onto one of coordinates, say the first one. Say this map is given by φ. Then the question becomes can we extend φ to $\varphi' : X' \to S^2$ in such a way that $\varphi'([0,1,0]) = \infty$. In other words, we have the following diagram

$$\begin{array}{ccccccc} \mathbb{C}^2 & \hookleftarrow & X & \subset & X' & \hookrightarrow & P^2(\mathbb{C}) \\ & \searrow^{\pi_1} & \downarrow \varphi & & \downarrow \varphi' & & \\ & & \mathbb{C} & \subset & S^2 & & \end{array}$$

where π_1 is the projection from $\mathbb{C}^2$ to $\mathbb{C}^1$ onto the first coordinate and $\varphi = \pi_1|_X$. We need to check that φ' is holomorphic around $[0,1,0]$. Therefore naturally, we use the appropriate coordinates around this point, namely we have

$$X' \cap U_1 = \{[x_1, 1, t_1]\}$$

where by above we know that

$$t_1 = 4x_1^3 - 320a_2 x_1^7 + \dots$$

Recall that X is a subset of U_0 so for the $t_1 \neq 0$ we know $[x_1/t_1, 1/t_1, 1] \in X \subset U_0$. Therefore

$$z := \varphi'([x_1, 1, t_1]) = \frac{x_1}{4x_1^3 - 320a_2 x_1^7 + \dots}$$

We want to check that defining this to be ∞ at $x_1 = 0$ is holomorphic. For this we switch to coordinates at infinity

$$1/z = \frac{4x_1^3 - 320a_2x_1^7 + \dots}{x_1}$$

which *does* extend holomorphically to $x_1 = 0$ and therefore φ extends to φ' holomorphically. Notice this also shows that φ' has a double pole at $[0, 1, 0]$.

We already knew by Theorem 3.5 that, ignoring the points of Γ, the map $z \mapsto [\wp(z), \wp'(z), 1]$ was an injective holomorphic map on $\mathbb{C}/\Gamma$. The work done above shows that this map can be extended holomorphically to the points of Γ by mapping them to $[0, 1, 0]$ and thus we have a biholomorphism between $\mathbb{C}/\Gamma$ and X'. We know that $\mathbb{C}/\Gamma$ is a torus which means in particular that the completed curve X' is isomorphic to $S^1 \times S^1$.

One might wonder whether there is an explicit formula for the inverse of this biholomorphism (which is of course only determined up to the addition of a constant in Γ). For this we take inspiration from the analogous situation that occurs with sin and cos.

Consider the curve

$$C : y^2 = 1 - x^2$$

in $\mathbb{R}^2$. We know this curve (the unit circle) is parameterised by $x = \cos\theta, y = \sin\theta$. What we would like to do is given a point (x, y) on the curve recover what θ is (which will only be unique up to integer multiples of 2π). We see that

$$dy = \sin'\theta d\theta = \cos\theta d\theta = x d\theta$$

Notice this implies that $d\theta = dy/x$ From the definition of the curve we know that

$$xdx + ydy = 0$$

so in particular

$$d\theta = \frac{dy}{x} = -\frac{dx}{y}$$

Then we can recover θ by

$$\theta = \int_{(1,0)}^{(\cos\theta,\sin\theta)} \frac{dy}{x} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}}$$

in a neighbourhood of $(1, 0)$ (i.e. where $x \neq 0$). This is of course how we defined arcsin in first year.

> **Remark 3.7.** The integral above is not technically well-defined since the unit circle is not simply connected. It will depend on the path chosen between $(1, 0)$ and $(\cos\theta, \sin\theta)$. However the value for different paths will only differ by integer multiples of 2π as we expect.

With this in mind we can go back to our curve X'. Notice we have

$$dx = \wp'(z)dz = ydz$$

so in particular

$$dz = \frac{dx}{y}$$

for $y \neq 0$. From the definition of the curve we have

$$2ydy = (12x^2 - 20a_2)dx$$

and hence

$$\frac{dy}{6x^2 - 10a_2} = \frac{dx}{y} = dz$$

Then just like before we can recover z by

$$z = \wp^{-1}(x) = \int_{[0,1,0]}^{[\wp(z),\wp'(z),1]} \frac{dx}{y} = \int_{[0,1,0]}^{[\wp(z),\wp'(z),1]} \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}$$

Again since a torus is not simply connected, the integral is not technically well-defined but answers will only differ by elements of Γ.

4 Functions with prescribed zeroes and poles

We want to explore how constrained (or not) the space of holomorphic functions/meromorphic functions is. One way we can try exploring this is to ask whether we can always construct a holomorphic/meromorphic function with a given set of zeroes/poles. The answer in both cases is yes. We will begin by considering the case for poles. The case for zeroes will require us to build some theory about infinite products.

Theorem 4.1 (Mittag-Leffler) *Given a set of poles $\{b_k\} \subset \mathbb{C}$ such that $\lim_{k\to\infty} b_k = \infty$ and $\{P_k(z)\}$ set of polynomials without constant term, we can find a meromorphic function with poles b_k and principal parts $P_k(1/(z - b_k))$. In fact the most general such meromorphic function on $\mathbb{C}$ is*

$$f(z) = \sum_{k=1}^{\infty} \left(P_k\left(\frac{1}{z - b_k}\right) - p_k(z) \right) + g(z)$$

where $p_k(z)$ are (well-chosen) polynomials to guarantee convergence and g is any entire function.

Remark 4.2. The assumption $\lim_{k\to\infty} b_k = \infty$ ensures that the b_k don't have a finite accumulation point. We know that the poles of meromorphic functions are isolated so this is certainly necessary.

Proof. We can assume that b_k are all non-zero. Then $P_k(1/(z - b_k))$ is holomorphic in $|z| < |b_k|$ and so we can expand it as a Taylor series at 0. Let $p_k(z)$ be sum of the first n_k terms where n_k is chosen so that

$$\left| P_k\left(\frac{1}{z - b_k}\right) - p_k(z) \right| \leq \frac{1}{2^k}$$

for $|z| \leq |b_k|/2$. Then we claim that

$$\sum_{k=1}^{\infty} P_k\left(\frac{1}{z-b_k}\right) - p_k(z)$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}$. In fact we will show we have convergence on $|z| \leq r$ for any r. In order to see this, choose m so that $|b_k| > 2r$ for $k \geq m$. Then for $|z| \leq r < |b_k|/2$ for such k we have

$$\sum_{k=m}^{\infty} \left| P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right| \leq \sum_{k=m}^{\infty} \frac{1}{2^k}$$

which we know converges.

Suppose we have have two functions with the given poles and principal parts. Then their difference is holomorphic on the complex plane and hence entire. This gives the second part of the theorem. □

4.1 Infinite products

Suppose b_k is a sequence of points in $\mathbb{C}$. Then naturally we want to say that

$$\prod_{k=1}^{\infty} b_k := \lim_{n\to\infty} \prod_{k=1}^{n} b_k$$

In other words, the infinite product 'should' converge if the partial products do. But of course the partial products might simply converge if one of the b_k is zero. Therefore we will also assert the the limit should be non-zero. But of course there are times when we want to allow 0 to be a point in the sequence (we are building of course to taking products of functions which may take the value of 0 at certain points and indeed our ultimate goal is to build holomorphic functions with a presecribed set of zeros). Therefore, we will say $\prod b_k$ converges if only finitely many of the terms are 0 and the partial products of the remaining terms converges to a non-zero finite complex number. Notice that a necessary condition for convergence is $b_k \to 1$ since

$$b_k = \frac{\prod_{j=1}^{k} b_j}{\prod_{j=1}^{k-1} b_j}$$

Therefore we often write $b_k = 1 + a_k$ where $a_k \to 0$.

We know a lot about convergence of series so it would be nice if we could translate the convergence of infinite products to the convergence of infinite sums. The way we will do this is by using log of course.

Theorem 4.3 *The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ with $1+a_n \neq 0$ converges if and only if $\sum_{n=1}^{\infty} \log(1+a_n)$ does.*

Proof. The above makes sense because we know for sufficiently large n, the b_n in the original product tend to 1 (which is to say a_n tend to 0). This means for sufficiently large n, $1+a_n$ is away from 0 so $\log(1+a_n)$ is well-defined and we can choose a consistent branch of log for all a_n. We will of course use the principal branch of log.

Let S_n denote the partial sum of the series and let P_n be the partial product of the infinite product. In particular we have $P_n = e^{S_n}$. Therefore if $S_n \to S$, it follows by continuity of the exponential that $e^{S_n} \to e^S =: P$ which in particular is non-zero. Therefore if the series converges then so does the product.

Now suppose the product converges so we have $P_n \to P$. We want to say of course that $S_n \to \log(P)$. In fact this might not be true (where recall we are taking log to be the principal branch of log). However the limit will differ from $\log(P)$ only by an integer multiple of 2π. In order to see this note that for every n there exists an integer h_n such that

$$\log\left(\frac{P_n}{P}\right) = S_n - \log(P) + h_n \cdot 2\pi i$$

We will show that all h_n are equal. We see that

$$(h_{n+1} - h_n)2\pi i = \log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) + \log(1 + a_{n+1})$$

Notice the left hand side is purely imaginary and the imaginary component of $\log(z)$ is simply the argument. Thus equating imaginary parts we get

$$(h_{n+1} - h_n)2\pi = \underbrace{\arg\left(\frac{P_{n+1}}{P}\right) - \arg\left(\frac{P_n}{P}\right)}_{\to 0} + \underbrace{\arg(1 + a_{n+1})}_{\leq \pi}$$

Since we are taking the principal branch of log we know that $|\arg(z)| \leq \pi$ for any z. Moreover by convergence we know that $\arg(P_{n+1}/P) - \arg(P_n/P) \to 0$. Therefore for large n the right hand side can be made smaller than $\pi + \epsilon$ (for any $\epsilon > 0$) in absolute value. However if h_{n+1}, h_n are different then the left hand side is at least 2π in absolute value (recall that the h_n are integers). Therefore we must have $h_{n+1} = h_n$ (for all n sufficiently large). Therefore

$$S_n \to \log P + 2\pi i h$$

□

We say that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if $\sum_{n=1}^{\infty} \log(1+a_n)$ converges absolutely. This series converges absolutely if and only if $\sum_{n=1}^{\infty} a_n$ converges absolutely. This is because

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1$$

Therefore

$$\left|\frac{\log(1+a_n)}{a_n} - 1\right| < \epsilon$$

for n large enough. This means that

$$(1-\epsilon)\,|a_n| < |\log(1+a_n)| < (1+\epsilon)\,|a_n|$$

Therefore if $\sum |a_n|$ converges we can compare it with $\sum |\log(1+a_n)|$ via the right inequality and if $\sum |\log(1+a_n)|$ converges we can compare it with $\sum |a_n|$.

Having discussed infinite products of complex number we naturally want to discuss infinite products of complex-valued functions.

Definition 4.4. Given a sequence of functions $f_n(z)$ defined on an open set Ω, we say that $\prod f_n(z)$ converges on compact $K \subset \Omega$ if

1. $f_n(z) \to 1$ uniformly on K
2. $\sum \log f_n$ is uniformly and absolutely convergent on K

By the previous theorem, the above conditions imply that the partial products converge uniformly on compact sets.

We state some trivial properties of infinite products.

Theorem 4.5 *Suppose $f_n(z)$ is a sequence of functions on an open subset Ω. Suppose $\prod f_n(z)$ converges absolutely and uniformly to the function $f(z)$ on compact subsets of Ω. Then*

1. *$f(z)$ is holomorphic on Ω and we have an 'associativity law'*

$$f = f_1 f_2 \dots f_p \prod_{n>p} f_n$$

2. *the set of zeroes of f, $Z(f)$ is*

$$Z(f) = \bigcup_{n=1}^{\infty} Z(f_n)$$

and the multiplicity of any zero of f is the sum of multiplicities of the point at all f_n

3. *the series of meromorphic functions $\sum f_n'/f_n$ converge uniformly and absolutely on compact subsets of Ω and*

$$\sum_{n=1}^{\infty} \frac{f_n'}{f_n} = \frac{f'}{f}$$

Proof. The first two statements are obvious. Let us then prove the third statement. Let K be a compact subset of Ω. Suppose we can choose a consistent branch of log for all the $f_n|_K$ (say because they only take values in a simply connected set not containing 0). Then we would have

$$\log f = \sum_{n=1}^{\infty} \log f_n$$

and differentiating both sides (recall we can different the series term by term) we would have

$$\frac{f'}{f} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n}$$

Of course it is possible that we do *not* have a consistent choice of log for all the f_n. For example, they might be 0 at some points. On the other hand, since we are working on a compact set, we know that $f_n \to 1$ uniformly on K. Therefore for sufficient large n we do have a consistent choice of log. Suppose p is such that for $n > p$ we have $|f_n(z) - 1| < 1$ for all $z \in K$. Then we define

$$g_p := \exp\left(\sum_{n>p} \log f_n\right)$$

In fact this is not quite sufficient. Because we want to work with holomorphic properties of g_p, we need to be working on an open set but g_p above is only defined on K. In order to fix this we will instead work on a slighly larger open set U which contains the given compact set K and whose closure is compact. Because the closure of U is compact, we can still pick a sufficiently large p to make g_p well-defined on U.

From above it follows that

$$\frac{g_p'}{g_p} = \sum_{n>p} \frac{f_n'}{f_n}$$

We have

$$f = f_1 \cdots f_p \cdot g_p$$

From the product rule and above equation it follows that

$$\frac{f'}{f} = \sum_{n=1}^{p} \frac{f_n'}{f_n} + \frac{g_p'}{g_p} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n}$$

□

Let us try express sin as an infinite product. Let us consider $\sin(\pi z)$ so that the zeroes lie on the integers. Then the natural choice is

$$f(z) = z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

which we wish to argue converges uniformly and absolutely on compact subsets of $\mathbb{C}$. But from above we know this is the same as showing that $\sum z^2/n^2$ converges absolutely and uniformly on compact sets and we know this holds true by comparison with $1/n^2$ (and the fact that on compact sets $|z|$ is bounded).

Therefore $f(z)$ is a holomorphic function with zeroes on the integers with each zero being simple. Then

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot(\pi z) = \frac{g'(z)}{g(z)}$$

where $g(z) = \sin(\pi z)$. Since their logarithmic derivatives are equal we conclude that $f(z) = cg(z)$ where c is some constant (this follows from consider $(f/g)'$ and concluding that it must be 0). We see that

$$\frac{f(z)}{z} \to 1$$

as $z \to 0$ and

$$\frac{\sin(\pi z)}{z} \to \pi$$

Hence the constant c must be $1/\pi$.

Now we can ask the natural extension of the Mittag-Leffler Theorem: given a sequence of complex numbers, can we find an entire function f where this sequence is exactly the zero set of f?

First suppose we want a function with no zeroes. We claim that any entire function f which is non-zero everywhere is of the form $e^{g(z)}$ where $g(z)$ is entire. In order to see this, consider f'/f which is also entire hence has an entire primitive $g(z)$. Then $f(z)e^{-g(z)}$ has 0 derivative since

$$(f(z)e^{-g(z)})' = f'(z)e^{-g(z)} - f(z)g'(z)e^{-g(z)} = 0$$

This means that $f(z) = Ae^{g(z)}$ and we can absorb the constant A into the exponent.

Naturally then if we want an entire function with a zero at the origin of order m (possibly zero) and zeroes at $a_1, \ldots, a_n$ (possibly with repetition), then the most general such function is

$$e^{g(z)} z^m \prod_{k=1}^{n} \left(1 - \frac{z}{a_k}\right)$$

For the case of infinitely many zeroes, we look to the following theorem by Weierstrass.

Theorem 4.6 (Weierstrass) *Given a sequence $\{a_k\}$ in $\mathbb{C}$ such that $\lim_{k\to\infty} a_k = \infty$, there exists an entire function with zeroes exactly a_k. The most general such function is of the form*

$$f(z) = e^{g(z)} z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{P_k(z)}$$

where a_k are all non-zero and where $P_k(z)$ are polynomials of the form

$$P_k(z) = \frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{a_k}\right)^{m_k}$$

Proof. As usual, we convert the question of infinite products to a question of infinite sums. We know the given product converges if and only if

$$\sum_{k=1}^{\infty} \log\left(1 - \frac{z}{a_k}\right) + P_k(z)$$

We will deal with this much like we did with the Mittag-Leffler Theorem, using terms from the Taylor series to ensure convergence.

Let us denote the terms of the above series by $g_k(z)$. Recall that

$$\log\left(1 - \frac{z}{a_k}\right) = -\frac{z}{a_k} - \frac{1}{2}\left(\frac{z}{a_k}\right)^2 - \cdots$$

We are going to choose $P_k(z)$ to be first few terms of this Taylor series. The main question of course is how many terms should we take. Suppose we take the first m_k terms (we will say precisely what m_k should be shortly). Then

$$g_k(z) = -\frac{1}{m_k+1}\left(\frac{z}{a_k}\right)^{m_k+1} - \frac{1}{m_k+2}\left(\frac{z}{a_k}\right)^{m_k+2} - \cdots$$

Suppose $|z| \leq r$ and consider a_k such that $|a_k| > r$ (we are going to show that this tail of the series is convergent for a suitable choice of m_k) Then

$$\begin{aligned}|g_k(z)| &\leq \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1} + \frac{1}{m_k+2}\left(\frac{r}{|a_k|}\right)^{m_k+2} + \dots \\ &\leq \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1}\left(1 + \frac{r}{|a_k|} + \left(\frac{r}{|a_k|}\right)^2 + \dots\right) \\ &= \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1}\left(1 - \frac{r}{|a_k|}\right)^{-1}\end{aligned}$$

Notice that we can bound $(1 - r/|a_k|)^{-1}$ by a constant since r is fixed and $a_k \to \infty$. Therefore if

$$\sum_{k=1}^{\infty} \frac{1}{m_k+1}\left(\frac{r}{|a_k|}\right)^{m_k+1} \tag{4.1}$$

converges then $\sum g_k$ also does. Therefore we need to choose m_k so (4.1) converges. A possible choice is $m_k = k$. □

Corollary 4.7 *Every meromorphic function on the plane is the quotient of two entire functions.*

Proof. Suppose h is a meromorphic function on the plane. Let g be an entire function which has zeroes at exactly the poles of h with the same multiplicities. Then $g(z)h(z)$ is entire on the plane. If we call this function f then

$$h(z) = \frac{f(z)}{g(z)}$$

□

5 Normal Families

Recall that a metric space is compact if and only if every (infinite) sequence has a convergent (infinite) subsequence. The metrizable space we are interested in is $\mathcal{C}(\Omega)$ or more precisely in its subspace $\mathcal{H}(\Omega)$. We will say that a family of continuous complex-valued functions $\mathscr{S} \subset \mathcal{C}(\Omega)$ is *normal* if every (infinite) sequence in $\mathscr{S}$ has an (infinite) subsequence that converges, although the limit may not lie in $\mathscr{S}$. Equivalently then, normal families are exactly the subsets of $\mathcal{C}(\Omega)$ with compact closure. An example of a normal family is $\mathscr{S} := \{f_n(z) = z^n : n \in \mathbb{N}\}$ on the unit disk. We know that the f_n converge (uniformly and absolutely) on compact subsets of the disk D but the limit does not lie in $\mathscr{S}$.

A nice way of checking that a family of functions is normal is the following.

Lemma 5.1 *A family of continuous functions $\mathscr{S} \subset \mathcal{C}(\Omega)$ is normal if and only if for every 'suitable' cover $\{E_i\}$ (which is to say that $\Omega = \bigcup_i E_i$) and every i we have that every infinite sequence in $\mathscr{S}$ has a subsequence which converges in E_i.*

Remark 5.2. What we mean by a suitable cover will be made clear in the proof.

Proof. The reverse direction is clear since we are given that in particular every sequence has a convergent subsequence. Thus we only need to show the converse.

We want to show that if the suitable cover condition holds then $\mathscr{S}$ is normal. So let $\{f_n\}$ be a a sequence in $\mathscr{S}$. Then we know by assumption that there is a subsequence $\{f_n^{(1)}\}$ which converges uniformly on E_1. Then there also exists a subsequence $\{f_n^{(2)}\}$ of $\{f_n^{(1)}\}$ that converges uniformly on E_2. Continuing in this manner we can construct a subsequence $\{f_n^{(k)}\}$ of $\{f_n^{(k-1)}\}$ that converges on E_k. Then the diagonal sequence $\{f_n^{(n)}\}$ converges on E_i for every i. Therefore a cover will be suitable if this allows us to conclude that we have convergence on all compact subsets of Ω. This means that convergence on one suitable cover immediately implies convergence on other suitable covers. □

A simple example of a suitable cover would be a covering by closed disks in Ω whose interiors cover Ω. If K is a compact subset, then it is contained in the union of finitely many of the disks so we have uniform convergence on K. Another example of a suitable cover is a family of compact sets $\{K_i\}$ with $K_1 \subset K_2 \subset \cdots$ so that $\Omega = \bigcup K_i$. Then any compact set K of Ω is contained in one of the K_i so convergence on K_i automatically implies convergence on K.

When talking of function spaces, the natural starting point is the Arzelà–Ascoli Theorem. For this we first need to discuss equicontinuity.

Definition 5.3 (Equicontinuity)**.** Let X be any subset of $\mathbb{C}$ and let $\mathscr{S} \subset \mathcal{C}(X)$ be a family of continuous functions. Then we say that $\mathscr{S}$ is equicontinuous at $a \in X$ if for every $\epsilon > 0$ there is some $\delta > 0$ such that for any $f \in \mathscr{S}$ we have $|f(z) - f(a)| < \epsilon$ for every $z \in X$ satisfying $|z - a| < \delta$. We say $\mathscr{S}$ is equicontinuous if $\mathscr{S}$ is equicontinuous at every point and we say it is uniformly equicontinuous if δ can be chosen independently of the point $a \in X$.

Remark 5.4. In particular δ only depends on ϵ and not on any particular $f \in \mathscr{S}$.

Example 5.5. An example of a (uniformly) equicontinuous family of functions is the set of holomorphic functions (on the unit disk say) with $|f'| \leq M$. This collection of functions satisfies $|f(z) - f(w)| \leq M\,|z - w|$ for $z, w \in D$ for every f so given any $\epsilon > 0$ we can take $\delta = \epsilon/M$.

The Arzelà–Ascoli Theorem typically says that a family of continuous functions has a convergent subsequence if it is equicontinuous and bounded. In our case, we need the functions to have a uniform bound on compact sets (although the bound may vary with the sets). In fact due to equicontinuity, it is sufficient to require boundedness at a single point as the following proposition demonstrates.

Proposition 5.6 *If Ω is a domain and $\mathscr{S}$ is an equicontinuous family of functions then the following are equivalent*

1. *There exists some $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathscr{S}\}$ is bounded*

2. *For every $z \in \Omega$, $\{f(z) : f \in \mathscr{S}\}$ is bounded*

3. *$\mathscr{S}$ is locally bounded, which is to say that for every $z_0 \in \Omega$ there is some open neighbourhood U of z_0 in Ω such that $|f(z)| \leq M$ for every $z \in U$.*

Proof. Equicontinuity implies that for every $w \in \Omega$, there exists a disk $D_w \subset \Omega$ centered at w such that $|f(z) - f(w)| < 1$ for all $z \in D_w$ for all $f \in \mathscr{S}$.

In order to see that 1) $\Rightarrow$ 2), let $U := \{z \in \Omega : \mathscr{S}$ is bounded at $z\}$. By above the statement about the implication of equicontinuity we see that U must be open. But this statement also shows that the complement of U must be open. Suppose $\mathscr{S}$ is unbounded at some w which is to say that the set $\{f(w) : f \in \mathscr{S}\}$ is unbounded. But then consider the above statement again, we know that $|f(z) - f(w)| < 1$ so that $|f(w)| - 1 < |f(z)| < |f(w)| + 1$ for all z sufficiently close to w implying for z, the set $\{f(z) : f \in \mathscr{S}\}$ is also unbounded. Hence U is both open and closed by 1) it is non-empty so $U = \Omega$.

We see that 2) $\Rightarrow$ 3) is immediate. Let z_0 be any point in Ω. We again use the above implication of equicontinuity to conclude that $\mathscr{S}$ is (uniformly) bounded on D_{z_0}. Finally 3) $\Rightarrow$ 1) is immediate. □

Thus we can state the theorem as follows.

Theorem 5.7 (Arzelà–Ascoli) *Let Ω be a domain in $\mathbb{C}$. Then $\mathscr{S} \subset \mathcal{C}(\Omega)$ is normal if and only if*

1. *$\mathscr{S}$ is equicontinuous and*

2. *There exists some $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathscr{S}\}$ is bounded*

Proof. Suppose first that $\mathscr{S}$ is normal. Further suppose there is some $z_0 \in \Omega$ such that $\mathscr{S}$ is not continuous at z_0. This means there exists some $\epsilon > 0$ such that there is a sequence of points $\{z_n\} \subset \Omega$ and a sequence of functions $|f_n(z_n)|$ where $|z_n - z_0| < 1/n$ but $|f_n(z_n) - f_n(z_0)| \geq \epsilon$.

Now choose n_0 so that the closed disk $|z - z_0| \leq 1/n_0$ is contained in Ω. Since $\mathscr{S}$ is normal, we know that $\{f_n\}$ contains a subsequence that converges on this disk. Passing to this subsequence and relabelling, we can assume that $\{f_n\}$ itself converges to some f on this disk. Then

$$\begin{aligned}\epsilon &\leq |f_n(z_n) - f_n(z_0)| \\ &\leq |f_n(z_n) - f(z_n)| + |f(z_n) - f(z_0)| + |f(z_0) - f_n(z_0)|\end{aligned}$$

By uniform convergence of f_n to f, for sufficiently large n, we can ensure that the first and last term are less than $\epsilon/3$. By continuity of f, for large n the middle term can be be made less than $\epsilon/3$. This leads to a contradiction. Therefore $\mathscr{S}$ is indeed an equicontinuous family.

Let z_0 be such that $\{f(z_0) : f \in \mathscr{S}\}$ is unbounded. Then there is a sequence of functions $\{f_n\}$ such that $f_n(z_0) \to \infty$. But normality implies that there is a subsequence that converges at z_0 leading to a contradiciton. Therefore $\{f(z_0) : f \in \mathscr{S}\}$ is bounded for every z_0.

Now suppose $\mathscr{S}$ is equicontinuous and bounded at a point. We will show that it is necessarily normal and we will use the usual diagonal argument one uses for the proof.

Let $T = \{z_k\}$ be a countable dense subset of Ω and let $\{f_n\}$ be any sequence in $\mathscr{S}$. We know by Proposition 5.6 that $\mathscr{S}$ is bounded at every point in Ω. In particular then $\{f_n(z_1)\}$ is a bounded sequence in $\mathbb{C}$ so there exists a subseqence $f_n^{(1)}$ so that $\{f_n^{(1)}(z_1)\}$ converges. Then $\{f_n^{(1)}(z_2)\}$ is bounded as well so there exists a subsequence of $\{f_n^{(1)}\}$ which we call $\{f_n^{(2)}\}$ so that $\{f_n^{(2)}(z_2)\}$ converges. We then continue in this manner. Notice that $\{f_n^{(n)}\}$ converges at z_k for all k. We will relabel this to be the sequence $\{f_n\}$ itself. We want to show that $\{f_n\}$ converges uniformly on compact subsets of Ω.

We will show that for any $\epsilon > 0$ there exists a natural number M such that

$$|f_p(z) - f_q(z)| < \epsilon$$

on K for all $p, q \geq M$. This will show that $\{f_n|_K\}$ is Cauchy and hence converges uniformly on K. Let K be any compact subset of Ω. Then in fact $\mathscr{S}$ is uniformly equicontinuous on K (exercise). Then there exists some $\delta > 0$ such that all $z, w \in K$ we have if $|z - w| < \delta$ then $|f(z) - f(w)| < \epsilon/3$. There exists a finite set $z_{k_1}, \ldots, z_{k_n} \in T \cap K$ such that δ disks centered at the z_{k_j} cover K. Now take $z \in K$ arbitrary. There is some z_{k_j} such that $\left|z - z_{k_j}\right| < \delta$. Then

$$|f_p(z) - f_q(z)| \leq \left|f_p(z) - f_p(z_{k_j})\right| + \left|f_p(z_{k_j}) - f_q(z_{k_j})\right| + \left|f_q(z_{k_j}) + f_q(z)\right|$$

We know the first and last term are less than $\epsilon/3$ by uniform equicontinuity. The central term can be made less than $\epsilon/3$ by taking p, q large enough (this is our choice of M) since f_n converge on all the $z_k \in T$. □

Arzelà–Ascoli is a general theorem about compact subsets in $\mathcal{C}(\Omega)$ but we are really interested in $\mathcal{H}(\Omega)$. Montel's (little) theorem tells us what the compact subsets in this space are.

Theorem 5.8 (Montel's (Little) Theorem) *Let $\Omega \subset \mathbb{C}$ be a domain and consider $\mathscr{S} \subset \mathcal{H}(\Omega)$. Then the following are equivalent:*

1. *$\mathscr{S}$ is normal*
2. *$\mathscr{S}$ is locally bounded*
3. *$\mathscr{S}' := \{f' : f \in \mathscr{S}\}$ is locally bounded and there is some $z_0 \in \Omega$ such that $\{f(z_0)\}$ is bounded.*

Proof. 1) $\Rightarrow$ 2) is an immediate application of Arzelà–Ascoli, since the theorem tells us that normal families are equicontinuous and locally bounded. For 2) $\Rightarrow$ 3) we use Cauchy's inequalities.

Let $z_0 \in \Omega$ be given. By local boundedness, we know there exists some $r > 0$ and $M < \infty$ such that $|f(z)| \leq M$ for all $|z - z_0| < r$ and all $f \in \mathscr{S}$. Then by Cauchy's inequalities for

$n = 1$ (see the discussion following Theorem 1.11) and considering the closed disk of radius $r/2$ we have

$$\left|f'(z_0)\right| \leq \frac{2M}{r}$$

for all $f \in \mathscr{S}$.

Finally for 3) $\Rightarrow$ 1), it is enough to show that $\mathscr{S}'$ being locally bounded implies that $\mathscr{S}$ is equicontinuous (the remainder of the statement follows from Arzelà–Ascoli). So given $w \in \Omega$, we know that $|f'(z)| \leq M$ in a disk D of radius r centered at w. Then $|f(z) - f(w)| \leq M\,|z - w|$ for $z \in D$ (by the generalised Mean Value Theorem). This holds for all f thus $\mathscr{S}$ is equicontinuous at w. □

We have the following immediate corollary.

Corollary 5.9 *A subset $\mathscr{S} \subset \mathcal{H}(\Omega)$ is compact if and only if $\mathscr{S}$ is closed and locally bounded.*

We will see that Arzelà–Ascoli holds in more general circumstances than we need above; in particular it holds for families of continuous functions with values in a complete metric space. A particular example of such a space is the Riemann sphere with the chordal metric, which is defined by

$$d(z, w) = \frac{2\,|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}$$

This is the Euclidean distance in $\mathbb{R}^3$ between the corresponding points on the unit sphere as given by stereographic projection. An important property of the chordal metric is that

$$d(z, w) = d\left(\frac{1}{z}, \frac{1}{w}\right)$$

Moreover, since norms in finite dimensions are equivalent, we have that the complex plane with the topology induced by the chordal metric is equivalent to the usual Euclidean topology. Notice also that we have $d(z, w) \leq 2$ for all z, w (this is the distance between antipodal points on the unit sphere). But then the Arzelà–Ascoli theorem tells us that a family of continuous functions with values in the Riemann sphere is normal (in the chordal metric) if and only if the they are equicontinuous in the chordal metric since all distances are already automatically bounded by 2.

6 Conformal Mappings

A mapping $f : \Omega \to \Omega'$ between open subsets of S^2 is *conformal* (or *biholomorphic*) if f is holomorphic and has a holomorphic inverse. The natural question that arises is if we are given Ω, Ω', how can we determine whether or not they are biholomorphic? And if so can we find all the biholomorphisms? An immediate remark to be made is that although it is necessary that biholomorphic maps be homeomorphic, it is not sufficient. For example, the (open) unit disk D is homeomorphic to $\mathbb{C}$ but cannot be biholomorphic since any holomorphic map from $\mathbb{C}$ to D is bounded and hence constant by Liouville's Theorem.

The biholomorphisms from a space to itself are called *automorphisms*. The collection of all automorphisms forms a group $\mathrm{Aut}(\Omega)$. Moreover, a biholomorphism $f : \Omega \to \Omega'$ induces a group isomorphism

$$\begin{aligned} \text{Aut}(\Omega) &\to \text{Aut}(\Omega') \\ g &\mapsto f \circ g \circ f^{-1} \end{aligned}$$

6.1 Automorphism Groups

A biholomorphism is a bijective holomorphic map $f : \Omega \to \Omega'$ with a holomorphic inverse (in fact every injective holomorphic map is automatically a biholomorphism onto its image because injectivity implies that f' is non-vanishing). A biholomorphism from a space to itself is also called an automorphism. The collection of all automorphisms of Ω forms a group and is sometimes denoted $\text{Aut}(\Omega)$.

6.1.1 Automorphisms of the Complex Plane

We claim that all automorphisms of $\mathbb{C}$ are given by linear transformations

$$w = az + b$$

with $a \neq 0$. In order to verify this we study the behaviour of a automorphism at ∞ (in general this is a nice way to study the behaviour of a holomorphic function). Any entire function f on $\mathbb{C}$ either has a removable singularity, essential singularity or a pole at infinity. If ∞ is a removable singularity, then f is an continuous function on a compact space S^2 so has a maximum. But this would mean that f is a bounded entire function and hence is constant.

Therefore we must have an essential singularity or a pole at infinity. However we know by Weierstrass's Theorem (see Theorem 1.19) that the image of any punctured neighbourhood of an essential singularity is dense in $\mathbb{C}$. Therefore if f has an essential singularity then $f(|z| < 1)$ and $f(|z| > 1)$ intersect despite being the images of disjoint sets (so in particular f could not be injective). Therefore f must have a pole at infinity which is to say that f must be a polynomial. But a polynomial of degree n has n roots (so in general a given value is achieved n times) therefore in order to be injective we must have f is a polynomial of degree 1 so $f(z) = az + b$ with $a \neq 0$.

6.1.2 Automorphisms of the Riemann Sphere

We claim that all automorphism are fractional linear transformations and hence are of the form

$$w = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$. It is clear that these fractional linear transformations certainly form a subgroup G of $\text{Aut}(S^2)$. In order to see that these are all the automorphisms, we use the following lemma.

Lemma 6.1 *Suppose G is a subgroup of $Aut(\Omega)$ such that G acts transitively on Ω and there exists $z_0 \in \Omega$ such that the fixed point subgroup $Aut(\Omega)_{z_0}$ (i.e. the stabiliser of z_0 in $Aut(\Omega)$) is contained in G. Then $G = Aut(\Omega)$.*

Remark 6.2. To say G acts transitively on Ω means that for any $z, w \in \Omega$ there exists $T \in G$ such that $T(z) = w$.

Proof. Let $S \in \text{Aut}(\Omega)$. Take $T \in G$ such that $T(z_0) = S(z_0)$. We know such a T exists because G acts transitively on Ω. Then $T^{-1} \circ S$ is an automorphism of Ω that fixes z_0. But then $T^{-1} \circ S \in G$ and so $S \in G$. □

We know that fractional linear transformations act transitively on S^2 (in fact any 3 (distinct) points can be sent to any 3 (distinct) points and this completely determines the transformation). Moreover the stabiliser of ∞ is exactly the automorphisms of $\mathbb{C}$ which we have seen above are of the form $az+b$ and are therefore contained in G (this is when $c = 0$). Therefore we conclude that $G = \text{Aut}(S^2)$.

6.1.3 Automorphisms of the Disk

We claim that automorphisms of the disk are give by fractional linear transformations of the form

$$w = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}$$

where $\theta \in \mathbb{R}$ and $|z_0| < 1$.

In order to verify this let $S \in \text{Aut}(D)$ and define

$$T = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}$$

where $S(z_0) = 0$ and $\theta = \arg(S'(z_0))$.

Now consider $f = T \circ S^{-1}$. Notice by construction that $f : D \to D$ and $f(0) = 0$ so by Schwarz's lemma we have $|f(z)| \leq |z|$. Applying the same argument to f^{-1} we conclude that $|f(z)| = |z|$ and hence $f(z) = e^{i\alpha} z$ (again by Schwarz). Thus

$$T(z) = e^{i\alpha} S(z)$$

In particular this means that $T'(z_0) = e^{i\alpha} S'(z_0)$ but by construction $T'(z_0)$ and $S'(z_0)$ have the same argument so we must have $\alpha = 0$.

6.1.4 Automorphisms of the Upper Half Plane

The automorphisms of the upper half plane $\mathbb{H}^+$ is (unsurprisingly) also given by fractional linear transformations. In this case they are characterised by

$$w = \frac{az + b}{cz + d}$$

where a, b, c, d are real and $ad - bc = 1$.

Automorphisms of $\mathbb{H}^+$ are easy to find once we have automorphisms of D since the two spaces are conformal. Therefore conjugating $\text{Aut}(D)$ by the biholomorphism $z \mapsto (z - i)/(z + i)$ gets us $\text{Aut}(\mathbb{H}^+)$. Since automorphisms of $\mathbb{H}^+$ will need to preserve $\mathbb{R}$, we conclude that a, b, c, d can be taken to be real. Further notice that

$$\text{Im}\left(\frac{ai + b}{ci + d}\right) = \frac{ad - bc}{c^2 + d^2}$$

Therefore if $(az+b)/(cz+d)$ is to be an automorphism of $\mathbb{H}^+$ we must have $ad-bc>0$ and hence by factoring out some appropriate constant from the numberator and denominator we can get $ad-bc=1$.

7 Riemann Mapping Theorem

Theorem 7.1 (Riemann Mapping Theorem) *Any simply connected open subset Ω of $\mathbb{C}$ except for $\mathbb{C}$ itself is biholomorphic to the unit disk D.*

Proof. Let Ω be a proper subset of $\mathbb{C}$ that is open and simply connected. First we show that there is a biholomorphism from Ω to a bounded open subset of $\mathbb{C}$.

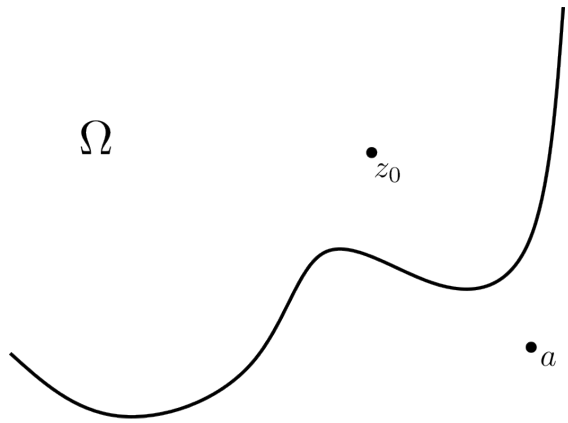


Figure 8: Since $\Omega \subsetneq \mathbb{C}$ there exists $a \notin \Omega$

Since $\Omega \subsetneq \mathbb{C}$ there exists some $a \in \mathbb{C} \setminus \Omega$. Then $dz/z-a$ has a holomorphic primitive $g(z)$ in Ω (because Ω is simply connected) and in fact this primitive is given by a branch of log. In particular this means that

$$z-a=e^{g(z)}$$

Notice that $g(\Omega)$ is open since holomorphic maps are open maps. So let E be an open disk contained in $g(\Omega)$ centered at $g(z_0)$ for some $z_0 \in \Omega$. Then $E+2\pi i$ (the translation of the disk by $2\pi i$) is disjoint from $g(\Omega)$. If it was not disjoint then there would be some $z_1, z_2 \in \Omega$ such that $g(z_2)=g(z_1)+2\pi i$. But this contradicts the fact that $e^{g(z)}=z-a$ is injective in Ω. Therefore

$$\frac{1}{g(z)-(g(z_0)+2\pi i)}$$

is holomorphic, 1-1 and bounded on Ω. Then by translating and scaling if necessary we can assume $0 \in \Omega$ and $g(\Omega) \subset D$. In fact we will relabel $\Omega = g(\Omega)$ and show that every simply connected set that contains 0 and is contained in D is biholomorphic to D.

We begin by defining a family of functions

$$\mathscr{A} := \{f \in \mathcal{H}(\Omega) : f \text{ is 1-1}, f(0)=0, f(\Omega) \subset D\}$$

We will prove the following two lemmas about this family which immediately give us the biholomorphism $g : \Omega \to D$ (the first lemma gives a criterion for when the image of a map in $\mathscr{A}$ is D and the second lemma tells us there is a function satisfying the criterion).

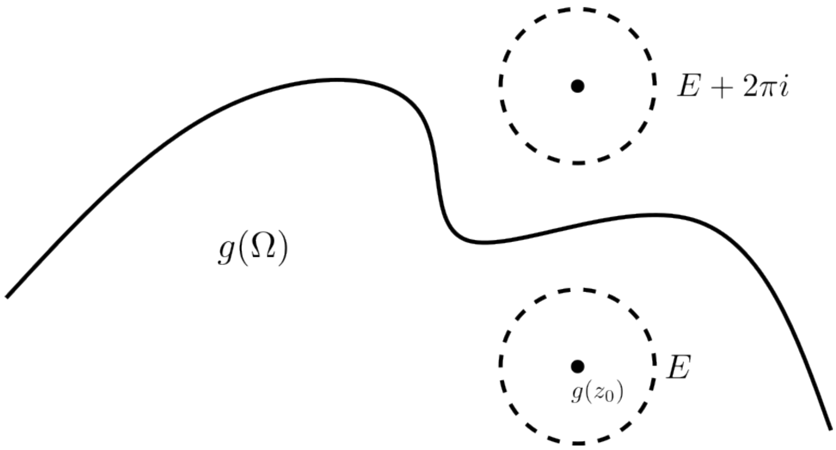


Figure 9: The discs E and $E + 2\pi i$ are disjoint

Lemma 7.2 *Let $g \in \mathscr{A}$. Then $g(\Omega) = D$ if and only if $|g'(0)| = \sup_{f \in \mathscr{A}} |f'(0)|$.*

Proof. Suppose $g(\Omega) = D$. Let $f \in \mathscr{A}$. Take $h = f \circ g^{-1}$ which is a map from the disk to itself which fixes the origin. Then by Schwarz's Lemma we have $|h'(0)| \leq 1$ and hence $|f'(0)| \leq |g'(0)|$.

For the converse suppose we have $f \in \mathscr{A}$ such that $f(\Omega) \subsetneq D$. Then we will find a $g \in \mathscr{A}$ such that $|g'(0)| > |f'(0)|$. Then let $a \in D \setminus f(\Omega)$. Define

$$\varphi(\zeta) = \frac{\zeta - a}{1 - \overline{a}\zeta}$$

Then

$$(\varphi \circ f)(z) = \frac{f(z) - a}{1 - \overline{a} f(z)}$$

is a non-vanishing function on a simply connected region Ω and hence has a well-defined holomorphic square root, say $F(z)$. This means that $(\varphi \circ f)(z) = F(z)^2$. Define $\theta(z) = z^2$ so that

$$\begin{aligned} f(z) &= (\varphi^{-1} \circ \theta \circ F)(z) \\ &= \underbrace{(\varphi^{-1} \circ \theta \circ \psi^{-1})}_{h} \circ \underbrace{(\psi \circ F)}_{g}(z) \end{aligned}$$

where

$$\psi(\eta) = \frac{\eta - F(0)}{1 - \overline{F(0)}\eta}$$

Then we define $g = \psi \circ F$ and $h = \varphi^{-1} \circ \theta \circ \psi^{-1}$. Notice that h is a holomorphic map from the disk to itself that fixes the origin. Then by Schwarz's lemma we have $|h'(0)| \leq 1$. If we had $|h'(0)| = 1$ then h would be a biholomorphism (in fact by Schwarz we would conclude that it is a rotation of the disk). But we know this cannot be the case since θ is not a biholomorphism while φ and ψ are. Therefore we must have $|h'(0)| < 1$. Writing $h = f \circ g^{-1}$, just as before we conclude via the chain rule that

$$\left|f'(0)\right| < \left|g'(0)\right|$$

□

Lemma 7.3 *There is some $g \in \mathscr{A}$ such that $|g'(0)| = \sup_{f\in\mathscr{A}} |f'(0)|$.*

Proof. We know that $\sup|f'(0)|$ must be at least 1 since $\mathscr{A}$ contains the identity. Therefore it suffices to show that

$$\mathscr{B} := \{f \in \mathscr{A} : |f'(0)| \geq 1\}$$

is closed in $\mathcal{H}(\Omega)$ since that would mean that in particular it contains a function which achieves the supremum. Therefore suppose we have a sequence of functions $f_n \in \mathscr{B}$ that converge to f. We want to verify that $f \in \mathscr{B}$. We immediately see that $f(0) = \lim f_n(0) = 0$. Moreover $|f'(0)| = \lim |f_n'(0)| \geq 1$. In particular this means that f is not constant and so by Hurwitz's lemma (see Corollary 2.5) we know that f must also be 1-1. Finally since $|f_n(z)| < 1$ we know that $|f(z)| \leq 1$ for $z \in \Omega$. However if $|f(z)| = 1$ for some z then the Maximum Modulus Principle would imply that f is constant. Therefore $f(\Omega) \subset D$. □

The two lemmas combined show there exists a biholomorphism from Ω to D. □

7.1 Boundary Behaviour

We have seen above that if Ω is a simply connected, proper open subset of $\mathbb{C}$ then there exists a biholomorphism from Ω to the unit disk D. The question then becomes how this biholomorphism behaves on the boundary. We have the following theorem from Carathéodory.

Theorem 7.4 (Carathéodory's Theorem) *A biholomorphism from a simply connected domain $f : \Omega \to D$ extends homeomorphically to the closures $\overline{f} : \overline{\Omega} \to \overline{D}$ if and only if $\partial\Omega$ is a Jordan curve (i.e. homeomorphic to S^1).*

We will not prove the general case but restrict ourselves to the case of a polygon and in fact construct an explicit formula for the inverse. So suppose $\partial\Omega$ is a closed polygonal curve and let $z_1, \ldots, z_n$ be the vertices (with $z_{n+1} = z_1$ due to cyclicity). Suppose we denote the inner angles of the polygons as $\alpha_k\pi$. Therefore we have

$$\alpha_k\pi = \arg\left(\frac{z_{k-1} - z_k}{z_{k+1} - z_k}\right)$$

Let $\beta_k\pi$ be the outer angles so $\beta_k\pi = \pi - \alpha_k\pi = \pi(1 - \alpha_k)$. We know that the sum of the exterior angles of a polygon is always 2π which means we have $\sum \beta_k = 2$. Ω is convex if and only if all $\beta_k > 0$.

First we want to show that any extension should map the boundary to the boundary. In particular, if $\{z_n\}$ is a sequence of points in Ω that approaches the boundary then $\{f(z_n)\}$ approaches the boundary of $f(\Omega)$ (in fact we don't even need f to be holomorphic for this. Simple continuity is sufficient). First we should make precise what it means to approach the boundary. We say that $\{z_n\} \subset \Omega$ approaches the boundary of Ω if it is eventually away from every point in Ω, i.e. for every $z \in \Omega$ there exists some $\epsilon > 0$ and n_0 such that $|z - z_n| \geq \epsilon$ for all $n \geq n_0$.

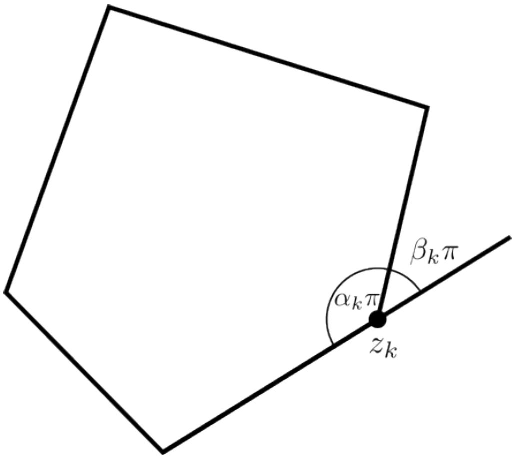


Figure 10: The interior angles of our polygon are $\alpha_k\pi$ and the exterior angles are $\beta_k\pi$

Lemma 7.5 *Suppose Ω, Ω' are regions whose boundary is a Jordan curve. Let $f : \Omega \to \Omega'$ be a continuous, surjective map. If $\{z_n\}$ approaches the boundary of Ω then $\{f(z_n)\}$ approaches the boundary of Ω'.*

Proof. Let K be a compact subset of Ω'. Then $f^{-1}(K)$ is compact (it's closed by continuity and bounded since Ω itself is bounded). For every $z \in \Omega$, there exists some ϵ such that the disk of radius ϵ centered at z does not contain the tail of $\{z_n\}$. Compactness of $f^{-1}(K)$ implies that it can be covered by finitely many such disks. Then there is a maximum n_0 such that z_n for $n \geq n_0$ are all outside the union of these disks. Therefore $f(z_n)$ for $n \geq n_0$ are outside of K. We finish the proof by taking K to be a closed disk (that is contained in Ω') centered at $w \in \Omega'$. □

Let f be the biholomorphism from the simply connected space Ω to the unit disk D. Let x_0 be a non-vertex point on the boundary. By rotating and translating if necessary we can assume that the edge that x_0 lies on is on the real axis and the polygon lies in the upper half-plane. Consider a small disk centered at x_0 so that f is never 0 on the disk (since f is a biholomorphism onto the unit disk, it is zero at exactly one point so by making the disk small enough we can avoid it). Then $\log f(z)$ has a holomorphic branch on the upper half of this disk. Notice that as z approaches the real axis, $f(z)$ approaches the unit circle by the above lemma. Therefore $\log |f(z)|$ approaches 0. Therefore the real part of the $\log f(z)$ extends continuously to the real axis (in particular it extends to be 0 there). We can then apply the reflection principle to conclude that $\log f(z)$ has a holomorphic extension onto the entire disk and therefore so does $f(z)$. This shows that f extends continuously to the open edges of the polygon and in fact even extends slightly beyond in a holomorphic manner (see Figure 11).

A priori, it is possible that because we are only checking for extensions locally on the boundary, we might get conflicting answers if a point lies in two of the above chosen disks. However the extensions will need to agree on Ω and therefore by the principle of analytic continuation they will agree everywhere on the intersection, in particular on $\partial\Omega$.

We are then only left with the vertices. We can deal with them in a similar manner but first we will need to 'open' them. Suppose we have a small disk D around z_k. Then its

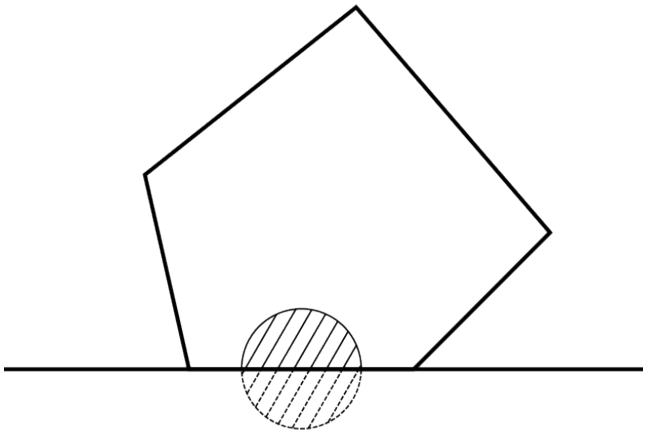


Figure 11: The map f can be extended across edges of the polygon

intersection with Ω form a small sector. We can map this sector to a half-disk via the map $\zeta = (z - z_k)^{1/\alpha_k}$. Equivalently the map $z = z_k + \zeta^{\alpha_k}$ maps a small half-disk to the sector (for an appropriately chosen branch of ζ^{α_k}). In particular then we have a map from the disk half-disk to the polygon given by $g(\zeta) = f(z_k + \zeta^{\alpha_k})$. By the same argument as before, using the reflection principle we can extend g to a function on the entire disk. Therefore f can be extended around the vertex z_k. The same argument holds for all the vertices allowing us to conclude that f extends continuously to the boundary (see Figure 12).

Finally we want to check that this extension is still 1-1 on the boundary. If we denote the boundary by γ, we have that

$$\int_{f\circ\gamma} \frac{1}{z} dz = \int_{\gamma} \frac{df}{f} = 1$$

where the final equality follows from the argument principle (there are no poles in Ω and exactly one zero since we map biholomorphically onto the unit disk). This means that $f \circ \gamma$ is a closed curved contained in the unit circle with winding number 1. Therefore $f \circ \gamma$ is homotopic to the unit circle. We want to argue that it is exactly the unit circle. For this we consider behaviour of $\arg f(z)$.

Recall that near any point on the open edge, we have a holomorphic branch of $\log f(z)$. Assuming that this neighbourhood lies in the upper half plane, we know that for as y decreases to 0, we have $\log|f(x+iy)|$ increases to 0. Then by the Cauchy-Riemann equations, we conclude

$$0 > \frac{\partial \log|f|}{\partial y} = -\frac{\partial \arg f}{\partial x}$$

In particular then as z travels along an open edge of Ω, $\arg f(z)$ is constantly increasing. Therefore $f \circ \gamma$ is a closed curved homoptopic to unit circle and is such that its argument is constant increasing. It must then be the unit circle exactly. Thus we see that the biholomorphic map from the polygon Ω to unit disk extends to a homeomorphism of the closures of the respective spaces.

7.2 Schwarz-Christoffel

Thus the Riemann Mapping Theorem gives us a biholomorphism between the unit disk and the interior of a polygon (which in fact extends to a homeomorphism on the closures of

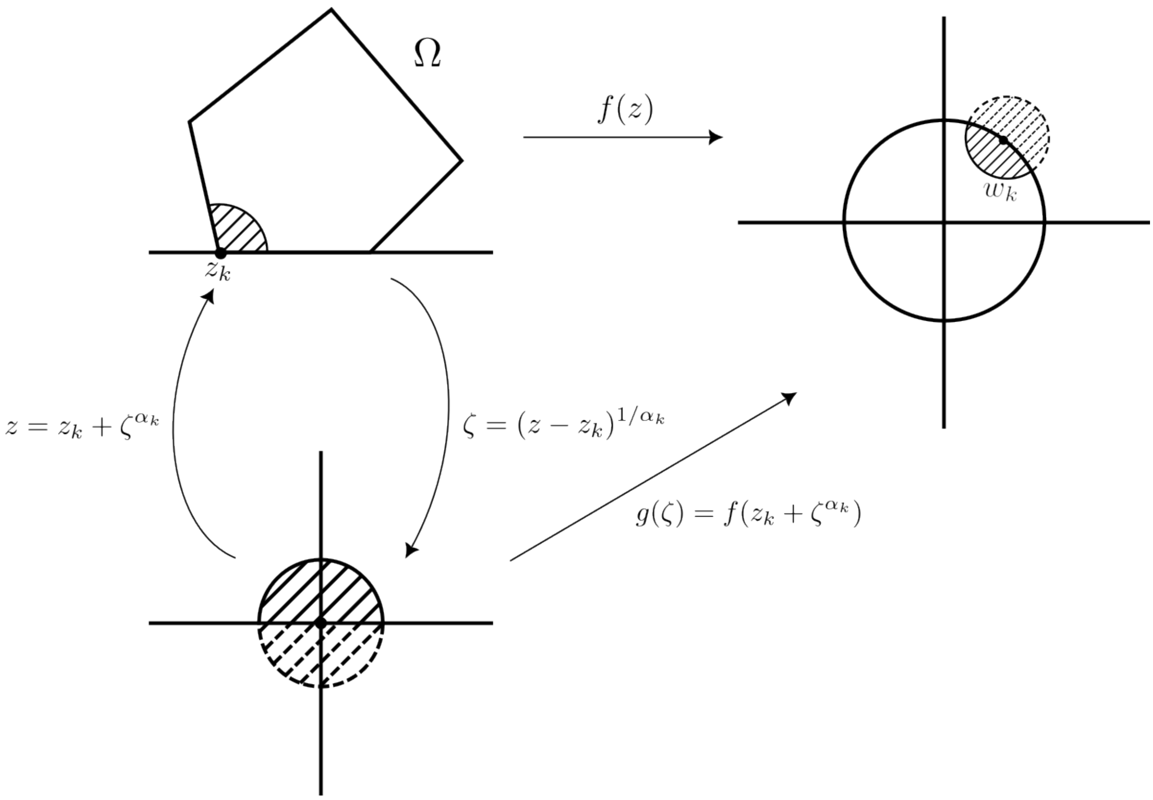


Figure 12: We open up the sector at a vertex and then use the same argument as before to extend below the upper half plane

these spaces). Now that we know such a function exists, we would like to know what it is. As it turns out, we can give an explicit formula for it or more precisely for its inverse.

Theorem 7.6 (Schwarz-Christoffel Formula) *The functions $z = F(w)$ which map $|w| < 1$ conformally to a polygon Ω with angles $\alpha_k\pi$ for $k = 1, \ldots, n$ are of the form*

$$F(w) = c\int_0^w \prod_{k=1}^{n}(w - w_k)^{-\beta_k}dw + c'$$

where c, c' are some complex constants (as one can guess they determine the scaling and translation) and the w_k are the images of the vertices z_k (hence w_k lie on the unit circle).

Remark 7.7. The integral is evaluated by integrating along any path from 0 to w. Because the disk is simply connected this is well-defined.

Proof. Let Ω be the interior of a polygon. In order to verify the formula, we want to show that if F is the inverse of a biholomorphism $f : \Omega \to D$ given by the Riemann mapping theorem then

$$F'(w) = c \prod_{k=1}^{n} (w - w_k)^{-\beta_k}$$

Consider $w = g(\zeta) = f(z_k + \zeta^{\alpha_k})$ as we did before. Notice that this is invertible near $\zeta = 0$ (the local extensions below the half-plane are injective). In particular there is a Taylor series expansion

$$\zeta = \sum_{n=0}^{\infty} b_n (w - w_k)^n$$

By construction, $b_0 = 0$ and $b_1 \neq 0$. Therefore

$$\zeta = \sum_{n=1}^{\infty} b_1 (w - w_k)^n = (w - w_k) \underbrace{\left(\sum_{n=0}^{\infty} b_{n+1} (w - w_k)^n \right)}_{g_k(w)}$$

where $g_k(w)$ is non-zero around $w = w_k$. Since $z = z_k + \zeta^{\alpha_k}$ we have

$$z = F(w) = z_k + (w - w_k)^{\alpha_k} g_k(w)^{\alpha_k}$$

Relabeling $g_k(w)^{\alpha_k} = g_k(w)$, we have

$$F'(w) = \alpha_k (w - w_k)^{\alpha_k - 1} g_k(w) + (w - w_k)^{\alpha_k} g_k'(w)$$

Since $\beta_k = 1 - \alpha_k$ we can write

$$F'(w)(w - w_k)^{\beta_k} = \alpha_k g_k(w) + (w - w_k) g_k'(w)$$

implying that $F'(w)(w - w_k)^{\beta_k}$ is non-vanishing around w_k. Notice that $F'(w)$ is non-zero away from the vertices since f is conformal at these points. Therefore

$$H(w) = F'(w) \prod_{k=1}^{n} (w - w_k)^{\beta_k}$$

is holomorphic and vanising in a neighbourhood of $\overline{D}$.

We claim that $H(w)$ is actually constant. We first show that $\arg H(w)$ is constant on S^1. We observe first that

$$\frac{d}{d\theta} F(e^{i\theta}) = F'(e^{i\theta}) i e^{i\theta}$$

Taking arguments of both sides we have

$$\arg\left(\frac{d}{d\theta} F(e^{i\theta}) \right) = \arg(F'(e^{i\theta})) + \left(\theta + \frac{\pi}{2} \right)$$

We claim the left hand side is constant. In order to see this, consider $e^{i\theta}$ between w_k and w_{k+1}. Notice that $F(e^{i\theta})$ describes a straight line so we can write

$$F(e^{i\theta}) = \alpha t(\theta) + \beta$$

where t is a real-valued function of θ and α, β are constants. Then

$$\arg\left(\frac{d}{d\theta} F(e^{i\theta}) \right) = \arg(\alpha t'(\theta)) = \arg(\alpha)$$

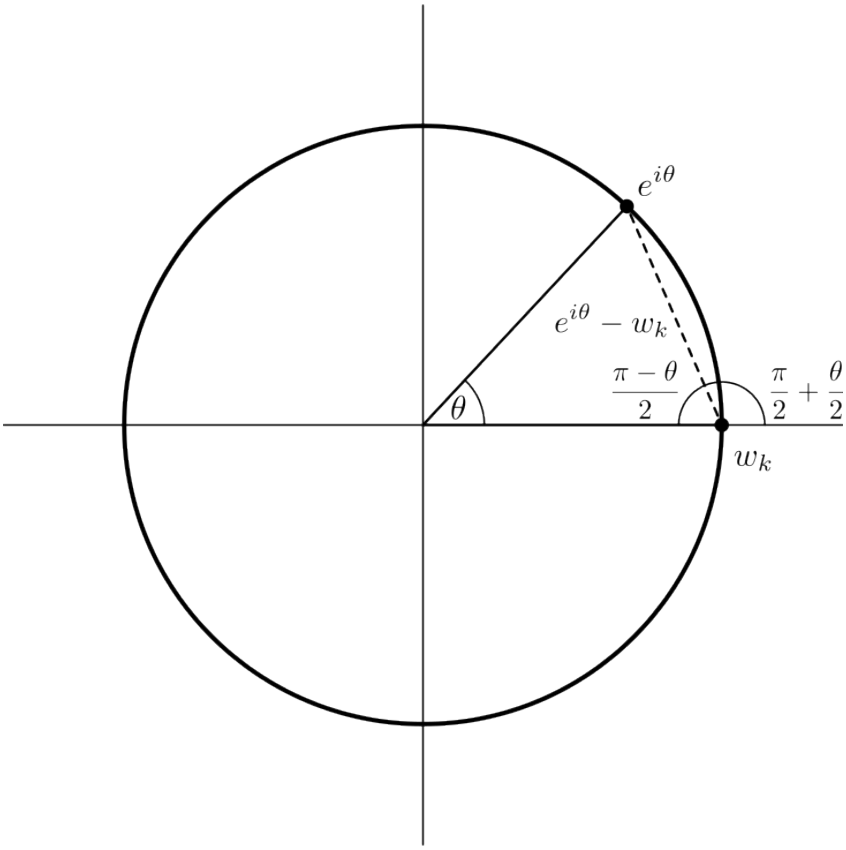


Figure 13: The argument of $e^{i\theta} - w_k$ is $\theta/2 + \text{const}$

Additionally by Figure 13 we see that $\arg(e^{i\theta} - w_k) = \theta/2 + \text{const}$. Then

$$\arg H(e^{i\theta}) = -\theta + \left(\sum_{k=1}^{n} \beta_k\right) \frac{\theta}{2} + \text{const}$$

Since $\sum \beta_k = 2$ by assumption, we see that $\arg H(e^{i\theta})$ is constant. By the mean value property, we conclude that $H(0)$ is exactly this constant. But then the maximum modulus principle implies that H must be constant on the entire closed disk. This means that

$$F'(w) = c \prod_{k=1}^{n} (w - w_k)^{-\beta_k}$$

Integrating both sides, we get the formula. □

7.3 Examples

Above we worked out the Schwarz-Christoffel formula for the disk. But since we have an explicit biholomorphism between the disk and the upper half-plane, we can translate the formula to the upper half-plane by performing a substitution $w = (\zeta - i)/(\zeta + i)$. In fact the formula remains the same after this substitution (things cancel out in a lovely way) with the w_k being real numbers.

We can then use this formula to give an explicit mapping from the upper half plane to a rectangle. First we need to choose 4 points on the real axis that will map to the four vertices of the rectangle. We will take these four points to be $1, -1, 1/k, -1/k$ for some $0 < k < 1$. Then according to the formula

$$z = F(w) = \int_0^w \frac{dw}{\sqrt{(1-w^2)}\sqrt{1-k^2w^2}}$$

maps the upper half-plane to the rectangle with vertices $K, K + iK', -K + iK', -K$ where

$$K = \int_0^1 \frac{dt}{(1-t^2)(1-k^2t^2)}$$

and

$$K' = \int_1^\infty \frac{dt}{(1-t^2)(1-k^2t^2)}$$

(see Figure 14 for reference). The inverse map $w = f(z)$ is a conformal map from the rectangle to the upper-half plane. We can extend this map beyond the rectangle to the entire plane by repeatedly reflecting across its sides. This defines a double periodic, meromorphic function on $\mathbb{C}$ with group of periods generated by $4K, 2iK'$

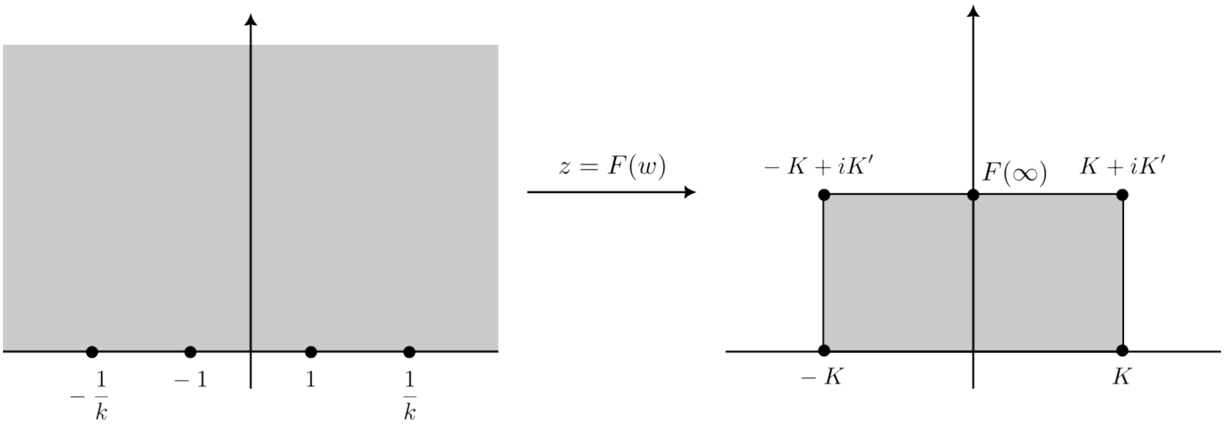


Figure 14: Mapping upper half-plane to a rectangle

We can do the same thing with a triangle with angles $\alpha_1\pi, \alpha_2\pi, \alpha_3\pi$. We will choose our three points on the real axis to be $0, 1$ and ∞. Then the map is

$$z = F(w) = \int_0^w w^{\alpha_1 - 1}(w-1)^{\alpha_2 - 1}$$

The inverse defines a map from the triangle to the upper half plane and we can ask again whether this map can be extended to the entire plane by reflection. In order for the reflections to line up, we would need to be able to able to tile the entire plane with these triangles in such a way that there are an even number of triangles around every vertex.

Therefore we have $\alpha_k \pi = 2\pi/2n_k$ for integers n_k. Then since the sum of the α_k is 1 in a triangle we have

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$$

One can then verify that there are only 3 sets of natural numbers satisfying this. Namely, $(3,3,3), (2,4,4)$ and $(2,3,6)$.

8 Normal families of Meromorphic Functions

Much like we did with holomorphic functions, we want to find the normal families (i.e. the (pre)compact subsets) of the collection of meromorphic functions. One can simply think of meromorphic functions as functions that take values in S^2 instead of just $\mathbb{C}$. A nice way to 'deal with' S^2 is the previously mentioned chordal metric. Recall that the chordal metric is the Euclidean distance in $\mathbb{R}^3$ between two points on the Riemann sphere under the usual stereographic projection map and is given by

$$d(z,w) = \frac{2\,|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

for $z, w \neq \infty$. As mentioned previously, an important property of the chordal metric is that $d(z,w) = d(1/z, 1/w)$ (which also tells us how the find the distance between finite points and the point at infinity).

Of course our starting point when studying normal families is the Arzelà–Ascoli theorem, which tells us that a family of continuous functions with values in the Riemann sphere *equipped with the chordal metric* is normal if and and only if it is equicontinuous (every function is automatically bounded since the chordal metric is itself at most 2).

Lemma 8.1 *Let $\{f_n\}$ be a sequence of meromorphic function on a domain Ω which converges uniformly on compact sets with respect to the chordal metric. Then the limit f is either meromorphic or identically ∞.*

Proof. Let $z_0 \in \Omega$ be arbitrary. Suppose $|f(z_0)| < \infty$. Then f is bounded in a neighbourhood of z_0. This means that $f_n \to f$ on compact sets in a neighbourhood of z_0 with respect to the Euclidean metric (because the two metrics are equivalent on bounded subsets of $\mathbb{C}$). This means that f is holomorphic on a neighbourhood of z_0.

Alternatively we might have $f(z_0) = \infty$, then we simply repeat the above argument with $\{1/f_n\}$ instead. In particular, $\{1/f_n\}$ are bounded in a neighbourhood of z_0 for n large enough. So $1/f$ is holomorphic in a neighbourhood of z_0 and $1/f$ is 0 at z_0. Either the zeroes of $1/f$ are isolated (in which case f is meromorphic) or $1/f$ is identically 0 in a neighbourhood of z_0. □

Example 8.2. The sequence $\{z_n\}$ converges uniformly on compact subsets in the complement of $\overline{D}$ to ∞.

Corollary 8.3 *Let $\{f_n\}$ be a sequence of holomorphic functions on a domain Ω which converges uniformly on compact sets with respect to the chordal metric. Then the limit f is either holomorphic or identically ∞.*

Proof. Done in an assignment. □

What we would like to do is generalise Montel's Theorem to work for meromorphic functions. For this we will need to introduce the spherical derivative.

8.1 Spherical Derivative

Definition 8.4 (Spherical Derivative)**.** If f is a meromorphic function defined on a domain $\Omega \subset \mathbb{C}$, then the *spherical derivative* of f is defined by

$$f^{\#}(z) := \lim_{w \to z} \frac{d(f(z), f(w))}{|z - w|}$$

where of course by d we mean the chordal metric.

Therefore the spherical derivative is always a non-negative real number. If we take Ω to be a subset of the Riemann sphere, then we use the chordal metric in the denominator as well. Notice that if z is not a pole then using the definition of the chordal metric we have

$$\begin{aligned} f^{\#}(z) &= \lim_{w \to z} \frac{d(f(z), f(w))}{|z - w|} \\ &= \lim_{w \to z} \frac{1}{|z - w|} \cdot \frac{2\,|f(z) - f(w)|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}} \\ &= \lim_{w \to z} \frac{|f(z) - f(w)|}{|z - w|} \cdot \frac{2}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}} \\ &= \frac{2\,|f'(z)|}{1 + |f(z)|^2} \end{aligned}$$

This is a very useful formula as it relates the spherical derivative with the true derivative. For example, we can use it to find a version of the chain rule for spherical differentiation.

$$(f \circ g)^{\#}(z) = \frac{2\,|(f \circ g)'(z)|}{1 + |(f \circ g)(z)|^2} = \frac{2\,|f'(g(z))|}{1 + |f(g(z))|^2}\,|g'(z)| = f^{\#}(g(z))\,|g'(z)|$$

Despite always being a real number (and always a non-negative one at that), the spherical derivative can still give us a fair bit of information. For example by the calculation above we see that if $f^{\#}(z) \neq 0$ then $f'(z) \neq 0$ so we know that f is locally 1-1. Moreover, by properties of the chordal metric, we have that $f^{\#} = (1/f)^{\#}$. This identity allows us to find the spherical derivative at poles. In fact this means that $f^{\#}$ is a continuous function on all of Ω, including the poles. The spherical derivative is what allows us to generalise Montel's Theorem to a statement about meromorphic functions.

Theorem 8.5 (Marty's Theorem) *A family of meromorphic functions $\mathscr{S}$ on a domain Ω is normal (with respect to the chordal metric) if and only if $\mathscr{S}^{\#} := \{f^{\#} : f \in \mathscr{S}\}$ is locally bounded.*

Proof. First suppose $\mathscr{S}$ is a normal family of meromorphic functions and suppose $\mathscr{S}^\#$ is not locally bounded. Because the Euclidean metric and chordal metric are equivalent on bounded subsets of $\mathbb{C}$, they share the same normal families (assuming no poles). We will use this along with the relationship between $f^\#$ and f' to get a contradiction.

Since $\mathscr{S}^\#$ is not locally bounded, there exists a point $z_0 \in \Omega$ such that $\mathscr{S}^\#$ is not bounded in any neighbourhood of z_0. This means there exists a sequence of functions $f_n \subset \mathscr{S}$ and a sequence of points $z_n \to z_0$ such that $f_n^\#(z_n) \to \infty$. By normality we can assume that this sequence of functions converges to some meromorphic function f (passing to a subsequence if necessary). Then we know by Lemma 8.1 that f is either meromorphic or identically infinity.

Suppose f is bounded at z_0. It is then bounded in a neighbourhood U of z_0. We can take U to be a bounded subset of $\mathbb{C}$ so that the f_n in fact converge to f on U in the Euclidean metric. Then $f_n'(z_0) \to f'(z_0)$. But then $f_n^\#(z_0) \to f^\#(z_0)$ leading to a contradiction. If f is not bounded at z_0, we can repeat the argument with $1/f$.

In order to see the converse, suppose $\mathscr{S}^\#$ is locally bounded. Let $z_0 \in \Omega$ be arbitrary and let D be a closed disk centered at z_0 so that $\{f^\# : f \in \mathscr{S}\}$ is bounded by M on D. By definition this means that

$$\lim_{z\to w} \frac{d(f(z), f(w))}{|z-w|} \le M$$

In particular for w, there exists a δ_w such that if $|z-w| < \delta_w$ then $d(f(z), f(w))/\,|z-w| < 2M$ or in other words $d(f(z), f(w)) < 2M\,|z-w|$. We cover D by such neighbourhoods (notice that the size of the neighbourhoods may vary with w). Let δ be a Lebesgue number for this covering (which exists by compactness of D) so that $\{|z-w| < \delta\} \subset \{|z-w| < \delta_w\}$ for all $w \in D$.

Now let w, w' be arbitrary points in D. Connect them via a line segment and partition the line $\{w_0, \dots, w_n\}$ with $w_0 = w$ and $w_n = w'$ such that $|w_{j+1} - w_j| < \delta$ (see Figure 15). Then

$$\begin{aligned} d(f(w'), f(w)) &\le \sum_{j=1}^{n} d(f(w_j), f(w_{j-1})) \\ &< 2M \sum_{j=1}^{n} |w_j - w_{j-1}| \\ &= 2M\,|w' - w| \end{aligned}$$

Therefore all the f in $\mathscr{S}$ are in particular Lipschitz with Lipschitz constant $2M$ and hence $\mathscr{S}$ is equicontinuous on D (with respect to the chordal metric) and hence form a normal family by the Arzelà–Ascoli Theorem. Since the collection of such disks cover D, we conclude that by the 'suitable cover lemma' (see Lemma 5.1) that $\mathscr{S}$ is itself a normal family on Ω. □

Theorem 8.6 (Picard's Big Theorem) *If $f(z)$ is holomorphic with an essential singularity at z_0, then there exists $\lambda \in \mathbb{C}$ such that in any neighbourhood z_0, f assumes every value except maybe λ (infinitely many times).*
Equivalently, if $f(z)$ is meromorphic in a punctured disk $0 < |z - z_0| < \delta$ and f omits 3 values in $\mathbb{C} \cup \{\infty\}$, then f is meromorphic in $|z - z_0| < \delta$.

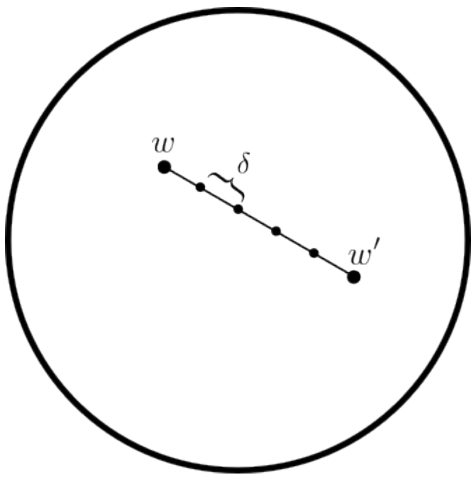


Figure 15: Partition the line segment connecting w and w'

Proof of equivalence. We first show why the above statements are equivalent. Suppose the first statement about holomorphic functions holds. Then by composing with an FLT if necessary we can assume that f omits ∞ and 2 other (finite) values. Then f on $0 < |z - z_0| < \delta$ is a holomorphic function that misses 2 values which means that z_0 cannot be an essential singularity of f and therefore must be a pole.

Now suppose the statement about meromorphic functions holds. Let f be an entire function and z_0 an essential singularity. Consider f on $0 < |z - z_0| < \delta$ (for any δ. Notice that such a punctured disk is contained in every punctured neighbourhood of z_0). Then f omits the value ∞ on this punctured disk (since it is holomorphic). Then if it missed 2 more distinct complex numbers we could conclude that f is meromorphic on the entire disk which we know is not true. Therefore f can miss at most one more value. □

We will see that Picard's Big Theorem is in fact a fairly straightforward consequence of Montel's Big Theorem.

Theorem 8.7 (Montel's Big Theorem) *A family of meromorphic functions on a domain Ω which omits 3 distinct values in $\mathbb{C} \cup \{\infty\}$ is normal in the chordal metric.*

However before proving Montel's Big Theorem, we need the following important result.

Lemma 8.8 (Zalcman's Lemma) *A family of meromorphic functions $\mathscr{S}$ on a domain Ω is not normal in the chordal metric if and only if there exists a sequence $a_n \to a_0 \in \Omega$, a sequence of positive numbers $\rho_n \to 0$ and a sequence function $f_n \in \mathscr{S}$ such that $g_n(z) = f_n(a_n + \rho_n z)$ converges uniformly on compact sets (in the chordal metric) to a non-constant function $g(z)$. This function $g(z)$ is meromorphic on all of $\mathbb{C}$ and is such that $g^{\#}(z) \leq 1$ for all z and $g^{\#}(0) = 1$.*

The statement seems like a strange one initially because we check for normality by finding a convergent limit. In fact the non-normality condition is exactly related to the non-constant condition of g. It is useful to consider an example where we apply the lemma.

Example 8.9. Suppose we have $\mathscr{S} = \{f_n\}$ where $f_n(z) = z^n$ which is normal on D and $\mathbb{C} \setminus \overline{D}$ but not on any domain containing the unit circle (because any neighbourhood of a point on the unit circle will contain points that lie in the interior of the disk and the exterior. Points in the interior tend to 0 and points in the exterior tend to infinity, so we can't have normality). We can verify this using the lemma above. For example if we have $a_n = 1$ and $\rho_n = 1/n$. Then

$$f_n(a_n + \rho_n z) = \left(1 + \frac{z}{n}\right)^n \to e^z = g(z)$$

Then notice that g is meromorphic on all of $\mathbb{C}$ and

$$g^\#(z) = \frac{2\,|g'(z)|}{1 + |g'(z)|^2} = \frac{2\,|e^z|}{1 + |e^z|^2} \leq 1$$

and $g^\#(0) = 1$.

Proof. Suppose $\mathscr{S}$ is normal. Suppose we have any sequence $\{f_n\} \subset \mathscr{S}$ and by normality we can assume that the f_n converge to f (uniformly on compact subsets with respect to the chordal metric). Now take any sequence $a_n \to a_0 \in \Omega$ and positive real numbers $\rho_n \to 0$. Then we see that

$$g_n(z) = f_n(a_n + \rho_n z) \to f(a_0)$$

and hence is constant. This follows from the fact that the f_n are equicontinuous (we know this from Arzelà–Ascoli). Therefore for any $\epsilon > 0$ there exists some δ such that if $|a_n + \rho_n z - a_0| < \delta$ then $|f_n(a_n + \rho_n z) - f_n(a_0)| < \epsilon$. Clearly for any z, we can find n large enough so that $|a_n + \rho_n z - a_0| < \delta$. But then if z is on a compact set, we can make this choice for n independently of z. Therefore g_n converge to a constant function.

Now suppose $\mathscr{S}$ is not normal. We will construct the appropriate data from this. First, since $\mathscr{S}$ is not normal, we know that $\mathscr{S}^\#$ is not locally bounded so just as we did before we find $b_n \to b_0 \in \Omega$ and $f_n \in \mathscr{S}$ such that $f_n^\#(b_n) \to \infty$ as $n \to \infty$. By translating if necessary we can assume that $b_0 = 0$ and that Ω contains a closed disk of radius r centered at 0. Then define

$$M_n := \sup_{|\zeta| \leq r} (r - |\zeta|) f_n^\#(\zeta)$$

Since $|\zeta| \leq r$ is a compact, we know the above sup is actually achieve so there is some a_n in this closed disk such that

$$M_n = (r - |a_n|) f_n^\#(a_n)$$

We notice that $M_n \to \infty$ as $n \to \infty$ (since we know that $f_n^\#(b_n) \to \infty$). Now we take $\rho_n = 1/f_n^\#(a_n)$ and consider

$$g_n(z) = f_n\left(a_n + \frac{z}{f_n^\#(a_n)}\right)$$

which is defined on $|z| \leq M_n$ since

$$\left|a_n + \frac{z}{f_n^\#(a_n)}\right| \leq |a_n| + \frac{|z|}{f_n^\#(a_n)} \leq |a_n| + \frac{M_n}{f_n^\#(a_n)} = |a_n| + r - |a_n| = r$$

Now fix $R < \infty$. Since $M_n \to \infty$, we know for sufficiently large n we have $R < M_n$. Then for $|z| \le R$ we have

$$\begin{aligned}
g_n^{\#}(z) &= \frac{f_n^{\#}(z_n + z/f_n^{\#}(a_n))}{f_n^{\#}(a_n)} && \text{(``chain rule'')} \\
&\le \frac{M_n}{r - \left|a_n + z/f_n^{\#}(a_n)\right|} \cdot \frac{r - |a_n|}{M_n} && \text{(definition of } M_n\text{)} \\
&\le \frac{r - |a_n|}{r - |a_n| - |z_n| / f_n^{\#}(a_n)} && \text{(Reverse triangle inequality)} \\
&= \frac{1}{1 - |z| / M_n} && \text{(factor out } r - |a_n|\text{)}
\end{aligned}$$

Then notice the last line converges to 1 as $n \to \infty$ (again since $M_n \to \infty$). Then by Marty's Theorem we know $\{g_n\}$ is a normal family so has a subsequence that converges uniformly on compact subsets in the chordal metric. Without loss of generality, we can assume this subsequence is $\{g_n\}$ itself and define $g := \lim_n g_n$. Notice by the above arguement that $g^{\#}(0) = 1$ and $g^{\#}(z) \le 1$. Then g is a meromorphic function and since $g^{\#}(0) \ne 0$, it is non-constant. Finally, we can have a_n converging to $a_0 \in \Omega$ by passing to a subsequence (recall they form a sequence in the compact set $|\zeta| \le r$). □

Theorem 8.10 (Montel's Big Theorem) *A family $\mathscr{S}$ of meromorphic functions in a domain Ω which omit 3 given distinct values a, b, c in $\mathbb{C} \cup \{\infty\}$ is normal in the chordal metric.*

Proof. First recall that we can check for normality on a domain by checking normality on all disks contained within the domain (see Lemma 5.1). Therefore without loss of generality we can assume that $\Omega = D$. Moreover, by composing with a fractional linear transformation if necessary we can assume that the values omitted at $0, 1$ and ∞. Therefore $\mathscr{S}$ is a family on holomorphic functions on the disk D that omits 0 and 1. Now define

$$\mathscr{S}_m = \{f \in \mathcal{H}(D) : f \text{ omits } 0 \text{ and } e^{2\pi i k/2^m}, k = 0, \ldots, 2^m - 1\}$$

In other words $\mathscr{S}_m$ is the collection of holomorphic functions on the disk that miss the 2^m-th roots of unity. Notice that $\mathscr{S} \subset \mathscr{S}_0$ so $\mathscr{S}_0$ is non-empty. Moreover every $f \in \mathscr{S}_m$ is non-vanishing so has a well-defined square root $f^{1/2}$ which is necessarily contained in $\mathscr{S}_{m+1}$. Therefore no $\mathscr{S}_m$ is empty.

Now suppose that $\mathscr{S}$ is not normal. Then there exists some $\{f_n\} \subset \mathscr{S} \subset \mathscr{S}_0$ that does not contain a convergent subsequence. Then $\{f_n^{1/2}\} \subset \mathscr{S}_1$ also has no convergent subsequence so $\mathscr{S}_1$ could not be normal. Continuing in this manner we conclude that none of the $\mathscr{S}_m$ are normal. Let g_m be the corresponding "g" given by Zalcman's Lemma for each $\mathscr{S}_m$. Zalcman only guarantees that g_m is meromorphic but by Corollary 8.3 we can conclude that g_m must be holomorphic (and hence entire) as well.

We also know by Zalcman's lemma that $\left|g_m^{\#}(z)\right| \le 1$ for all z and for all m. Therefore we can apply Marty's Theorem to conclude that the g_m form a normal family. As usual then we can assume that the g_m are convergent and let g be the limit. Notice that g is non-constant because $g^{\#}(0) = 1$ since each $g_m^{\#}(0) = 1$. Moreover, being the limit of the g_m

we must have that g omits all the 2^m roots of unity. Then $g(\mathbb{C})$ is either contained within the disk or contained in $\mathbb{C} \setminus \overline{D}$. In either case either g is bounded or $1/g$ is bounded which would imply by Liouville that g is constant, leading to a contradiciton. $\square$

Theorem 8.11 (Picard's Big Theorem) *A meromorphic function in a punctured disk which omits 3 distinct values in $\mathbb{C} \cup \{\infty\}$ extends to a meromorphic function on the entire disk.*

Proof. As usual we can assume that the disk is centered at 0 and the values omitted are $0, 1$ and ∞. Let ϵ_n be a sequence of positive real numbers that strictly decrease to 0. Then consider $\mathscr{S} := \{f_n(z) := f(\epsilon_n z)\}$ which is a normal family on $\Omega := \{0 < |z| < 2\}$ (we choose ϵ_n to be make this a valid domain) by Montel's Big Theorem. Let g be the limiting function. Since each $f(\epsilon_n z)$ is holomorphic, by the problem set we know that g is either holomorphic on Ω or identically ∞.

Suppose g is holomorphic on Ω. Let M be such that $|g(z)| \leq M < \infty$ for $|z| = 1$ which means that $|f_n(z)| \leq M$ for $|z| = \epsilon_n$. In fact by convergence, there is some n_0 such that for all $n \geq n_0$ we have $|f_n(z)| \leq M + 1$ on $|z| = \epsilon_n$. Therefore we can apply the maximum modulus principle (see Theorem 1.13) which roughly says that the modulus of a non-constant holomorphic function f can only achieve a maximum on the boundary of its domain of definition. Considering f_n restricted to the annulus $\epsilon_{n+1} \leq |z| \leq \epsilon_n$, we then conclude that $|f_n(z)| \leq M + 1$ on this annulus (again for $n \geq n_0$). This means that $|f(z)| \leq M + 1$ on $0 < |z| \leq \epsilon_{n_0}$. Since f is bounded in a neighbourhood of 0, we conclude that 0 is a removable singularity and hence f extends to be holomorphic on the entire disk.

On the other hand if g is identically infinity, then we can apply the same argument to $1/f(\epsilon_n z)$ to conclude $1/f$ extends to be holomorphic at 0 and hence f is meromorphic on the disk. $\square$

Theorem 8.12 (Picard's Little Theorem) *Any non-constant entire function omits at most 1 value.*

Proof. An entire function either has a pole or an essential singularity at infinity. If we have a pole, then the function is a polynomial so we hit every value in $\mathbb{C}$. Otherwise we have an essential singularity at infinity and since $\mathbb{C}$ is a neighbourhood of this essential singularity, we know it's image can omit at most 1 value of $\mathbb{C}$. $\square$

9 Riemann Surfaces

9.1 Complex Manifolds

A manifold with a complex structure of dimension n is a Hausdorff topological space with a countable basis such that every point has a neighbourhood which is homeomorphic to an open subset of $\mathbb{C}^n$. Moreover, for any two so-called coordinate charts (φ_i, U_i) and (φ_j, U_j) we have that the map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is holomorphic. We will mostly be interested in the case of $n = 1$.

Example 9.1 (Riemann Sphere)**.** An important example of a manifold with a complex structure, and one we have already discussed in a fair bit of detail, is the Riemann sphere. In this case we can cover the entire sphere with 2 coordinate charts. Let $U = S^2 \setminus \{N\}$

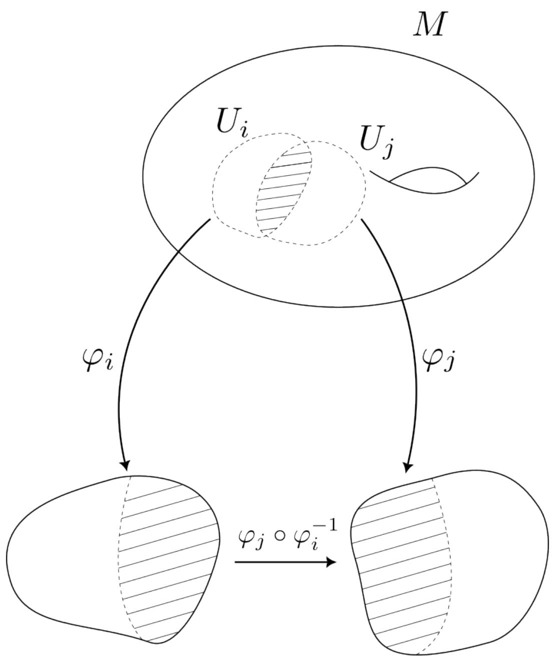


Figure 16: A manifold is covered by a collection of compatible charts

and define

$$\begin{aligned} \varphi_U : U &\to \mathbb{C} \\ (x, y, t) &\mapsto \frac{x + iy}{1 - t} \end{aligned}$$

and $V := S^2 \setminus \{S\}$ with

$$\begin{aligned} \varphi_V : V &\to \mathbb{C} \\ (x, y, t) &\mapsto \frac{x - iy}{1 + t} \end{aligned}$$

Recall that the transition map $\varphi_V \circ \varphi_U^{-1} : \mathbb{C} \setminus \{0\} = \varphi_U(U \cap V) \to \varphi_V(U \cap V) = \mathbb{C} \setminus \{0\}$ is given by $z \mapsto 1/z$ which is certainly holomorphic on $\mathbb{C} \setminus \{0\}$.

A map $f : M \to N$ between manifolds with complex structures is holomorphic if $\psi_j \circ f \circ \varphi_i^{-1}$ (defined on $\varphi_i(U_i \cap f^{-1}(V_j))$) are holomorphic for every i and j, see Figure 17.

A holomorphic map $f : M \to N$ is an *isomorphism* (or *biholomorphism*) if it is a homeomorphism with a holomorphic inverse. We say two complex structures are equivalent if the identity map is a biholomorphism. Now we can finally given the proper definition of a complex manifold.

Definition 9.2 (Complex Manifold)**.** A *complex manifold* is a manifold with an equivalence class of complex structures.

A complex manifold of dimension 1 is called a complex curve or more commonly a(n abstract) Riemann surface.

The simplest examples of complex manifolds are of course $\mathbb{C}$ and S^2 or open subsets of

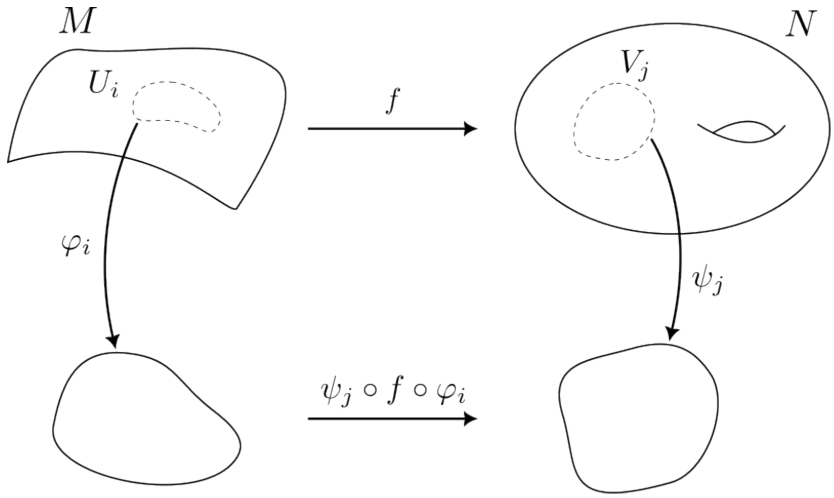


Figure 17: Map between manifolds with complex structure is holomorphic if it's holomorphic when viewed through charts

them. A slightly more interesting example is $\mathbb{C}/\mathbb{Z}$ where we say that $z \sim z'$ if $z - z' \in \mathbb{Z}$. We can easily give this a complex structure. Let $p : \mathbb{C} \to \mathbb{C}/Z$ be the projection map on the quotient. Then for every $z_0 \in \mathbb{C}$, we can find an open neighbourhood V of z_0 such that $p|_V$ is injective (for example we could take V to be a disc of radius $1/2$). Then p^{-1} acts as a coordinate chart on $p(V)$.

Another example in a similar vein that we will explore more thoroughly is $\mathbb{C}/\Gamma$ where Γ is a discrete subgroup of $\mathbb{C}$ (viewed as an additive group of course). Then we know that topologically $\mathbb{C}/\Gamma$ is a torus but different choices of Γ may lead to torii with different complex structures (in particular $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$ need not be biholomorphic).

Importantly, the local properties of holomorphic functions holds for complex manifolds as well, practically by definition. Examples of such properties are the principle of analytic continuation, the maximum modulus principle, the mean value property, etc. A meromorphic function on a complex manifold M is a holomorphic map from M to S^2. For example, we have a 1-1 correspondence between meromorphic functions on $\mathbb{C}/\Gamma$ and meromorphic functions on $\mathbb{C}$ with Γ as group of periods.

9.2 Differential Forms

The next step of course is to determine how we integrate on complex manifolds. As usual, we integrate forms. A holomorphic differential (1-)form ω on a complex manifold M assigns a covector of the tangent space to every point on the manifold in a holomorphic manner. Another way to say this is that when viewed through coordinate charts ω is a holomorphic differential form on an open subset of $\mathbb{C}$ (as we have dealt with earlier).

To be precise suppose (φ_i, U_i) is a coordinate charts on M. Then we get a differential form ω_i on $\varphi_i(U_i)$ by pushing forward ω (we can do this because φ_i is a biholomorphism onto its image). Since ω_i is a differential form on an open subset of $\mathbb{C}$ we know it is of the form $\omega_i = f_i(z)dz$ where f_i is a holomorphic map.

Now suppose (φ_j, U_j) is another coordinate chart where we denote the coordinates by w (which is to say $w = \varphi_j(x)$ for $x \in U_j$). Then as before we get a differential form on $\varphi_j(U_j)$, which we denote $\omega_j = f_j(w)dw$. We of course want the two forms to agree on the overlap,

$\varphi_i(U_i \cap U_j)$. What does it mean to agree on the overlap? Let g_{ij} be the biholomorphism from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$ so that $w = g_{ij}(z)$. Therefore we have

$$\omega_j = f_j(w)dw = f_j(g_{ij}(z))d(g_{ij}(z)) = f_j(g_{ij}(z))g'_{ij}(z)dz$$

Then to say that the forms agree on the overlap we must have

$$f_j(g_{ij}(z))g'_{ij}(z)dz = f_i(z)dz$$

As before if ω is a holomorphic differential form on a complex manifold M, then by Cauchy's Theorem we know that a holomorphic differential form ω is closed and therefore has a local primitive. This means for every point, there exists a neighbourhood of it such that there is a holomorphic function g defined on this neighbourhood such that $\omega = dg$. In general, ω will not have a global primitive. As before, a differential form has a global primitive (i.e. is exact) if and only if

$$\int_\gamma \omega = 0$$

for every closed curve γ. In particular this means that every holomorphic differential form on a simply connected complex manifold has a global primitive. Even if the integral of ω over closed curves is not zero, it provides important information about the form and the geometry of the surface. We call the integral of ω over closed curves the periods of ω.

One of the most important tools for evaluating complex integrals is the Residue Theorem. The theorem still holds on general complex manifolds and hence serves as a very powerful tool for studying the geometry of these surfaces.

Suppose ω is a holomorphic differential form defined on the complement on a discrete set $E \subset M$ and let $a \in E$. Recall that the residue of ω at a is then defined to be

$$Res(\omega, a) = \frac{1}{2\pi i}\int_\gamma \omega$$

where γ is a simple closed curve with winding number 1 (with respect to a). We can compute it using local coordinates. Let z be local coordinates at a and we can assume that $z(a) = 0$. Then

$$\omega = \left(f(z) + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right) dz$$

where f is holomorphic near a. Then we compute that the residue of ω at a is c_1. With our understanding of residues, the Residue Theorem exactly as it did before.

Theorem 9.3 (Residue Theorem) *Let Ω be an open subset of a complex manifold M and $f(z)$ a holomorphic function on the complement of a discrete set in Ω. Let K be a compact subset of Ω with piecewise C^1 boundary Γ. Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{z_k \in K} Res(f, z_k)$$

where S is the set of singular points of f in K.

9.3 Riemann Surfaces

Let Y be a complex curve (most often $\mathbb{C}, S^2$ or an open subset of these). Then a Riemann surface over Y is a connected complex curve X along with a nonconstant holomorphic map $\varphi : X \to Y$.

A branch point of φ (or of X) is a point where φ has multiplicity greater than 1 (equivalently where φ' is 0). Branch points are isolated and the preimage of any point of Y is discrete (both statement follow form the fact that zeroes of a non-zero holomorphic function are isolated). A Riemann surface without branch points is called unramified. Importantly, φ need not be injective even if the the surface is unramified.

Example 9.4. A simple example to start with is to take $Y = \mathbb{C} \setminus \{0\}$ and $X = \mathbb{C}$ with $\varphi : X \to Y$ given by $\varphi(z) = e^z$.
In this case not only is X a Riemann surface over Y, it in fact forms a *covering space* of Y. This means that X is an unramified surface and every point of Y has an open neighbourhood V such that $\varphi^{-1}(V)$ is a disjoint union of open sets U_i where each U_i is mapped homeomorphically to V via φ. In this case with X, Y, φ as above, for $b \in Y = \mathbb{C} \setminus \{0\}$ we can take $V = \{|z - b| < |b|\}$.

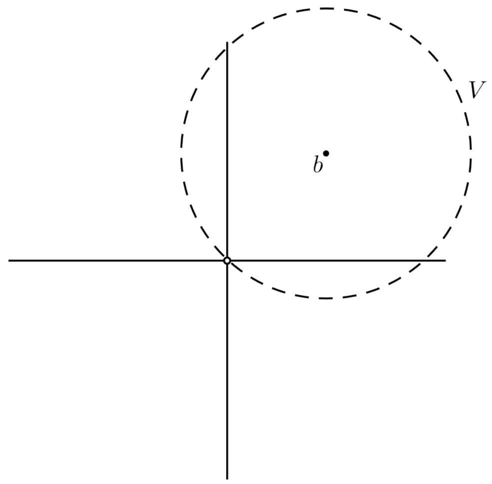
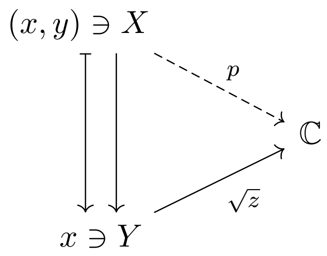


Figure 18: log has a holomorphic branch on V serving as a coordinate map

Example 9.5. A very important use of Riemann surfaces is to make multi-valued functions single-valued. Consider for example the square root function $y = x^{1/2}$. In order to make this single-valued, consider

$$X := \{(x, y) \in \mathbb{C} \times \mathbb{C} | x = y^2\}$$

This forms a Riemann surface over $Y = \mathbb{C}$ with $\varphi : X \to \mathbb{C}$ given by $\varphi(x, y) = x$. Then the square root function $Y \to \mathbb{C}$ given by $z \mapsto \sqrt{z}$ lifts to the map $p(x, y) = y$ from X to $\mathbb{C}$.



9.4 Worked Example

9.4.1 Manifold construction

Consider the multivalued function

$$y = (1 - x^3)^{1/3}$$

We want to find a Riemann surface so that this function is single-valued over it. As we did with $\sqrt{z}$ we take

$$X := \{(x, y) \in \mathbb{C} \times \mathbb{C} : x^3 + y^3 = 1\}$$

We first show that X is a manifold. Consider the function

$$f(x, y) = x^3 + y^3$$

Notice that

$$\frac{\partial f}{\partial y}(x_0, y_0) = 3y_0^2$$

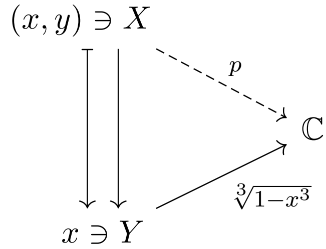
Therefore if $y_0 \neq 0$ then $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ so by the Implicit Function Theorem (see Theorem 3.6), we can express y as a function of x implying that x serves as a coordinate in this region. In particular, near (x_0, y_0) X is given as the graph of the function $x \mapsto \sqrt[3]{1 - x^3}$ with a choice of branch which is equal to y_0 when $x = x_0$. Hence, to be precise, for every $(x_0, y_0) \in X$ where $y_0 \neq 0$, we have that the projection onto the first coordinate x (along with a choice of choice of cube root) serves locally as a coordinate chart. Similarly if $x_0 \neq 0$ then y (i.e. projection onto the second coordinate) serves as a local coordinate on X.

We want to check that the transition maps, when $x_0 \neq 0$ and $y_0 \neq 0$, are holomorphic. The first coordinate chart φ_1 is given by $(x, \sqrt[3]{1 - x^3}) = (x, y) \mapsto x$ and the second chart φ_2 by $(\sqrt[3]{1 - y^3}, y) = (x, y) \mapsto y$. Then

$$(\varphi_1 \circ \varphi_2^{-1})(y) = \sqrt[3]{1 - y^3}$$

which is holomorphic since we chose a holomorphic branch of $\sqrt[3]{1 - y^3}$ (to be specific we chose a branch so that $x_0 = \sqrt[3]{1 - y_0^3}$). The analogous argument holds for $\varphi_2 \circ \varphi_1^{-1}$. Therefore X is indeed a manifold.

Now we want to consider the original function $y = \sqrt[3]{1 - x^3}$ and its lift to X. The commutative diagram is then the exact same as above



with $Y = \mathbb{C}$ and $p(x, y) = y$, as we had before.

In general, for a given y, there are 3 distinct points in X such that $p(x, y) = y$. These correspond to the three choices of the cube root of course. However, these points coincide when x is a cube root of unity since then we are taking the cube root of 0. Therefore $(1, 0), (j, 0)$ and $(j^2, 0)$, where $j = e^{2\pi i/3}$ are branch points of X (although quite often we identify the branch points with their images in Y, so one might simply say that the branch points of the Riemann surface are $1, j$ and j^2).

9.4.2 Extension of Riemann Surface

Currently X is simply a Riemman surface over $\mathbb{C}$. One might wonder whether it extends to be a Riemann surface over S^2. This is analogous to how the real line is a smooth (real) curve in $\mathbb{C}$ but it can be compactified and extended to be a curve (and in fact a closed curve) in S^2.

Therefore first we will need to compactify X and find the 'points at infinity'. Then we will verify whether these points can be mapped to $\infty \in S^2$ holomorphically.

We compactify the curve by considering it as a subset of $P^2(\mathbb{C})$. Recall that we have

$$P^2(\mathbb{C}) = \{[x, y, t] : t \neq 0\} \sqcup \{[x, y, t] : t \neq 0\}$$

The first set can be identified with $\mathbb{C} \times \mathbb{C}$ with the coordinate chart $[x, y, t] \mapsto (x/t, y/t)$. Then the line (notice line, not point) through infinity is exactly the second set, where $t = 0$. With respect to the identification above, the equation of the curve X is given by

$$\left(\frac{x}{t}\right)^3 + \left(\frac{y}{t}\right)^3 = 1$$

which we can rearrange to

$$x^3 + y^3 = t^3$$

This is exactly what it means to homogenise the equation. Let X' be the points in $P^2(\mathbb{C})$ that satisfy this equation (this is the compactification of X). The points at infinity are exactly when $t = 0$. Therefore they are given by $[x, y, 0]$ satisfying

$$x^3 + y^3 = 0$$

Of course the point $(0, 0, 0)$ is not an element of $P^2(\mathbb{C})$ so in particular y must be non-zero (in fact both x and y must be non-zero). Finally since $[x, y, 0] = [x/y, 1, 0]$, we can conclude that the 3 points at infinity are $[-1, 1, 0], [-j, 1, 0]$ and$[-j^2, 1, 0]$ where recall $j = e^{2\pi i/3}$. Therefore

$$X' = X \cup \{[-1, 1, 0], [-j, 1, 0], [-j^2, 1, 0]\}$$

Let $\varphi' : X' \to S^2$ be an extension of $\varphi : X \to \mathbb{C}$ so that it agrees with φ on X and maps the 3 points at infinity to $\infty \in S^2$. We want to check whether not φ is holomorphic. We already know this is the case for points away from the points at infinity so we only need to check at those three points.

Notice that at the three points at ∞ y is never 0. Therefore we can verify holomorphicity in this chart. In this chart the curve is given by dehomogenising with respect to y so that we are looking at

$$x'^3 - t'^3 + 1 = 0$$

where $x' = x/y$ and $t' = t/y$. Since the partial derivative of the left hand side with respect to x is non-zero at $(-j^k, 0)$ for $k = 0, 1, 2$ we conclude that t' acts as a local coordinate with $x' = \sqrt[3]{1-(t')^3}$. Then we compute what this map looks like at the level of coordinate charts.

$$\begin{array}{ccccccc} \mathbb{C} \setminus \{0\} \supset U & \longrightarrow & X' & \longrightarrow & S^2 \setminus \{S\} & \longrightarrow & \mathbb{C} \\ & & & & & & \\ t' & \longmapsto & [\sqrt[3]{1-(t')^3}, 1, t'] & \longmapsto & \varphi([\frac{\sqrt[3]{1-(t')^3}}{t'}, \frac{1}{t'}, 1]) & \longmapsto & \frac{t'}{\sqrt[3]{1-(t')^3}} \end{array}$$

where for the final mapping we use the coordinates at infinity in S^2 (so although as a map into $\mathbb{C}$ we had $\varphi([x, y, 1]) = x$, when we switch to the coordinates at ∞ this becomes $[x, y, 1] \mapsto 1/x$). Therefore overall the map is given by

$$t' \mapsto \frac{t'}{\sqrt[3]{1-(t')^3}}$$

which is holomorphic in a neighbourhood of $t' = 0$. In fact these are 3 different functions that arise from the 3 choices of cube root but this aligns with the fact that we had 3 points at infinity (so each choice of cube root corresponds to one of the points at infinity).

Recall that the Riemann surface was introduced to better understand the function $y = (1-x^3)^{1/3}$. Thus one thing we may want to do is check how this function acts on closed curves. Of course this function is multivalued on the complex plane so we will instead try to understand this behaviour by "pulling back" to the Riemann surface.

Suppose γ is a closed curve in $\mathbb{C}$ that encloses (the images of) the three branch points. Then let δ be a lift of γ. In other words, δ is a curve on X such that $\varphi \circ \delta = \gamma$. Notice that δ itself need not be closed.

Since $y = (1-x^3)^{1/3}$ lifts to the map $(x, y) \mapsto y$ on the Riemann surface X, we want to compute

$$\frac{1}{2\pi i} \int_\delta \frac{dy}{y}$$

Differentiating the defining equation of X we get

$$3x^2 dx + 3y^2 dy = 0$$

Therefore

$$\frac{dy}{y} = \frac{-x^2 dx}{y^3} = \frac{-x^2 dx}{1-x^3}$$

Now we observe the remarkable fact that

$$\begin{aligned}\frac{1}{2\pi i}\int_\delta \frac{dy}{y} &= \frac{1}{2\pi i}\int_\delta \frac{-x^2}{1-x^3}dx \\ &= \frac{1}{2\pi i}\int_\delta \varphi^*\left(\frac{-z^2}{1-z^3}dz\right) \\ &= \frac{1}{2\pi i}\int_{\varphi(\delta)} \frac{-z^2}{1-z^3}dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{-z^2}{1-z^3}dz\end{aligned}$$

where φ^* in the second line is the pullback.

We can use the Residue Theorem to compute the final integral. The poles of the form are exactly the cube roots of 1 and the residues at each of them is $1/3$. Therefore

$$\frac{1}{2\pi i}\int_\delta \frac{dy}{y} = \frac{1}{2\pi i}\int_\gamma \frac{-z^2}{1-z^3}dz = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$$

Since the integral evaluates to an integer, we conclude that the image of δ under the map is indeed a closed curve (in particular the argument of y changes by exactly 2π). However, it is clear that this only occurs because γ encircles all 3 branch points/poles. If γ only contained one of the branch points then the argument would change by $2\pi/3$ and with two branch points the argument would change by $4\pi/3$. In either of these cases, the result would not be a closed curve. This corresponds exactly with the fact that $y = (1-x^3)^{1/3}$ is multivalued on $\mathbb{C}$. By the above discussion, we can make this single-valued by introducing cuts on $\mathbb{C}$ as in Figure 19.

By adding these cuts, any closed curve needs to encircle either none or all 3 branch points (if a curve doesn't encircle any of the branch points in one orientation it encircles all of them once the orientation is flipped) which in particular means that the image of any closed curve will be a closed curve. Hence the function $y = (1-x^3)^{1/3}$ is well-defined. However, we have 3 different choices of the cube root and each choice gives a well-defined holomorphic branch of this function. Therefore the Riemann surface for this function will 'look like' 3 copies of the plane with these cuts but these so-called sheets are glued together along these cuts. This is analogous to how the Riemann sphere 'looks like' two copies of $\mathbb{C}$ that are glued along $\mathbb{C} \setminus \{0\}$.

> **Remark 9.6.** The cuts made above are somewhat arbitrary. Any choice of cuts that forces closed curves to encircle all 3 branch points is valid.

9.4.3 Evaluation of a real integral

One very nice use of the above construction(s) is that it allows us compute real integrals. For example, suppose we want to evaluate

$$\int_0^1 \frac{dx}{(1-x^3)^{1/3}}$$

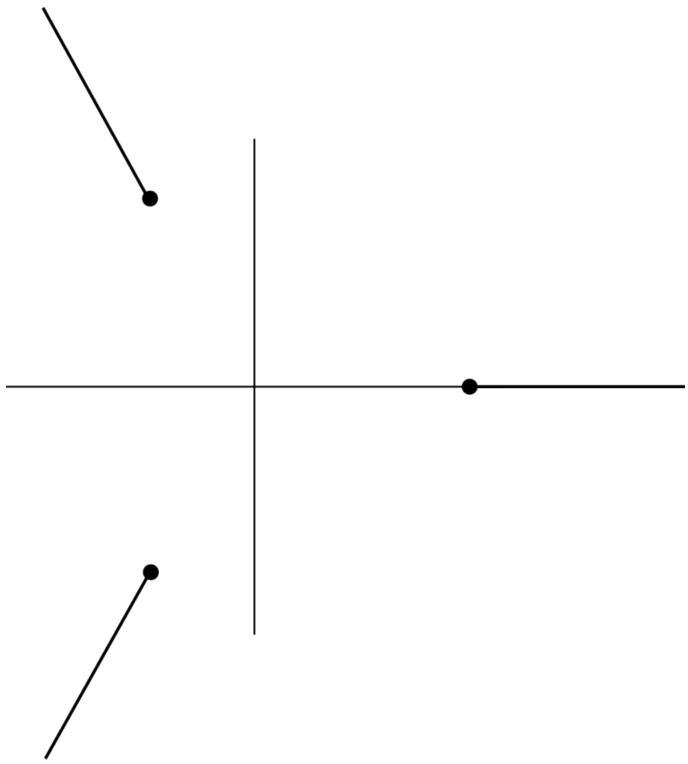


Figure 19: Any closed curve avoiding the cuts either encircles none of the branch points or all of them (remember curves can also extend to ∞)

We will do so by viewing this integral in the complex numbers. Of course, the function is multi-valued over $\mathbb{C}$ so we will go to the Riemann surface so that we have a well-defined holomorphic function.

Since we want to integrate from 0 to 1, we will introduce suitable cuts and integrate along a contour as in Figure 20.

To be precise, we want to know what happens as we let the countour approach the cut. We can break this integral as the sum of the 6 straight lines (the integrals over the circular arcs have negligible contribution in the limit). However, notice that every time we go around a branch point, the argument of the denominator increases by $2\pi/3$. For example, let us denote the integral over γ_1, the integral from 0 to 1, by I (this is what we are trying to evaluate). The integral over γ_2 is the same integral from 1 to 0 so it should be the same as I except we pick up a negative sign due reversing the orientation and the denominator picks up a factor of $j = e^{2\pi i/3}$. Thus we have

$$\int_{\gamma_1} \frac{dz}{(1-z^3)^{1/3}} + \int_{\gamma_2} \frac{dz}{(1-z^3)^{1/3}} = I - \frac{1}{j} I = I - j^2 I$$

Now we want to compute the integral over γ_3. Notice that $\gamma_3(t) = j \cdot \gamma_1(t)$. Moreover, since the denominator picked up a factor of j when going around 1, we have

$$\int_{\gamma_3} \frac{dz}{(1-z^3)^{1/3}} = \int_{\gamma_1} \frac{d(jz)}{j(1-(jz)^3)^{1/3}} = \int_{\gamma_1} \frac{dz}{(1-z^3)^{1/3}} = I$$

Similarly we get

$$\int_{\gamma_4} \frac{dz}{(1-z^3)^{1/3}} = -j^2 I$$

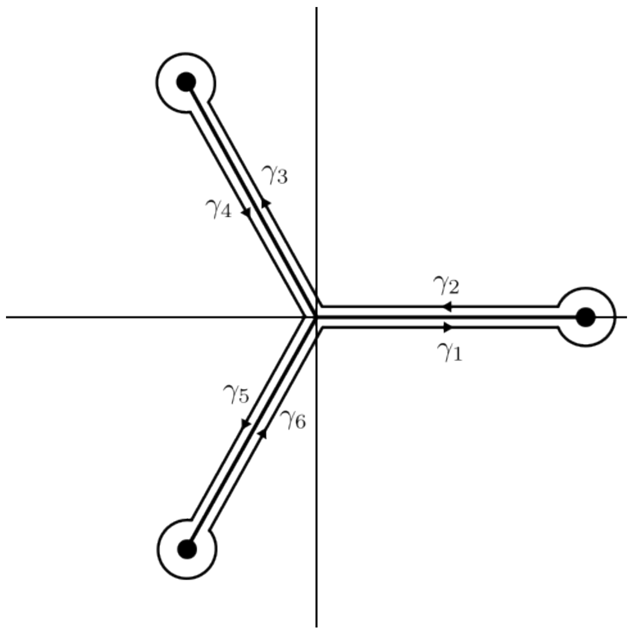


Figure 20: We introduce cuts from the origin to branch points and integrate just slightly around them

and of course the same thing will hold for the remaining two curves due to the same reasoning

$$\int_{\gamma_5} \frac{dz}{(1-z^3)^{1/3}} = I$$
$$\int_{\gamma_6} \frac{dz}{(1-z^3)^{1/3}} = -j^2 I$$

Then if we denote the entire contour by δ we have

$$\int_{\delta} \frac{dz}{(1-z^3)^{1/3}} = 3(I - j^2 I)$$

On the other hand, we can also use the Residue Theorem to compute the left hand side. In particular we will use the residue at ∞.

In order to compute the residue at ∞, we will simply switch to coordinates at ∞ so let $u = 1/x$. Then

$$\frac{dx}{(1-x^3)^{1/3}} = \frac{d(1/u)}{(1-(1/u)^3)^{1/3}} = -\frac{du}{u^2(1-1/u^3)^{1/3}} = -\frac{du}{u(u^3-1)^{1/3}}$$

The residue of this form at $u = 0$ is $-1/(-1)^{1/3}$. Notice we have 3 different answers corresponding to the three distinct lifts of δ or the three distinct points at ∞. Therefore

$$3(I - j^2 I) = \int_{\delta} \frac{dz}{(1-z^3)^{1/3}} = 2\pi i \cdot \{\text{one of } 1, j, j^2\}$$

In order to determine what the value of the integral is, we use the fact that I is a real integral and hence the answer should be real. Trying the three different options, we determine that the only possible choice is j which gives

$$I = \int_0^1 \frac{dx}{(1-x^3)^{1/3}} = \frac{2\pi}{3\sqrt{3}}$$

9.5 Riemann surfaces and Elliptic Curves

Suppose we have an equation of the form

$$y^2 = P(x)$$

where P is a polynomial of degree 3 with 3 distinct roots. Without loss of generality we can assume that the coefficient of x^2 is 0 by completion of the cubic (for example if we have $P(x) = x^3 + ax^2 + bx + c$ then we can consider $P(x - a/3)$ which has no quadratic term). Therefore we can write

$$y^2 = 4x^3 - 20a_2x - 28a_4 \tag{9.1}$$

where a_2, a_4 are just suggestively labeled constants.

We then get a Riemann surface $\varphi : X \to \mathbb{C}$ where X is the curve defined by the given equation in $\mathbb{C}\times\mathbb{C}$ where φ is the projection onto the x coordinate (exactly as we had earlier). We can then complete this to the curve $X' \subset P^2(\mathbb{C})$. We saw in Subsection 3.2 that there is a single point at infinity $[0, 1, 0]$ and the extension $\varphi' : X' \to S^2$ is holomorphic at this point.

Consider the form dz which is given by

$$\frac{dx}{y}$$

when x is a local coordinate (i.e. $y \neq 0$). Computing the differential of both sides of (9.1), we see that

$$2ydy = 12x^2 - 20a_2dx$$

This means that

$$\frac{dx}{y} = \frac{dy}{6x^2 - 10a_2}$$

Therefore when y is a local coordinate (i.e.when $x \neq 0$) we can use the right hand side to evaluate the integral.

The holomorphic differential form ω has a local primitive at every point of X. Globally the primitive is a multi-valued function given by the integral of $\omega = dx/y$. Notice that any branch of z serves as a coordinate in a neighbourhood of any point of X.

At $[0, 1, 0]$, the chart is given by $[x', 1, t']$ which in our usual coordinates is $[x'/t', 1/t', 1]$ so that $x = x'/t'$ and $y = 1/t'$. Thus

$$\begin{aligned}\omega &= \frac{dx}{y} \\ &= t'd(x'/t') \\ &= dx' - \frac{x'}{t'}dt' \\ &= dx' - x' \cdot \frac{12x'^2 + \cdots}{4x'^3 + \cdots}dx' \\ &= -2(1 + g(x'))dx'\end{aligned}$$

where g is a holomorphic function satisfying $g(0) = 0$.

Recall from our discussion of $\wp$ that if Γ is a discrete group and

$$a_2 = 3 \sum_{\omega \in \Gamma \setminus \{0\}} 1/\omega^4 \text{ and } a_4 = 5 \sum_{\omega \in \Gamma \setminus \{0\}} 1/\omega^6$$

then the meromorphic transformation $x = \wp(z)$ and $y = \wp'(z)$ induces a biholomorphism $\mathbb{C}/\Gamma \to X'$ given by $z \mapsto [\wp(z), \wp'(z), 1]$. Notice that $dx = \wp'(z)dz = ydz$ so $dz = dx/y$. Thus in particular the inverse of the biholomorphism is given by the multivalued function

$$z = \int dz = \int \frac{dx}{y} = \int \omega$$

The branches of z differ by the constants in Γ. Abel's Theorem below tells us that this result has something of a converse. Namely given an elliptic curve, we can recover the discrete group.

Theorem 9.7 (Abel's Theorem) *Suppose we are given constants a_2, a_4 such that $P(x) = 4x^3 - 20a_2x - 28a_4$ has 3 distinct roots. Then there exists a discrete group Γ such that $a_2 = 3\sum 1/\omega^4$ and $a_4 = 5\sum 1/\omega^6$. It follows that $y^2 = P(x)$ has a parameterisation given by $x = \wp(z)$ and $y = \wp'(z)$.*

We will give a sketch of the proof as the complete proof requires some algebraic topology and other results. The proof requires 2 lemmas.

Lemma 9.8 *Let $z = \int \omega = \int dy/x$ by the multivalued function arising from the curve $y^2 = P(x)$. Then branches of z differ from each other by constants that form a discrete group Γ of $\mathbb{C}$ where Γ is generated by two complex numbers e_2 and e_2 which are linearly independent over $\mathbb{R}$.*

Proof. Notice that $z = \int \omega$ is well definied up to the addition of a period, i.e. up to the addition of $\int_\gamma w$ where γ is a C^1 closed curve (technically a a 1-cycle or a 1-chain with 0 boundary) in X'. If γ is the boundary of a 2-chain, then $\pi(\gamma) = \int_\gamma \omega = 0$ by Stokes' Theorem.

Then π induces a homomorphism from the first homology group $H_1(X, \mathbb{Z})$ to $\mathbb{C}$. Thus $z : X' \to \mathbb{C}/\Gamma$ where $\Gamma = \{\pi(\gamma) : \gamma \in H_1(X', \mathbb{Z})\}$ is the group of periods. By the Riemann-Hurwitz formula, we compute that $H_1(X', \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. In particular then Γ is generated by two complex numbers e_1 and e_2. They are either linearly independent over the reals or they are not. If they are then we get a lattice as claimed. If not then Γ is contained in a 1-dimensional subspace of $\mathbb{C}$. We show this cannot happen.

Suppose Γ is contained in a 1-dimensional subspace of $\mathbb{C}$. Then by applying an appropriate rotation, i.e. by multiplying by an appropariate unit complex number α we get that $\alpha\Gamma$ is contained in the imaginary axis. In particular then $\text{Re}(\alpha\pi(\gamma))$ for all $\gamma \in H_1(X', \mathbb{Z})$ is 0. Recall we argued above that the branches of z can only differ by an element of Γ. Therefore if $\text{Re}(\alpha\pi(\gamma))$ is 0 for all γ then $\text{Re}(\alpha z)$ is a (single-valued) harmonic function on X'. Since X' is a compact, the function attains a maximum. But then by the maximum modulus principle we conclude that $\text{Re}(\alpha z)$ is constant. Since a holomorphic function is constant if and only if its real part is constant, this would imply that z is constant, leading to a contradiciton.

□

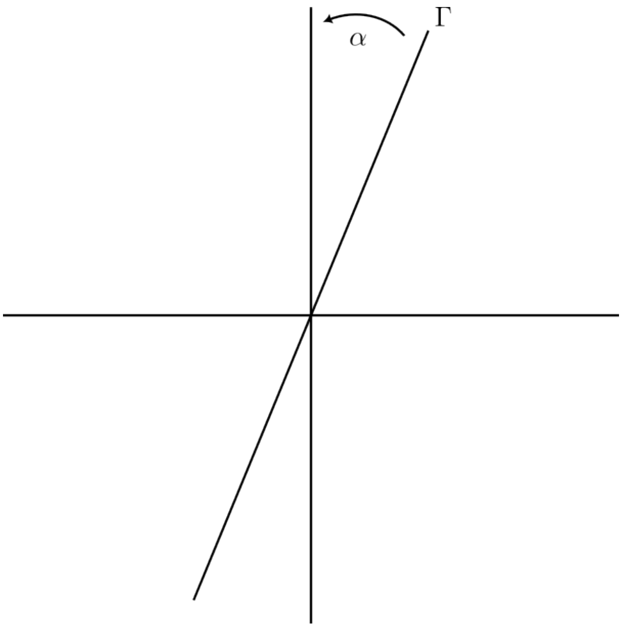


Figure 21: If Γ is contained in a 1-dimensional subspace, we can rotate it to be contained in the imaginary axis

Now that we have found a lattice we want to show that this lattice gives rise to the same elliptic curve.

Lemma 9.9 *The map $X' \to \mathbb{C}/\Gamma$ given by*

$$z(p) = \int_{p_0}^{p} \omega$$

(where p_0 is the point at infinity $[0,1,0]$) is a biholomorphism and in fact the composition

$$\begin{aligned} X' \to \mathbb{C}/\Gamma &\to X'' \\ p \mapsto z &\mapsto [\wp(z), \wp'(z), 1] \end{aligned}$$

is the identity.

Proof. The first part of the lemma about z being a biholomorphism requires some algebraic topology so we will simply believe it.

Notice that x is a meromorphic function on X' with a pole of order 2 at $p_0 := [0,1,0]$. This follows from work we've done previously. The coordinates near ∞ of X' are given by $[x', 1, t']$ where we know from Subsection 3.2 that

$$t' = 4x'^3 - 320a_2x'^7 + \cdots$$

In our usual $[x, y, 1]$ coordinates then we have

$$x = \frac{x'}{t'} = \frac{x'}{4x'^3 - 320a_2x'^7 + \cdots}$$

Then via the biholomorphism z, we can pull back x to a function on $\mathbb{C}/\Gamma$. Since z maps p_0 to 0, we know that x has a pole of order 2 at $z = 0$. Therefore

$$x(z) = \frac{c}{z^2} + \frac{d}{z} + e + fz + \cdots$$
$$x'(z) = -\frac{2c}{z^3} - \frac{d}{z^2} + f + \cdots$$

Then since $dz = dx/y$ we have $x'(z) = y$. Therefore

$$x'(z)^2 = y^2 = 4x^3 - 20a_2x - 28a_4$$

Substituting the above series and equating coefficients we conclude that $c = 1$, $d = 0$ and $e = 0$. This means that $x(z)$ and $\wp(z)$ (with respect to the lattice Γ) have the same principal part so $x(z) - \wp(z)$ is a doubly periodic, entire function so must be constant. Since the difference is 0 at 0, we conclude that $x(z) = \wp(z)$ everywhere. □