What is K-theory and what is it good for?

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1 Introduction

The below is an adaptation of a lecture delivered by Professor Paul Baum in September 2021. A recording of this lecture can be found here: `https://www.youtube.com/watch?v=YXcSTY7l46s`

2 The basic definition of $K-$theory

Let J be an abelian semi-group, which is to say that J is a set with a closed operation that is both commutative and associative. We define a relation on $J \oplus J$ where

$$(\xi, \eta) \sim (\xi', \eta') \Leftrightarrow \exists \theta \in J \text{ with } \xi + \eta' + \theta = \xi' + \eta + \theta$$

Then one can show that $\widehat{J} = (J \oplus J)/\sim$ forms an abelian group. For example $\widehat{\mathbb{N}} = \mathbb{Z}$ (this is of course how one constructs the integers from the natural numbers).

Now suppose Λ is a ring with unit 1_Λ. Let $M_n(\Lambda)$ denote the set of all $n \times n$ matrices with entries from Λ. Then with the operations of matrix addition and multiplication $M_n(\Lambda)$ is again a ring with unit (the unit being the so-called identity matrix with 1_Λ down the diagonal and 0's everywhere else).

We define $GL(n, \Lambda)$ to be the set of all invertible elements of $M_n(\Lambda)$. We additionally define

$$P_n(\Lambda) = \{\alpha \in M_n(\Lambda) : \alpha^2 = \alpha\}$$

The elements are called idempotent (a word stemming from the Latin for "same", *idem* and "powerful", *potentum*). We can then define a relation on $P_n(\Lambda)$, namely:

$$\alpha \sim \beta \Leftrightarrow \exists \gamma \in GL(n, \Lambda) \text{ with } \beta = \gamma^{-1}\alpha\gamma$$

We can then define

$$P(\Lambda) = P_1(\Lambda) \cup P_2(\Lambda) \cup \dots$$

where we extend the matrices by adding rows and columns of 0's when necessary.

Then we impose yet another relation, this time of stable similarity on $P(\Lambda)$. Namely we say that $\alpha \in P_n(\Lambda)$ and $\beta \in P_m(\Lambda)$ are *stably similar* if and only if there exists non-negative integers r and s such that $n+r=m+s$ and $\alpha+[0]_r$ is similar to $\beta+[0]_s$ (where $[0]_k$ is used to denoted the $k\times k$ zero matrix). That is to say that if we extend α and β in the natural way and find that they are similar, then they are stably similar.

Now we define

$$J(\Lambda) = P(\Lambda)/(\text{stable similarity})$$

and claim that this is an abelian semi-group. The addition is given by

$$\alpha+\beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

written as a block diagonal matrix. The quotienting of stable similarity ensures that this addition is indeed commutative (indeed if we take γ to be the matrix that permutes the columns of β with the columns of α we can see that $\alpha+\beta$ is similar to $\beta+\alpha$).

Finally, we can apply the aforementioned trick to convert this abelian semi-group into a full fledged group. This leads us to the basic definition of $K-$theory. Namely

$$K_0\Lambda = \widehat{J(\Lambda)}$$

We note that this construction is functorial. That is, a ring homomorphism $\varphi : \Lambda \to \Omega$, induces a homomorphism of abelian groups $\varphi_* : K_0\Lambda \to K_0\Omega$ wherein we simply apply φ to each of the matrix entries.

Theorem 2.1. *If Λ is a field, then $K_0\Lambda = \mathbb{Z}$.*

Proof. We recall from linear algebra that two matrices are stably similar if and only if they have the same rank, where rank denotes the dimension spanned by the columns (or equivalently the rows) of a matrix. This means that $J(\Lambda)$ is then completely characterised by the set of non-negative integers $\mathbb{N}$. As we know, $\widehat{\mathbb{N}} = \mathbb{Z}$ which means that $K_0\Lambda = \mathbb{Z}$. □

Now suppose X is a compact, Hausdorff space. We obtain a ring with unit from X as well by defining

$$C(X) = \{\alpha : X \to \mathbb{C} | \alpha \text{ is continuous}\}$$

The operations of addition and multiplication are pointwise, that is,

$$(\alpha+\beta)(x) = \alpha(x)+\beta(x), (\alpha\beta)(x) = \alpha(x)\beta(x)$$

and the unit is the constant function 1. Atiyah and Hirzebruch then define

$$K^0(X) = K_0C(X)$$

3 A brief history of $K-$theory

3.1 Hirzebruch-Riemann-Roch Theorem

An underlying unity between K-theory for C^*-Algebras and Algebraic K-theory! Consider the following: M is a non-singular projective algebraic variety over $\mathbb{C}$. E is an algebraic vector bundle on M. $\underline{E}$ is the sheaf of germs of algerbaic sections of E. $H^j(M, E) :=$ j-th cohomology of M using $\underline{E}$, $j = 0, 1, 2, 3, ...$

Using the lemma, for all $j = 0, 1, 2, ...dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$ and for all $j > dim_{\mathbb{C}}(M)$, $H^j(M, \underline{E}) = 0$.

The HRR theorem states that: Suppose M is a non-singular projective algebraic variety over $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then
$\chi(M, E) = (ch(E) \cup Td(M))[M]$.

On a high-level this is a generalization of what Riemann and Roch did on the Riemann surface when it came to their work on K-theory in algebraic geometry.

3.2 Tangent Vector Fields on Spheres

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Then a continuous tangent vector field is defined as a continuous map

$$V : S^{n-1} \to \mathbb{R}^n$$

such that $\langle p, V(p) \rangle = 0$, using the standard inner product. We can then consider linearly independent continuous tangent vector fields, were a list of continuous tangent vector fields $V_1, \ldots, V_r$ is said to linearly independent if $V_1(p), \ldots, V_r(p)$ are linearly independent for each $p \in S^{n-1}$. The natural question that arises then is given S^{n-1}, what is the maximum number of linearly independent continuous vector fields one can have.

In the case for $S^2, S^4, S^6, \ldots$ (i.e. for odd n) the answer is 0. This follows from the Hairy Ball Theorem which asserts that for any continuous vector field V on an even-dimensional sphere, there exists a point p where $V(p) = 0$.

The case for odd-dimensional sphere was proven by John Frank Adams in 1962. In particular, Adams showed the following

Theorem 3.1. *Given $n \in \mathbb{N}$, decompose it as $n = 2^{c(n)} 16^{d(n)} u$ where u is an odd integer. Define $\rho(n) = 2^{c(n)} + 8d(n)$. Then S^{n-1} admits $\rho(n) - 1$ linearly independent continuous tangent vector fields and does not admit $\rho(n)$ linearly independent continuous tangent vector fields.*

Adams' proof of this statement used $K-$theory quite intimately and provided strong evidence for the usefulness of $K-$theory. We make a few remarks about this theorem. We see that the statement does capture the case for even-dimensional spheres (if n is odd then $c(n) = d(n) = 0$ implying that $\rho(n) = 1$ as desired). Additionally there are relatively few linearly independent continuous vectors fields on any sphere as the exponent on 16 becomes a linear term of $\rho(n)$ implying that there is something logarithmic in nature going on. Finally we note that there is a certain periodicity going on as $\rho(n)$ is independent of u.

3.3 C^*−algebras

Definition 3.1 (Banach Algebras)**.** *A Banach algebra is an algebra A over $\mathbb{C}$ with a norm, $\|\cdot\|$ with respect to which A is complete. Additionally, A satisfies the following axioms:*

1. $\|\lambda a\| = |\lambda| \, \|a\|$ *for all* $\lambda \in \mathbb{C}, a \in A$
2. $\|a + b\| \leq \|a\| + \|b\|$ *for all* $a, b \in A$
3. $\|ab\| \leq \|a\| \, \|b\|$ *for all* $a, b \in A$
4. $\|a\| = 0 \Leftrightarrow a = 0$

Definition 3.2 (C^* algebras)**.** *A C^* algebra is a Banach algebra with the additional structure of an involution denoted by a * that satisfies:*

1. $(a + b)^* = a^* + b^*$
2. $(ab)^* = b^* a^*$
3. $(\lambda a)^* = \overline{\lambda} a^*$
4. $\|a^* a\| = \|a\|^2$

Given C^*−algebras A and B, a $*$−homomorphism is an algebra homomorphism $\varphi : A \to B$ such that $\phi(a^*) = \phi(a)^*$ for all $a \in A$.

Lemma 3.1. *If $\varphi : A \to B$ is a $*$−homomorphism, then $\|\phi(a)\| \leq \|a\|$ for all $a \in A$.*

Proof. It is a well-known fact in C^*−algebras that for any $a \in A$,

$$\|a\| = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a^* a)\}^{1/2}$$

(this strong correspondence between the norm of an element and its spectrum, an algebraic property, is part of what makes the subject so interesting!). Then all we need show is that $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$.

We can assume A, B to be unital, where we adjoin a unit if necessary. Now suppose $\lambda \in \mathrm{sp}(\varphi(a))$ for some $a \in A$. This means that $\varphi(a) - \lambda 1_B = \varphi(a - \lambda 1_A)$ is not invertible. But then $a - \lambda 1_A$ cannot be invertible as if it were, then its inverse would get mapped to the inverse of $\varphi(a - \lambda 1_A)$. Hence $\lambda \in \mathrm{sp}(a)$. □

3.4 Examples of C^*-algebras

Suppose X is a locally compact, Hausdorff space. Let $X^+ = S \cup \{p_\infty\}$ denote the one-point compactification of X. We define

$$C_0(X) = \{\alpha : X^+ \to \mathbb{C} | \alpha \text{ is continuous and } \alpha(p_\infty) = 0\}$$

The addition and (both kinds of) multiplication on $C_0(X)$ are pointwise. The norm is the supremum norm given by

$$\|\alpha\| = \sup_{p \in X} |\alpha(p)|$$

The involution is given by conjugation

$$\alpha^*(p) = \overline{\alpha(p)}$$

One remark to be made is that X is compact, then $C_0(X) = C(X)$

The above is an important example of C^*-algebras as the Gelfand-Naimark Theorem for commutative C^*-algebras tells us that any commutative C^*-algebra is isometrically $*-$isomorphic to an algebra of the above form.

The second important example of C^*-algebras comes from Hilbert spaces and in particular the bounded operators on Hilbert spaces. Let $\mathcal{L}(H)$ be the set of bounded operators on H, where we say an operator T is bounded if $\{\|Tu\| : u \in H, \|u\| \leq 1\}$ is bounded (for operators on a Hilbert space, this is equivalent to saying that T is continuous). This allows us to define a norm on $\mathcal{L}(H)$, namely

$$\|T\| = \sup\{\|Tu\| : u \in H, \|u\| \leq 1\}$$

The addition and scalar multiplication are pointwise. The multiplication is now composition, that is $(TS)(u) = T(S(u))$. Finally the star operation is that of the adjoint, i.e. T^* is the unique operator in $\mathcal{L}(H)$ satisfying $\langle Tu, v\rangle = \langle u, T^*v\rangle$ for all $u, v \in H$.

4 Algebraic vs Topological $K-$theory

4.1 Reduced C^*-algebras

Let G be a locally compact, second countable (i.e. the topology has a countable basis), Hausdorff topological group. Some examples include $p-$adic groups, adelic groups, discrete groups, etc. Given such a G we fix a left-invariant Haar measure dg. By left-invariant, we mean that

$$\int_G f(\gamma g) dg = \int_G f(g) dg \forall \gamma \in G$$

Haar's Theorem guarantees that such a measure always exists on a locally compact, Hausdorff topological group. Given a measure, we can now define

$$L^2G = \left\{ u : G \to \mathbb{C} \middle| \int_G |u(g)|^2 dg < \infty \right\}$$

which is a Hilbert space with the inner product given by

$$\langle u, v\rangle = \int_G u(g)\overline{v(g)}dg$$

We can then define

$$C_cG = \{f : G \to \mathbb{C} | f \text{ is continuous and has compact support}\}$$

where scalar multiplication and addition are given as usual (that is pointwise) but multiplication is given by convolution

$$(f * h)g_0 = \int_G f(g)h(g_0^{-1}g)dg$$

Then one forms an injection of algebras

$$0 \to C_cG \to \mathcal{L}(L^2G)$$

given by $f \mapsto T_f$, where $T_f(u) = f * u$ for $u \in L^2G$. Let us denote C_r^*G denote the closure (in the operator norm) of the image of C_cG in $\mathcal{L}(L^2G)$. Then C_r^*G is a sub C^*-algebra of $\mathcal{L}(L^2G)$ and is known as the reduced C^*-algebra.

4.2 Higher $K-$groups

Suppose that A is a C^*-algebra (or even a Banach algebra) with unit 1_A. Recall from before that $GL(n, A)$ is the set of invertible $n \times n$ matrices with entries in A. Then we can use the topology on A arising from the norm to topologise $GL(n, A)$ (in particular by viewing it as a subspace of A^{n^2}). Additionally we see that $GL(n, A)$ embeds into $GL(n + 1, A)$ in a natural way:

$$GL(n, A) \hookrightarrow GL(n + 1, A)$$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 1_A \end{pmatrix}$$

This allows us to consider

$$GL(A) = \lim_{n\to\infty} GL(n, A) = \bigcup_{n=1}^{\infty} GL(n, A)$$

And giving $GL(A)$ the direct limit topology, this topology in which a set $U \subset GL(A)$ is open if and only if $U \cap GL(n, A)$ is open in $GL(n, A)$ for all $n = 1, 2, 3, ...$ And that is how A can be topologized.

4.3 Working towards Baum-Connes Conjecture

Now suppose we have a C^*-algebra (or a Banach algebra) A with unit 1_A and the higher K-theory groups K_1A, K_2A, K_3A, we can define $K_jA := \pi_{j-1}(GLA)$, $j = 1, 2, 3, ...$ Note that GLA is topologized, and thus we can take its homotopy group π_{j-1}.

Now for the Bott Periodicity, one uses the remarkable fact that if we take the loop space twice on $GL(A)$ then it is homotopically equivalent to $GL(A)$ i.e. $\Omega^2 GL(A) \sim GL(A)$. We also note that the higher K-theory groups repeat i.e. $K_jA \cong K_{j+2}A$, where $j = 0, 1, 2, ...$ And thus, due to Bott Periodicity, we only have 2 groups, the even group K_0A and the odd group K_1A.

And thus we can note that for a C^* algebra (or a Banach algebra) A with unit 1_A, $K_0A = K_0^{alg}A = \widehat{J(A)}$ as we have done before where $A = (A, \|\|, *)$. In other words, for K_0A we can forget $\|\|$ and $*$ and view A as a ring with unit. We can then define K_0A as above using idempotent matrices. For K_1A however, we cannot forget $\|\|$ and $*$. And thus this is the K-theory including higher K-theory for $C*$-Algebra or Banach Algebra with unit 1_A.

Now given the same conditions for the C^*-Algebra (or Banach Algebra) A, we can find the Bott Periodicity isomorphism to be $K_0A = \widehat{J(A)} \to K_2A = \pi_1 GLA$, the fundamental group of GLA is just K_2A. As such, the Bott periodicity isomorphism assigns to $\alpha \in P_n(A)$ the loop of n x n invertible matrices $t \to I + (e^{2\pi it} - 1)\alpha$, where $t \in [0, 1]$ and I is the n x n identity matrix.

Extending the K-theory including higher K-theory for C^* algebras or Banach algebras A without unit 1_A, then we note that we have to adjoin a unit such that:

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{C} \longrightarrow 0$$

And thus we define:
$K_jA = K_j\tilde{A}$ such that $j = 1, 3, 5, ...,$
$K_jA = \text{Ker}(K_j\tilde{A} \longrightarrow K_j\mathbb{C})$, such that $j = 0, 2, 4, ...$
and we maintain the same $K_jA \equiv K_{j+2}A$, $j = 0, 1, 2, 3....$ And this is how the theory is extended to A that does not have a unit 1_A whilst maintaining Bott Periodicity.

Lastly, we note that this is a functorial construction of K-theory where for A, B that are C^*-algebras, then we have:
$\delta : A \longrightarrow B$ which is a $*$- homomorphism
$\delta^* : K_jA \longrightarrow K_jB$, where $j = 0, 1$
which is heavily involved the classification of C^*-algebras which we will go into in the next essay.

4.4 Baum-Connes Conjecture

Now given a topological group G which is a locally compact Haursdoff space that is second countable (second countable = topology of G has a countable base), then for a reduced C^* algebra of G C_r^*G, we can pose the problem... What is the K-theory of the reduced C^*-algebra? In other words, $K_jC_r^*G =?$ for $j = 0, 1$

Herein is the Baum-Connes Conjecture which states that $\mu : K_j^G(\underline{E}G) \to K_jC_r^*(G)$ is an isomorphism, for $j = 0, 1$.

We note that if G is compact or abelian, then the conjecture is true. However, there are many open problems if otherwise which can be solved if the conjecture is true which we will not go into in this essay but possibly in a future essay.

5 The Unity of $K-$theory

5.1 Algebraic $K-$Theory

Let Λ be a ring with unit 1_Λ. We already have $K_0\Lambda$. Once again we take $GL\Lambda = \lim_{n\to\infty} GL(n, \Lambda)$, however this time there is no topology on $GL\Lambda$. We then define

$$K_1^{alg}\Lambda = GL\Lambda/[GL\Lambda, GL\Lambda]$$

in order to abelianise $GL\Lambda$ (as usual $[GL\Lambda, GL\Lambda]$ denotes the set of commutators). We now have a lemma

Lemma 5.1. $[GL\Lambda, GL\Lambda]$ *is perfect, which is to say it has no non-trivial abelian quotients.*

This property allows us to use the Quillen +-construction on $BGL\Lambda$, the classifying space of $GL\Lambda$. We can then define

$$K_j^{alg}\Lambda = \pi_j(BGL\Lambda)_+$$

where $(BGL\Lambda)_+$ denotes the Quillen +-construction of $BGL\Lambda$.

We can extend this to the case of when Λ has no unit by unitising it to get $\tilde{\Lambda}$ and defining

$$K_j^{alg}\Lambda = \mathrm{Ker}(K_j^{alg}\tilde{\Lambda} \to K_j^{alg}\mathbb{Z})$$

5.2 K-theory for C^*-Algebras

For a C^*-Algebra A, we can apply a trivial move known as stabilizing A such that when applied:

$$M_n(A) \hookrightarrow M_{n+1}(A)$$
$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 1_A \end{pmatrix}$$

which is a one-to-one $*$-homomorphism where the mapping is norm preserving. By taking the limit:

$$\begin{aligned} M_\infty(A) &= \lim_{\rightarrow} M_n(A) \\ &= \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \end{pmatrix} \middle| \text{ almost all } a_{ij} = 0 \right\} \end{aligned}$$

And thus we note that: $\dot{A} = \overline{M_\infty(A)}$ and we denote the C^*-algebra $\dot{A}$ as the stabilization of A. Often, this is called the tensor-product of A with the compact operators. And so, $K_j(A) = K_j(\dot{A})$, for $j = 0, 1$.

5.3 Karoubi Conjecture

This leads us to the Karoubi conjecture which unifies both C^*-algebra K-theory and algebraic K-theory. The conjecture states that for C^* algebra A, $K_j(\dot{A}) = K_j^{alg}(\dot{A})$, for $j = 0, 1, 2, 3, ...$ This theorem unifies K-theory and says that C^*-algebra K-theory is a subdiscipline of algebraic K-theory in which Bott periodicity is maintained and certain basic examples are easy to compute.