Topology

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Section 1 Introduction to point-set topology

This section includes material covered in Lectures 1 through —.

1.1 Defining topological spaces

Definition 1.1.1 — Topology (in class's notation). Given a set X, we can define a topology $\mathcal{T}$ on X as any collection of subsets of X such that:

1. $\phi, X \in T$
2. The union of the elements of any sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.

R The professor recommends thinking of topology as a 'collection' rather than as a set, but also made a remark that, for the purposes of this course, the notion of considering topology as a set would also be acceptable. Hence, for the remainder of these notes we consider $\mathcal{T} \subseteq \mathcal{P}(X)$. In exams, however, it would be best to use the notation of collections instead.

Definition 1.1.2 — Topology. Given a set X, we can define a topology $\mathcal{T}$ on X as $\mathcal{T} \subseteq \mathcal{P}(X)$ such that:

$$1. \phi, X \in \mathcal{T};$$

$$2. (\forall \alpha \in \Lambda, U_\alpha \in \mathcal{T}) \implies \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T};$$

$$3. \{U_i \in \mathcal{T} | 1 \leq i \leq n\} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}$$

Definition 1.1.3 — Topological space. A topological space $(X, \mathcal{T})$ is an ordered pair (2-tuple) consisting of a set X and a topology $\mathcal{T}$ defined on X.

Definition 1.1.4 — Open set. For a set X and a topology $\mathcal{T}$ defined on X, we say that a subset $U \subseteq X$ is an open set of X if $U \in \mathcal{T}$.

Note using this terminology, we can restate the above axioms as: the empty set and the set itself are open; any arbitrary union of open sets is open; any finite intersection of open sets is open.

■ **Example 1.1** Let $X = \{1, 2, 3\}$. We will define a few topologies on X.

1. Consider $\mathcal{T} = \{\phi, X\}$. Is $\mathcal{T}$ a topology? It satisfies the first requirement by definition. Furthermore, $\phi \cup X = X \in \mathcal{T}$ so it satisfies the second requirement too. Next, $\phi \cap X = \phi$, and so it satisfies the third requirement as well. Hence, $\mathcal{T}$ is indeed a topology.

2. Consider $\mathcal{T} = \{\phi, X, \{1\}, \{2\}\}$. Is $\mathcal{T}$ a topology? No, because note that $\{1\} \cup \{2\} = \{1,2\} \notin \mathcal{T}$, hence failing the second requirement.

■

1.2 Introducing different topologies

Definition 1.2.1 — Discrete and Indiscrete topology. Let X be some set.
1. The collection $\mathcal{T} = \mathcal{P}(X)$ is defined to be the discrete topology.
2. The collection $\mathcal{T} = \{\phi, X\}$ is defined to be the indiscrete topology. It is also sometimes referred to as the trivial topology.

Lemma 1.2.1 For any set X, the discrete topology and indiscrete topology are well-defined topologies on X.

Proof. First we consider the discrete topology. By definition of the power set, the discrete topology contains the empty set and the set X itself. Additionally any union and intersection of subsets of X are still subsets of X, hence they remain in $\mathcal{P}(X)$ proving that the discrete topology is indeed a topology. The case for the indiscrete topology has been shown above. ■

Definition 1.2.2 — Finite complement topology. Let X be a set. The finite complete topology on X, $\mathcal{T}_f$, is defined as the collection of all subsets $U \subseteq X$ such that $X - U$ is either finite or all of X. This is also known as the cofinite topology.

Definition 1.2.3 — Countable complement topology. Let X be a set. The countable complement topology on X, $\mathcal{T}_c$, is defined as the collection of all subsets $U \subseteq X$ such that $X - U$ is either countable, or all of X. This is also known as the cocountable topology.

Lemma 1.2.2 For any set X, $\mathcal{T}_f$ and $\mathcal{T}_c$ on X are indeed well-defined topologies.

Proof. Let us show that $\mathcal{T}_f$ is a topology. First we note that $\phi \in \mathcal{T}_f$ since $X - \phi$ is all of X. Additionally, we note $X - X = \phi$ which is finite. Thus $\mathcal{T}_f$ satisfies the first condition for being a topology.
Suppose $U_\alpha \in \mathcal{T}_f$ for α in index set Λ. Then

$$X - \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcap_{\alpha\in\Lambda} (X - U_\alpha)$$

If all $X - U_\alpha$ are just X then their intersection is also all of X, so $\bigcup_{\alpha\in\Lambda} U_\alpha$ would be in $\mathcal{T}_f$. If this is not the case, then at least one of the $X - U_\alpha$ must be finite so the above would have to be a subset of a finite set so must be finite itself. This implies the union is open in $\mathcal{T}_f$.
Suppose $U_i \in \mathcal{T}_f$ for $1 \le i \le n$ for some $n \in \mathbb{N}$. Then

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

If even one of $X - U_i$ is all of X then $U_i = \phi$ so the intersection is just empty which is indeed in $\mathcal{T}_f$ by above. If this is not the case, then all $X - U_i$ are finite so the above is a finite union of finite subsets so is itself finite, implying that $\bigcap_{i=1}^{n} U_i$ is in $\mathcal{T}_f$. Thus we conclude that $\mathcal{T}_f$ is a topology (the proof for $\mathcal{T}_c$ being a topology is entirely analogous). ■

Definition 1.2.4 — Comparing topologies. Let X be a set with topologies $\mathcal{T}$ and $\mathcal{T}'$. We say that $\mathcal{T}$ is *finer* than $\mathcal{T}'$ if $\mathcal{T} \supseteq \mathcal{T}'$. If $\mathcal{T} \subseteq \mathcal{T}'$, we say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

1.3 Bases and subbases

1.3.1 Bases

Definition 1.3.1 — Basis. Let X be a set. A *basis* $\mathcal{B}$ of subsets of X is a collections of subsets that satisfies:

- For every $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$
- If x is an element of $B_1 \cap B_2$ for two basis elements B_1, B_2 from $\mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_3$

Definition 1.3.2 — Topology defined by basis. Let X be a set and $\mathcal{B}$ be a basis. Then a set U is open in the topology generated by $\mathcal{B}$ if for every $x \in U$ there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Immediately, this means that every subset in a basis is open.

Lemma 1.3.1 The topology generated by a basis is indeed a topology.

Proof. Let $\mathcal{T}$ be the collection of sets that satisfy the above requirement. We will show that $\mathcal{T}$ is a topology.
The empty set ϕ is in $\mathcal{T}$ as it vacuously satisfies the statement. The set X itself also clearly satisfies the statement since every basis element is a subset of X and hence X is also in the topology. This means that $\mathcal{T}$ satisfies the first requirement for being a topology.
Let U_α for $\alpha \in \Lambda$ be a collection of sets in $\mathcal{T}$. Let x in $\bigcup_{\alpha\in\Lambda} U_\alpha$. Then x is an element of some U_α (at least one). By definition there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$ which means that the union is open in $\mathcal{T}$.
Let $U_1, U_2 \in \mathcal{T}$ and $x \in U_1 \cap U_2$. Then by definition there must exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. By the second property of basis, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. This implies that $U_1 \cap U_2$ is in $\mathcal{T}$. We can then use induction to prove that this holds for all finite intersections of open sets. ■

Lecture 1 content ends here.

■ **Example 1.2** Here, we look at examples of bases.

1. Show that the collection of all open disks in a plane forms a basis for the plane $X = \mathbb{R}^2$.
   - Let $x \in X$. Then, for any $r > 0$, we know that $x \in B(x,r) \in B$ where $B(x,r)$ is the (open) ball of radius r centered at x. In other words $B(x,r) = \{y \in \mathbb{R}^2 | d(x,y) < r\}$
   - For arbitrary $a_1, a_2 \in X, r_1, r_2 \in \mathbb{R}$, let $B_1 = B(a_1, r_1)$ and $B_2 = B(a_2, r_2)$ be the open disks of radius r_1 and r_2 respectively, centered at a_1 and a_2 respectively. Let $x \in B_1 \cap B_2$.
     We define $r = \min\{r_1 - d(x,a_1), r_2 - d(x,a_2)\}$. Let $B_3 = B(x,r)$. Clearly, $B_3 \in B$, by definition. We want to show that $B_3 \subseteq B_1 \cap B_2$.
     Let $y \in B_3$ be arbitrary. Then, we have:

$$\begin{aligned} d(y,a_1) &\le d(y,x) + d(x,a_1) \\ &\le r + d(x,a_1) \\ &= \min\{r_1 - d(x,a_1), r_2 - d(x,a_2)\} + d(x,a_1) \\ &\le r_1 \end{aligned}$$

Hence, $y \in B_1$ and by symmetry, $y \in B_2$.
2. The same approach shows that the collection of all rectangular regions in a plane forms a basis.

■

■ **Example 1.3** $B = \{\{x\} : x \in X\}$ a basis for the discrete topology $\mathcal{T}$ on a set X. ■

Lemma 1.3.2 Let X be a set, and let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collections of all unions of elements of $\mathcal{B}$.

Proof. Let X be a set, and let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$ on X. Let $\mathcal{T}'$ be the collection of all unions of elements of $\mathcal{B}$.

1. Let $U \in \mathcal{T}$. For any $x \in U$, denote by B_x the element in $\mathcal{B}$ such that $x \in B_x \subset U$. Hence, we note that $U = \bigcup_{x \in U} B_x \in \mathcal{T}'$. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$.
2. Let $U \in \mathcal{T}'$, which implies that we can write $U = \bigcup_{\alpha \in \Lambda} B_\alpha$ for some index set Λ where for all $\alpha \in \Lambda$, $B_\alpha \in \mathcal{B}$ is open with respect to $\mathcal{T}$. Thus, U is open with respect to $\mathcal{T}$ also, and so, $U \in \mathcal{T}$.

■

Lemma 1.3.3 Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for every open set $U \subseteq X$ and every $x \in U$, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then, $\mathcal{C}$ is a basis for $\mathcal{T}$.

Proof. We first show that $\mathcal{C}$ is indeed a basis. Clearly, the first requirement for being a basis is satisfied by $\mathcal{C}$. Now, let $x \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are both open with respect to $\mathcal{T}$, it follows that $C_1 \cap C_2$ is also open. Hence, by definition of $\mathcal{C}$, this implies the existence of some C_3 such that $x \in C_3 \subseteq C_1 \cap C_2$. This shows that $\mathcal{C}$ is indeed a basis.
Since $\mathcal{C}$ is a basis, it generates some topology $\mathcal{T}'$. We now show that $\mathcal{T}' = \mathcal{T}$.

1. To begin with, suppose that $U \in \mathcal{T}$ and let $x \in U$ be arbitrary. By definition, there exists some $C \in \mathcal{C}$ such that $x \in C \subseteq U$ and thus, $U \in \mathcal{T}'$.
2. Next, let $U \in \mathcal{T}'$. From the previous lemma, $U = \bigcup_{\alpha \in \Lambda} C_\alpha$. However the C_α are also open in $\mathcal{T}$ by assumption. Since the arbitrary union of open sets is open, $U \in \mathcal{T}$.

■

Lemma 1.3.4 Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}, \mathcal{T}'$, respectively, on the set X. Then, the following statements are equivalent.

1. $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. For each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. First, we prove that the first statement implies the second, and then we prove the converse.

1. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$, ie. $\mathcal{T} \subseteq \mathcal{T}'$. This implies that, if U is open with respect to $\mathcal{T}$, then it follows that U is also open with respect to $\mathcal{T}'$.
   Now, let $x \in X$ and $B \in \mathcal{B}$ be given. Since B is open with respect to $\mathcal{T}$, it is also open with respect to $\mathcal{T}'$. Since B is open in $\mathcal{T}'$ there must be a basis element in $\mathcal{B}'$ such that $x \in B' \subseteq B$.
2. Suppose that for each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.
   Now, let $U \in \mathcal{T}$ and $x \in U$ be arbitrary. By definition of $\mathcal{B}$, there exists some $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Furthermore, by the assumption, we know that there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B \subseteq U$. Thus, U is also open in $\mathcal{T}'$.

■

Lecture 2 content ends here. Lecture 3 content starts from the next definitions.

1.3.2 Sub-bases

Definition 1.3.3 — Sub-basis. A sub-basis $\mathcal{S}$ for a topology X is a collection of subsets of X whose union is equal to X. The topology generated by $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all possible finite intersections of elements of $\mathcal{S}$.

Lemma 1.3.5 The topology generated by a subbasis $\mathcal{S}$ is indeed a topology.

Proof. It suffices to prove that the collection of finite intersections of the elements of $\mathcal{S}$ forms a basis. Let us denote this collection as $\mathcal{B}$. Since the union of all elements of $\mathcal{S}$ is X, there is some $S \in \mathcal{S}$ such that $x \in S$. By definition $S \in \mathcal{B}$, hence the first condition for basis is satisfied. Now suppose $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$. We note that $B_1 \cap B_2 \in \mathcal{B}$ since B_1 and B_2 are both finite intersections of the elements of S and hence their intersection is also a finite intersection of elements of S. Hence there does indeed exist $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$, namely $B_1 \cap B_2$ itself. Thus $\mathcal{B}$, the collection of all finite intersections of elements of $\mathcal{S}$, is indeed a basis. ■

Exercise 1.1 Show that the topology generated by a subbasis $\mathcal{S}$ is equal to the intersection of all the topologies that contain $\mathcal{S}$. ■

R The exercise above shows that we can equivalently define the topology generated by a subbasis $\mathcal{S}$ to be the weakest/coarsest topology on X containing $\mathcal{S}$.

1.4 Topologies on $\mathbb{R}$

Definition 1.4.1 — Standard topology on the real line. This is the topology generated where we take the basis to be the collection of all open intervals on $\mathbb{R}$. The standard topology is usually simply denoted as $\mathbb{R}$.

Exercise 1.2 Show that the standard topology has a countable basis. ■

Definition 1.4.2 — Lower Limit Topology on the real line. This is the topology generated where we take the basis to be the collection of half-open intervals of the form $[a,b)$. The lower limit topology is usually denoted as $\mathbb{R}_l$.

Exercise 1.3 Show that the basic sets defined above for $\mathbb{R}_l$ are closed in $\mathbb{R}_l$. (See section 1.6).
(Harder) Exercise: Show that $\mathbb{R}_l$ does *not* have a countable basis (*Hint:* Show that for every $x \in \mathbb{R}$ there must exist a basis element B such that $x = \inf(B)$). ■

Definition 1.4.3 — K-topology on the real line. We define $K = \{\frac{1}{n} | n \in \mathbb{N}\}$. Then we take as basis all open intervals (a,b) as well as sets of the form $(a,b) - K$. The $K-$topology is usually denoted $\mathbb{R}_K$.

Lemma 1.4.1 The lower limit topology $\mathbb{R}_l$ and the $K-$topology $\mathbb{R}_K$ are both strictly finer than the standard topology but they are not comparable with one another.

Proof. Let $x \in \mathbb{R}$ be arbitrary and (a,b) be some interval containing x. Then $[x,b) \subset (a,b)$ is a basis element of $\mathbb{R}_l$ hence by Lemma 1.3.4 $\mathbb{R}_l$ is finer that $\mathbb{R}$. The proof is even simpler for $\mathbb{R}_K$ since all open intervals are defined to be open in $\mathbb{R}_K$ as well.
Now we show that $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable. Let $x \in \mathbb{R}$ be arbitrary and consider the basis element $[x,b)$ for some $b > x$. Then there is no basis element B in $\mathbb{R}_K$ such that $x \in B \subset [x,b)$. Thus $\mathbb{R}_K$ cannot be finer than $\mathbb{R}_l$. Conversely, consider some basis element of $\mathbb{R}_K$ around 0 of the form $(a,b) - K$. Then there is no basis element in $\mathbb{R}_l$ that contains 0 and is a subset of $(a,b) - K$. ■

1.5 Important topologies

1.5.1 Product topology

Definition 1.5.1 — Product Topology. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two topological spaces. The product topology on $X \times Y$ (usual Cartesian product) is topology generated by basis $\mathcal{B} = \{U \times V | U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$.

Lemma 1.5.1 Given two topological spaces $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$, the collection $\mathcal{B} = \{U \times V | U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$ is indeed a basis.

Proof. Let $(x,y) \in X \times Y$ be arbitrary. Then there is an open $U \subset X$ and open $V \subset Y$ such that $x \in U$ and $y \in V$. Therefore $(x,y) \in U \times V$. Now suppose $(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. We simply note, using elementary set theory, that $(U_1 \times V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. Since the finite intersection of open sets is open, the second requirement for a basis is also satisfied. ■

R It might be tempting to say that $\mathcal{B}$ defined above should be the topology on $X \times Y$ rather than simply a basis, however this collection fails to be a topology (for example, the union of two arbitrary rectangles is in general not a rectangle). Therefore we go for the next best thing.

Theorem 1.5.2 If $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ is a basis for the topology on Y, then the collection $\mathcal{D} = \{B \times C | B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.

Proof. Let (x,y) be some point in $X \times Y$. We know by definition of basis, that there is some $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B$ and $y \in C$ thus $(x,y) \in B \times C \in \mathcal{D}$.
Furthermore, let $(x,y) \in (B_1 \times C_1) \cap (B_2 \times C_2)$. Then $x \in B_1 \cap B_2$ and $y \in C_1 \cap C_2$. Then there are $B_3 \in \mathcal{B}$ and $C_3 \in \mathcal{C}$ such that $x \in B_3 \subseteq B_1 \cap B_2$ and $y \in C_3 \subseteq C_1 \cap C_2$. Thus $(x,y) \in B_3 \times C_3 \subseteq (B_1 \times C_1) \cap (B_2 \times C_2)$.
Now we need show that that this basis indeed generates the product topology. Using Lemma 1.3.3, all we need show is that for every open set U in the product topology and $x \in U$, we have set $D \in \mathcal{D}$ such that $x \in D \subseteq U$. By Lemma 1.3.2, we know that all open sets are just unions of elements of basis, so without loss of generality, we can assume that the open set U is of the form $V \times W$ where V is open in X and W is open in Y. Choosing any $(x,y) \in V \times W$ we know there is a basis element $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset V$ and $y \in C \subset \mathcal{C}$. Thus $(x,y) \in B \times C \subseteq V \times W = U$.

■

Definition 1.5.2 — Projections. Given the product of two spaces $X \times Y$, there are two very natural maps onto the individual spaces. We define $\pi_1 : X \times Y \to X$ where $\pi_1(x,y) = x$. Similarly we have $\pi_2 : X \times Y \to Y, \pi_2(x,y) = y$. These maps are called projections.

Theorem 1.5.3 The collection $\mathcal{S} = \{\pi_1^{-1}(U) | U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) | V \in \mathcal{T}_Y\}$ is a subbasis for the product topology on $X \times Y$.

Proof. Let $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$ which is a basis element for the product topology. ■

Exercise 1.4 Show that $\Delta = \{(x,y) \in \mathbb{R}^2 | x + y = 0\}$ is closed in $\mathbb{R}_l \times \mathbb{R}_l$ (endowed with the product topology of course). ■

1.5.2 Order topology

Definition 1.5.3 — Order Topology. Suppose X is a set with an order relation. Then there is a topology that is induced by this order relation. We define the basis to be the collection of open intervals $(a,b) := \{x \in X : a < x < b\}$ for all $a, b \in X$ such that $a < b$. Alternatively we can take as subbasis the collection of all open rays $(b, \infty) := \{x \in X : x > b\}$ and $(-\infty, b) = \{x \in X : x < b\}$ for $b \in X$.

1.5.3 Subspace topology

Definition 1.5.4 — Subspace Topology. If $(X, \mathcal{T})$ is a topological space and we have $Y \subset X$, the $\mathcal{T}$ induces a topology on Y. Namely, we consider $(Y, \mathcal{T}_Y)$ as a topological space where $\mathcal{T}_Y = \{U \cap Y | U \in \mathcal{T}\}$. If $Y \subset X$ is endowed with the subspace topology, we say Y is a subspace of X.

Exercise 1.5 Consider $\mathbb{R}$ with the standard topology. Show that $\mathbb{N} \subset \mathbb{R}$ as a subspace has the discrete topology. Does the same hold true for $\mathbb{Q} \subset \mathbb{R}$? ■

1.6 Closed sets

Definition 1.6.1 — Closed sets. Given a topological space X, we say $D \subset X$ is closed if $X - D$ is open

■ **Example 1.4** The closed intervals $[a,b]$ are closed in the standard topology. ■

■ **Example 1.5** Finite sets are closed in the finite complement topology on any set. ■

■ **Example 1.6** All subsets are closed in the discrete topology. ■

(R) As the final example shows, being closed does not prevent a set from being open.

Proposition 1.6.1 Let X be a topological space. Then the following statements hold:

1. ϕ, X are closed
2. Finite unions of closed sets is closed
3. Arbitrary intersections of closed sets is closed

Proof. This is just an exercise in applying De Morgan's laws to the properties of open sets in 1.1. ■

Theorem 1.6.2 Let Y be a subspace of X. Then a set A is closed in Y if and only if there is a closed set F in X such that $A = F \cap Y$.

Proof. Suppose A is closed in Y. Then $Y - A$ is open in Y. Hence there exists a $U \subset X$ open such that $Y - A = U \cap Y$. Then $X - U$ is closed and $A = (X - U) \cap Y$.
Conversely suppose F is closed in X and $A = F \cap Y$. We wish to show that A is closed in Y or in other words that $Y - A$ is open in Y. This is true because $Y - A = (X - F) \cap Y$ which is open by definition. ■

Theorem 1.6.3 Let Y be a subspace of X. If A is closed in Y and Y is closed in X then A is closed in X.

Proof. Let $A \subset Y$ be closed in Y. Then there exists $F \subset X$ closed in X such that $A = F \cap Y$. Since F and Y are closed in X, $F \cap Y = A$ is closed in X. ■

Definition 1.6.2 — Interior & Closure. Let A be a subset of X. The union of all open sets contained in A is the *interior* of A. This is usually denoted $\text{int}(A)$. The intersection of all the closed sets that contain X is the *closure* of A, which is usually denoted $\overline{A}$.

Properties:

- Interior of a set is always open
- Closure of a set is always closed
- $\text{int}(A) \subset A$
- $A \subset \overline{A}$
- A is open if and only if $\text{int}(A) = A$
- A is closed if and only if $A = \overline{A}$

(R) If $Y \subset X$ and $A \subset Y$, then the closure of A with respect to Y is not necessarily the same as the closure of A with respect to X.

The above remark tells us that closures depends on the space they sit in. This of course makes sense since closed sets in subspaces are not the same as closed sets in the larger space. However the following theorem tells us that the second natural statement we would want for closures in subspaces does hold.

Theorem 1.6.4 Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ be the closure of A in X. Then $\overline{A} = \overline{A} \cap Y$.

Proof. Proof by "seems correct enough". TODO ■

Theorem 1.6.5 Let A be a subspace of a topological space X. Then

1. $x \in \overline{A}$ if and only if every open set containing x intersects A
2. If the topology is generated by a basis $\mathcal{B}$, then $x \in \overline{A}$ if and only if every basis element containing x intersects A.

Proof.

1. We will show the converse of the statement. Namely that $x \notin \overline{A}$ if and only if there is an open set containing x that does not intersect A.
Suppose $x \notin \overline{A}$. Then $x \in X - \overline{A}$ which is open and cannot intersect A since $A \subset \overline{A}$ so we are done. On the other hand let $x \in X$ be such that there is an open set U containing x that does not

intersect A. Therefore $X-U$ is closed set that contains A. Since $x \notin X-U$, it cannot be in the closure which is defined to the intersection of all closed sets that contain A.

2. Suppose $x \in \overline{A}$. Then by above, every open sets containing x intersects A. Since basic sets are open, this also holds for all basic sets that contain x. Conversely, let $x \in X$ be such that every basis element containing x intersects A. Let U be any open set containing x. There is then a basic set B that contains x and is contained in U. By assumption B intersects A, therefore U intersects A. As this holds for all U, $x \in \overline{A}$ by above.

■

■ **Example 1.7** $X = \mathbb{R}$ and $A = (0,1], B = \{\frac{1}{n} | n \in \mathbb{N}\}, C = \mathbb{Q}$. Then $\overline{A} = [0,1], \overline{B} = B \cup \{0\}, \overline{C} = \mathbb{R}$. ■

1.7 Limit Points

Definition 1.7.1 Let X be a topological space. Let $A \subset X$ and $x \in X$. Then x is a limit point of A if every neighbourhood of x intersects A in some point other than x. In other words, x is a limit point of A if $x \in \overline{A - \{x\}}$. The set of limit points of a set A is often denoted A'.

■ **Example 1.8** If $X = \mathbb{R}, A = (0,1)$ then $A' = [0,1]$ ■

■ **Example 1.9** If $X = \mathbb{R}, B = \{\frac{1}{n} | n \in \mathbb{N}\}$, then $B' = \{0\}$ ■

Exercise 1.6 It might seem as if $x \in \overline{A}$ means that x *must* be a limit point of A. Give an example where this is not true. In other words, give an example where $x \in \overline{A}$ but $x \notin A'$ ■

Exercise 1.7 Prove or disprove: any subset of a topological space has limit points. ■

Theorem 1.7.1 Let A be a subset of a topological space X. Then

$$\overline{A} = A \cup A'$$

Proof. We will first show that $\overline{A} \subset A \cup A'$ and then show the reverse inclusion holds.
Let $x \in \overline{A}$. If $x \in A$, we are done. So suppose $x \notin A$. Then every neighbourhood of x intersects A and since $x \notin A$, it must intersect at a point different from x. Thus x is a limit point of A.
Conversely suppose $x \in A \cup A'$. If $x \in A$, then $x \in \overline{A}$ since $A \subset \overline{A}$. If $x \in A'$ then every neighbourhood of x intersects A at a point different than x. In particular, every neighbourhood of x intersects A hence $x \in \overline{A}$ by Theorem 1.6.5. ■

1.8 Hausdorff Spaces

Definition 1.8.1 — Hausdorff Spaces. A topological space X is Hausdorff if for every pair of distinct $x, y \in X$ we have open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Theorem 1.8.1 Every finite point set is closed in a Hausdorff space.

Proof. It is sufficient to show that the singleton sets are open since the finite union of closed sets is closed (see Proposition 1.6.1).

Suppose X is a Hausdorff space and let $x \in X$ be arbitrary. For every $y \in X - \{x\}$, we have an open set U_y that does not contain x by Hausdorff hence is contained in $X - \{x\}$. Therefore $X - \{x\}$ is open so $\{x\}$ is closed. ■

Theorem 1.8.2 X is Hausdorff if and only if $\Delta = \{(x,x) | x \in x\} \subset X \times X$ is closed in $X \times X$.

Proof. Suppose X is Hausdorff. We will show that $X \times X - \Delta$ is open. So let $(x,y) \in X \times X - \Delta$ be arbitrary. Note x and y must be distinct so by Hausdorff there exist open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$. Then $(x,y) \in U \times V \subset X \times X - \Delta$. Hence the complement of Δ is open so Δ itself is closed.
Suppose $\Delta \subset X \times X$ is closed for some topological space X. Let x,y be distinct elements in X. Then $(x,y) \in X \times X - \Delta$. Since the complement of Δ is open, we can find a basic set $U \times V \subset X \times X - \Delta$ such that $x \in U \times V$ where U and V are open in X. Then $x \in U$ and $y \in V$ and $U \cap V$ must be empty since if it wasn't, their product would intersect the diagonal. ■

Exercise 1.8 Show that if X is Hausdorff then so is $X \times X$. ■

1.9 Separation Axioms

We want to be able to distinguish points in our topologies (consider how in the trivial topology all points 'look' the same). Hausdorff spaces give us one such way of doing this, however there are also other axioms we could have, some stronger and some weaker. One such axiom is the T_1 axiom.

Definition 1.9.1 — T_1 axiom. A topological space satisfied the T_1 axiom if for every pair of distinct points $a,b \in X$, we have open sets O_a and O_b such that $a \in O_a$ and $b \in O_b$ but $a \notin O_b$ and $b \notin O_a$.

The T_1 axiom is equivalent to saying that singleton sets are closed. When proving that singletons are closed in Hausdorff spaces, we in fact only used the T_1 property. Hausdorff itself is the T_2 axiom. Note that clearly Hausdorff spaces satisfy the T_1 axiom.

Exercise 1.9 Show that if X is a T_1 space, then singletons are closed in X. What does this say about topologies on finite sets that satisfy the T_1 axiom? ■

Theorem 1.9.1 Let X be a space satisfying the T_1- axiom. Let A be a subspace of X. Then x is limit point of A if and only if every neighbourhood of x contains infinitely many points from A.

Proof. Let x be a limit point of A and U be some neighbourhood of x. Then by definition of limit point, there exists some $x_1 \in U \cap (A - \{x\})$. Note that $U - \{x_1\} = U \cap (X - \{x_1\})$ which is open since singleton sets are closed in T_1 spaces. Hence $U - \{x_1\}$ is another open neighbourhood of x and so there must exists some x_2 in $U - \{x, x_1\}$. We then simply proceed inductively. ■

R It seems like we are implicitly assuming that X is an infinite set. In fact the theorem is also true for finite sets. The exercise above should help explain why. (*Hint:* this is something of a trick question).

Exercise 1.10 Give an example of a space that satisfies the T_1 axiom but is not Hausdorff. ■

Section 2 Continuity and Metrizability

2.1 Continuity

2.1.1 Sequences

Although there is little to say about sequences at the moment, they certainly play an incredibly vital role in topology (we may see later how sequences provide an alternate way of thinking about open and closed sets, given sufficiently nice conditions on the topology). For now we limit ourselves to definitions and simple theorems.

Definition 2.1.1 — Sequence. A sequence of points in X is a map $f : \mathbb{N} \to X$. We often denote sequences as $(x_n)_{n\in\mathbb{N}}$, where $x_n = f(n)$.

Definition 2.1.2 — Convergence. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to $x \in X$ if given any open neighbourhood U of x, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in U$.

Note that with our simple definition of topology and our definition above, a sequence can in fact converge to more than one point. Consider any sequence on a set X endowed with the trivial topology. Then any sequence will converge to every point in X. This is of course not ideal, which is where our separation axioms come in.

Theorem 2.1.1 If X is a Hausdorff space, then a sequence of points of of X converges to at most one point of X.

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$. Let $y \neq x$ be arbitrary. We wish to show that the sequence cannot converge to y.
Since X is Hausdorff and x and y are distinct points, we know there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x, there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in U$. Since U and V are disjoint this means that for $n \geq N$, $x_n \notin V$. Thus $(x_n)_{n\in\mathbb{N}}$ cannot converge to y. ■

Exercise 2.1 Show that the T_1 axiom is not enough to guarantee uniqueness of limits. ■

2.1.2 Continuous Functions

Definition 2.1.3 — Continuous Functions. Let X, Y be topological spaces. A function $f : X \to Y$ is continuous if for every open set $V \subset Y$, we have that $f^{-1}(V)$ is open in X.

■ **Example 2.1** Consider the identity function $i : \mathbb{R} \to \mathbb{R}_l$ (where the domain is $\mathbb{R}$ with the standard topology). Note that this function is in fact *not* continuous since $i^{-1}([0,1)) = [0,1)$ which is not open in the standard topology on $\mathbb{R}$. ■

We have a few equivalent characterisations of continuous functions that can sometimes be more useful.

Theorem 2.1.2 Let X and Y be topological spaces and let $f : X \to Y$ be a map. Then the following are equivalent:

1. f is continuous (i.e. preimages of open sets are open)
2. $\forall A \subset X$, we have $f(\overline{A}) = \overline{f(A)}$
3. $\forall B \subset Y$, where B is closed, $f^{-1}(B)$ is closed
4. $\forall x \in X$ and for every neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subset V$

Proof. $(1 \Rightarrow 2)$ Let $x \in \overline{A}$. Then we need to show that $f(x) \in \overline{f(A)}$. Let U be an open neighbourhood of $f(x)$. Then $f^{-1}(U)$ is an open neighbourhood of x by continuity of f. Since $x \in \overline{A}$, we know that $f^{-1}(U) \cap A$ is non-empty. Let y be an element in this intersection. Then $f(y) \in U \cap f(A)$ implying $f(A) \cap U$ is non-empty. Hence $f(x) \in \overline{f(A)}$.
$(2 \Rightarrow 3)$ Let B be closed in Y and let $A = f^{-1}(B)$. We will show that $\overline{A} \subset A$ which will allow us to conclude that $A = \overline{A}$ hence it is closed. Let $x \in \overline{A}$. Then

$$f(x) \in f(\overline{A}) = \overline{f(A)} = \overline{f(f^{-1}(B)} \subset \overline{B} = B$$

Since $f(x) \in B$, this means that $x \in f^{-1}(B) = A$.
$(3 \Rightarrow 1)$ Let U be some open set in Y. Then $Y - U$ is closed so $f^{-1}(Y - U) = f^{-1}(Y) - f^{-1}(U)$ is closed. Since $f^{-1}(Y) = X$, $f^{-1}(U)$ is open.
$(1 \Rightarrow 4)$ Let $x \in X$ be arbitrary and let V be some neighbourhood of $f(x)$. Then we can just take $U = f^{-1}(V)$.
$(4 \Rightarrow 1)$ Let V be some open set in Y. Then for every $x \in f^{-1}(V)$ we have an open neighbourhood U_x of x such that $f(U_x) \subset V$. Then

$$U = \bigcup_{x \in f^{-1}(V)} U_x$$

■

Definition 2.1.4 — Homeomorphism. Let X, Y be topological spaces. We say that $f : X \to Y$ is a *homeomorphism* if

1. f is bijective
2. f is continuous
3. f^{-1} is continuous

We certainly require f^{-1} to be continuous. Consider for example the identity map from $\mathbb{R}_l$ to $\mathbb{R}$, which, as we have seen before, does not have a continuous inverse.
For topologists, if there is a homeomorphism between two spaces, then the two spaces are essentially the same. Note by definition, homeomorphisms provide a $1-1$ correspondence between the open sets in the two spaces. However the data for a topology is exactly the collection of all open sets. So the existence of a homeomorphism tells us that we can go from one topological space to another by simply 'relabeling' the elements in some way.

Exercise 2.2 Explain why there can never be a homeomorphism between $\mathbb{R}$ and $\mathbb{R}_l$ or between $\mathbb{R}$ and $\mathbb{R}_K$. ■

Exercise 2.3 Show that any two circles in the plane are homeomorphic (with the induced topology from the standard topology on $\mathbb{R}^2$ of course). ■

Definition 2.1.5 — Topological Embedding. Suppose we have $f : X \to Y$ injective and continuous. Consider $Z = f(X)$ as a subspace (i.e. endowed with the subspace topology). If $f : X \to Z$ is a homeomorphism, then f is an embedding.

Exercise 2.4 (**Hard**) Let $\mathbb{T}$ denote the unit circle in the real plane, i.e. $\mathbb{T} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Show that there is an embedding of $\mathbb{T}^2$ in $\mathbb{R}^3$ (*Hint*: To every point on a circle, you are associating another circle. Think geometrically about what such a shape would look like. If you're getting tired of thinking, go grab a snack!). ■

Properties of continuous functions

Let X, Y, Z be topological spaces.

1. Constant functions, $f : X \to Y, f(x) = y_0$ for all $x \in X$, are always continuous
2. If $A \subset X$ is a subspace, then inclusion map from A to X given by $a \mapsto a$ is continuous (and injective)
3. If $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is continuous
4. If $f : X \to Y$ is continuous and $A \subset X$ is a subspace then $f|_A$ is continuous
5. Let $f : X \to Y$ be a map and $\{U_\alpha\}$ be a collection of sets such that $X = \bigcup_\alpha U_\alpha$. If $f|_{U_\alpha}$ is continuous for each α then f is continuous.

Theorem 2.1.3 — The Pasting Lemma. Let X be a topological space and $A, B \subset X$ closed such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions to another topological space Y. Suppose additionally that $f(x) = g(x)$ for all $x \in A \cap B$. Then $h : X \to Y$, given by

$$h(x) = \begin{cases} f(x), x \in A \\ g(x), x \in B \end{cases}$$

is well-defined and continuous.

Proof. The fact h is well-defined follows from f and g agreeing on $A \cap B$. We will show that h is continuous by showing that the preimages of closed sets are closed.

Let C be some closed set of Y. Then $h^{-1}(C) = (f^{-1}(C) \cap A) \cup (g^{-1}(C) \cap B)$ where $f^{-1}(C)$ is closed in A (with the subspace topology) and $g^{-1}(C)$ is closed in B (again in the subspace topology). Then by Theorem 1.6.3, $h^{-1}(C)$ is closed in X. ■

2.2 Product Topology

We have already looked at taking finitely many products of topological spaces and feel reasonably confident in working with them. Now we wish to generalise this to being able to take infinitely many products (both countable and uncountable).

Recall that we have two different definitions for the product topology in the finite case. We can consider the collection of all products of open sets as basis or we can consider the collection inverse images of open sets under projection maps as a subbasis. Although both of these generate the same topology in the finite product case, this does not remain true when we take infinite products.

Definition 2.2.1 — $j-$tuple. Let J be any index set. Then we can define a $j-$tuple of elements of X as a function $x : J \to X$. We often denote $x(\alpha)$, for some $\alpha \in J$, as x_α and write the $j-$tuple as $(x_\alpha)_{\alpha \in J}$.

Definition 2.2.2 Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Then let $X = \bigcup_{\alpha \in J} A_\alpha$. We define the Cartesian product as

$$\prod_{\alpha \in J} A_\alpha = \{x : J \to X | x(\alpha) \in A_\alpha\} = \{(x_\alpha)_{\alpha \in J} | x_\alpha \in A_\alpha\}$$

Now that we have some idea of what it means to 'multiply' infinitely, we can generalise our previous definitions for the product topology.

R It might seem a bit strange to consider such spaces but what we are doing here is studying the so called 'function spaces' (since the elements are functions) which is an incredibly important subject, for example in the study of differential equations.

Definition 2.2.3 — Box topology. Suppose $\{X_\alpha\}$ is an indexed family of topological spaces. Let

$$\mathcal{B} = \left\{ \prod_\alpha U_\alpha : U_\alpha \text{ is open in } X_\alpha \right\}$$

Then $\mathcal{B}$ is a basis and the topology it generates is called the box topology.

Definition 2.2.4 — Product topology. Suppose $\{X_\alpha\}_{\alpha \in J}$ is an indexed family of topological spaces. For $\beta \in J$, define

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\}$$

Then the topology generated by $\{\mathcal{S}_\beta\}_{\beta \in J}$ as a subbasis is called the product topology.

Let us take a moment to consider what the basis for the product topology looks like, which will also make it clear why the two topologies are different. In order to get a basis from a subbasis, we simply take all possible finite intersections of the subbasis elements. Thus we can express a basis element B of the product topology as

$$B = \prod_{\alpha \in J} V_\alpha$$

where for all but finitely many $\alpha \in J$, we have $V_\alpha = X_\alpha$. From this perspective, it is clear that the box topology is in fact finer than the product topology. While it may seems like we would want the finer topology (since it contains the other one), it turns out the box topology is in fact a bit too fine for our tastes (think about how little use the discrete topology gets despite being the finest possible topology). The exercise below might help illustrate why the increased fineness of the box topology is not all that nice.

Exercise 2.5 Let $X = \prod_{\alpha \in J} X_\alpha$. Show that a sequence $(x_n)_{n \in \mathbb{N}}$ converges in the product topology if and only if $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges for each $\alpha \in J$. Show that the same does not hold true for the box topology. ■

The two spaces aren't *entirely* dissimilar, however. Below we note some of the things they share in common.

Theorem 2.2.1 Let A_α be a subspace for X_α for every $\alpha \in J$. Then $\prod_{\alpha\in J} A_\alpha$ is a subspace of $\prod_{\alpha\in J} X_\alpha$ in both the product and box topology.

Theorem 2.2.2 If each X_α is Hausdorff then, $\prod_\alpha X_\alpha$ is Hausdorff in both the product and box topology.

Proof. Let $(x_\alpha), (y_\alpha)$ be distinct points in $\prod X_\alpha$. There exists some $\tilde{\alpha}$ such that $x_{\tilde{\alpha}} \neq y_{\tilde{\alpha}}$. Since $X_{\tilde{\alpha}}$ is Hausdorff, there exist disjoint open neighbourhoods U, V of $x_{\tilde{\alpha}}$ and $y_{\tilde{\alpha}}$ respectively. Then $\pi_{\tilde{\alpha}}^{-1}(U)$ and $\pi_{\tilde{\alpha}}^{-1}(V)$ are disjoint neighbourhoods (in either topology) of (x_α) and (y_α) respectively. ■

Theorem 2.2.3 Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Suppose for each $\alpha \in J$, we have A_α as a subset of X_α. Then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Proof. As is typical, we will show that we have inclusion in both directions.
Let $x \in \prod \overline{A_\alpha}$ and let $U \subset \prod X_\alpha$ be a neighbourhood of x. For every α, we know that $\pi_\alpha(U)$ intersects A_α (since projections maps open sets to open sets in both topologies). Thus we can choose a y_α from each $\pi_\alpha(U) \cap A_\alpha$ to see that $y = (y_\alpha) \in U \cap \prod A_\alpha$. Therefore $x \in \overline{\prod A_\alpha}$.
Now suppose $x \in \overline{\prod A_\alpha}$. Let U_α be a neighbourhood of x_α. We know that $\pi_\alpha^{-1}(U_\alpha)$ intersects $\prod A_\alpha$. Thus $U_\alpha \cap A_\alpha$ is non-empty. ■

Given any map $f : A \to \prod X_\alpha$, we can express this map in terms of its 'coordinate' functions. Namely $f(a) = (f_\alpha(a))$ where each f_α is a map from A to X_α. This leads us to (one of) the properties that makes the product topology so appealing.

Theorem 2.2.4 Let $f : A \to \prod X_\alpha$ be given. Then f is continuous if and only if each $f_\alpha : A \to X_\alpha$ is continuous in the product topology.

Proof. Suppose f is continuous. Then $f_\alpha = \pi_\alpha \circ f$. Since the composition of continuous maps is continuous, f_α is continuous.
Now suppose that each f_α is continuous. In order to check continuity of f, it is sufficient to check that the preimages of open sets is open. In fact, it suffices to show that the preimage of subbasis elements is open since for *any* function $g : A \to B$ and $C, D \subset B$, we have $g^{-1}(C \cap D) = g^{-1}(C) \cap g^{-1}(D)$ (this has nothing to do with continuity or topologies at all. This is a purely set-theoretic statement).
Let $\pi_\alpha^{-1}(U_\alpha)$ be a subbasis element of the product topology. Then $f^{-1}(\pi^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f_\alpha^{-1}(U_\alpha)$. ■

The above does not hold for the box topology. Consider for example $f : \mathbb{R} \to \mathbb{R}^\omega$ (where $\mathbb{R}^\omega = \prod_{n\in\mathbb{N}} \mathbb{R}$ given by $f(t) = (t, t, t, \ldots)$. Clearly each f_i is continuous so f is continuous if we endow $\mathbb{R}^\omega$ with the product topology. However this does not hold for the box topology. Consider, for example, $B = \prod_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n})$ which is open in the box topology. However $f^{-1}(B) = \{0\}$ which is not open in $\mathbb{R}$.

2.3 Metric Topology

Definition 2.3.1 — Metric. Given a set X, a metric is a function $d : X \times X \to \mathbb{R}$ such that
1. (Positive definite) $d(x,y) \geq 0$ where we have equality if and only if $x = y$
2. (Symmetric) $d(x,y) = d(y,x)$
3. (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$

This allows us to define the notion of open balls on a set. Namely, we get $B_d(x,\varepsilon) = \{y \in X | d(x,y) < \varepsilon\}$.

Definition 2.3.2 — Metric Topology. The topology generated by considering the collection of all open balls as a basis is called the metric topology.

■ **Example 2.2** On any set X we can define $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. This generates the discrete topology ■

■ **Example 2.3** $\mathbb{R}$ with the standard topology is also generated by a metric, $d(x,y) = |x-y|$. ■

Definition 2.3.3 — Metrizability. A topological space X is said to metrizable if there exists a metric d that induces the topology on X.

Definition 2.3.4 — Bounded Sets. Given a metric space (X,d), we say $A \subset X$ is bounded if there exists some $M \in \mathbb{R}$ such that $d(x_1,x_2) \leq M$ for all $x_1, x_2 \in M$.

Although metric are very useful, one should keep in mind what properties arise from a metric and distinguish them from the properties of the topology. Boundedness, for example, is a property of the metric rather than the topology.

Proposition 2.3.1 Let (X,d) be a metric space. Then there exists a metric $\overline{d}$ that induces the topology on X and relative to which every subset of X is bounded.

Proof. We define

$$\overline{d}(x,y) = \min\{d(x,y), 1\}$$

and show that it is indeed a metric.
Symmetry and positive definiteness are clear. All that remains is to check whether $\overline{d}$ satisfies the triangle inequality. In particular, we wish to show that

$$\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$$

for all $x,y,z \in X$. Thus let us take x,y,z to be arbitrary points of X. If $d(x,y) \geq 1$ or $d(y,z) \geq 1$ we are done since $\overline{d}(x,z)$ is at most 1. So suppose $d(x,y) < 1$ and $d(y,z) < 1$. Then

$$\overline{d}(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \overline{d}(x,y) + \overline{d}(y,z)$$

Hence $\overline{d}$ does indeed satisfy the triangle inequality.
The topology induced is the same the "small balls" in both metrics are the same (i.e. those with radius less than 1). ■

Lemma 2.3.2 — Comparing metric topologies. Let d, d' be two metrics on X that generate topologies $\mathcal{T}, \mathcal{T}'$ respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$.

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $B_d(x,\varepsilon)$ be an open ball in $\mathcal{T}$. We wish to show there exists some $\delta > 0$ such that $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$. Since $\mathcal{T}'$ is finer, we know by Lemma 1.3.4 that there exists some $y \in X$ and $\delta' > 0$ such that $B_{d'}(y,\delta') \subset B_d(x,\varepsilon)$. Then we simply take $\delta = \delta' - d'(x,y)$
Now suppose the given condition holds. Let $B_d(x,\varepsilon)$ be arbitrary and let y be some point in this ball. Take $\varepsilon' = \varepsilon - d(x,y)$. Then $B_d(y,\varepsilon') \subset B_d(x,\varepsilon)$. By assumption, there exists some $\delta > 0$ such that $B_{d'}(y,\delta) \subset B_d(y,\varepsilon') \subset B_d(x,\varepsilon)$. Once again we invoke Lemma 1.3.4 to complete the proof. ■

2.3.1 Uniform Metric

Definition 2.3.5 Given $x,y \in \mathbb{R}^J$, with $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha\in J}$ we define,

$$\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha,y_\alpha) : \alpha \in J\}$$

This is known as as the uniform metric.

Although the name is rather suggestive, we should confirm that $\overline{\rho}$ is indeed a metric.

Proposition 2.3.3 $\overline{\rho}$ is a metric on $\mathbb{R}^J$.

Proof. Positive definiteness and symmetry are clear. All that remains to check is that it satisfies the triangle inequality. Let $(x_\alpha)_{\alpha\in J}, (y_\alpha)_{\alpha\in J}, (z_\alpha)_{\alpha\in J}$ be points in $\mathbb{R}^J$. We know that for any given α, we have

$$\overline{d}(x_\alpha,z_\alpha) \le \overline{d}(x_\alpha,y_\alpha) + \overline{d}(y_\alpha,z_\alpha) \le \overline{\rho}(x,y) + \overline{\rho}(y,z)$$

Thus $\overline{\rho}(x,y) + \overline{\rho}(y,z)$ is an upper bound for $\overline{d}(x_\alpha,z_\alpha)$ as we vary α implying that

$$\overline{\rho}(x,z) \le \overline{\rho}(x,y) + \overline{\rho}(y,z)$$

as desired. ■

As with any topology, it is useful to know what the basis sets looks like. In this case, we wish to study what the open balls in the uniform metric look like.

Proposition 2.3.4 If we have $\mathbb{R}^J$, where J is any indexing set then

$$B_{\overline{\rho}}(x,\varepsilon) = \bigcup_{\delta<\varepsilon} \prod_{\alpha\in J} (x_\alpha - \delta, x_\alpha + \delta)$$

Proof. Suppose we have $y \in \bigcup_{\delta<\varepsilon} \prod_{\alpha\in J}(x_\alpha - \delta, x_\alpha + \delta)$. Then $y \in \prod_{\alpha\in J}(x_\alpha - \delta, x_\alpha + \delta)$ for some $\delta < \varepsilon$. This means that $|x_\alpha - y_\alpha|$ for any α is less than δ. Thus $\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha,y_\alpha) : \alpha \in J\} \le \delta < \varepsilon$. Hence $y \in B_{\overline{\rho}}(x,\varepsilon)$.
On the other hand, suppose $y \in B_{\overline{\rho}}(\varepsilon)$. We define $\delta = \sup\{\overline{d}(x_\alpha,y_\alpha) : \alpha \in J\}$ which we know is less than ε. Then we know that $y \in \prod_{\alpha\in J}(x_\alpha - \delta, x_\alpha + \delta)$. ■

Now that we have topology when taking products (at least those of $\mathbb{R}$), we should compare it with our previous topologies.

Theorem 2.3.5 The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. These topologies are all different when J is infinite.

Proof. Let B be a basis element of the product topology containing a point $x = (x_\alpha)_{\alpha\in J}$ We recall that $B = \prod_{\alpha\in J} U_\alpha$ where all but finitely many $U_\alpha = \mathbb{R}$ and the remaining are open sets of $\mathbb{R}$. Let $\alpha_1, \dots, \alpha_n$ be the index values where $U_{\alpha_i} \neq \mathbb{R}$. Since these are open, we can find ε_i for each α_i such that $B_{\overline{d}}(x_{\alpha_i}, \varepsilon_i) \subset U_{\alpha_i}$. We take $\varepsilon = \min\{\varepsilon_i\}$. Then it holds that $B_{\overline{\rho}}(x, \varepsilon) \subset B$. It is also clear that the two must be unequal when J is infinite.
Now we show that the box topology is finer than the uniform topology. Let $B_{\overline{\rho}}(x, \varepsilon)$ be an open ball in the uniform topology and $y = (y_\alpha)_{\alpha\in J}$ be an element of it. Then by the previous proposition, $y \in \prod_{\alpha\in J}(x_\alpha - \delta, x_\alpha + \delta)$ for some $\delta < \varepsilon$. As this set is open in the box topology we are done.
In order to see that the inequality is strict in the infinite case, we simply note that

$$B = \prod_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

cannot be open in the uniform topology. ■

Theorem 2.3.6 Let X, Y be metrizable spaces with metrics d_X and d_Y and $f : X \to Y$ be a map. Then f is continuous if and only if for every $x \in X$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \varepsilon$.

Lemma 2.3.7 — The Sequence Lemma. Let X be a topological space and let $A \subset X$. if there is a sequence of points of A that converge of x, then $x \in \overline{A}$. The converse holds if X metrizable.

Proof. Suppose x_n is a sequence of points in A that converges to some $x \in X$. Let U be some neighbourhood of x. Then we know that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$. Thus $U \cap A$ is non-empty so $x \in \overline{A}$.
Now suppose X is metrizable and $x \in \overline{A}$. For every n, we choose a point from $A \cap B(x, \frac{1}{n})$ which we know to be non-empty. This gives the desired sequence. ■

R The converse holds if we loosen the restriction and simply require X to be *first countable*. We will see what this means later.

The sequence lemma is useful for deciding what spaces are not metrizable, by using the contrapositive. We see an example below.

Proposition 2.3.8 $\mathbb{R}^\omega$ with the box topology is not metrizable.

Proof. We will show that $\mathbb{R}^\omega$ does not satisfy the sequence lemma. Let $A = \{(x_1, x_2, x_3, \dots : \forall i. x_i > 0\}$. It is clear that $0 = (0, 0, 0, \dots) \in \overline{A}$. However no sequence of elements in A converges to 0. Suppose we have a sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = (a_{1n}, a_{2n}, a_{3n}, \dots)$ converges to 0. Then we take $U = (-a_{11}, a_{11}) \times (-a_{22}, a_{22}) \times \dots$ which is a neighbourhood of 0 but by construction contains none of the elements of the sequence. ■

Theorem 2.3.9 $\mathbb{R}^\omega$ with the product topology is metrizable with

$$D(x,y) = \sup\left\{\frac{\overline{d}(x_i,y_i)}{i} : i \in \mathbb{N}\right\}$$

Proof. The fact that D is positive-definite and symmetric is follows from $\overline{d}$ being positive-definite and symmetric. Then we only need show that it satisfies the triangle inequality. Let $x,y,z \in \mathbb{R}^\omega$. Then we wish to show that

$$D(x,z) \leq D(x,y) + D(y,z)$$

We know for any $i \in \mathbb{N}$ that

$$\frac{\overline{d}(x_i,z_i)}{i} \leq \frac{\overline{d}(x_i,y_i)}{i} + \frac{\overline{d}(y_i,z_i)}{i} \leq D(x,y) + D(y,z)$$

Thus $D(x,y)+D(y,z)$ is an upper bound for all $\frac{\overline{d}(x_i,z_i)}{i}$ hence $D(x,z) \leq D(x,y)+D(y,z)$.
Now we show that this metric does indeed generate the product topology. Let $B_D(x,\varepsilon)$ be an open ball in $\mathbb{R}^\omega$. We know there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then we define

$$U = \prod_{i=1}^{\infty} U_i$$

where

$$U_i = \begin{cases} B_{\overline{d}}(x, i\varepsilon), & i < N \\ \mathbb{R}, & i \geq N \end{cases}$$

Note that U is open in product topology (since all but finitely many of the terms in the product are $\mathbb{R}$). Let $y \in U$. We claim that $D(x,y) < \varepsilon$.

$$\begin{aligned} D(x,y) &= \sup\left\{\frac{\overline{d}(x_i,y_i)}{i} : i \in \mathbb{N}\right\} \\ &= \max\left\{\sup\left\{\frac{\overline{d}(x_i,y_i)}{i} : i < N\right\}, \sup\left\{\frac{\overline{d}(x_i,y_i)}{i} : i \geq N\right\}\right\} \end{aligned}$$

Recall that $\overline{d}$ is at most 1 so the second term in max is at most $\frac{1}{N}$ which is strictly smaller than ε. On the other hand, by construction for $i < N$ we have

$$\overline{d}(x_i,y_i) < i\varepsilon$$

Therefore $D(x,y)$ is certainly less than ε. Thus $U \subset B_D(x,\varepsilon)$.
Now suppose B is basis element of the product topology. In other words

$$B = \prod_{i=1}^{\infty} B_i$$

where there exists some $N \in \mathbb{N}$ such that $B_n = \mathbb{R}$ for $n \geq N$. ■

Theorem 2.3.10 $\mathbb{R}^J$ with the product topology is not metrizable if J is uncountable.

Proof. We will show that $\mathbb{R}^J$ is not metrizable by showing that is does not satisfy the sequence lemma. Let $A := \{x \in \mathbb{R}^J : x_\alpha = 1 \text{ for all but finitely many } \alpha\}$. Then we claim that $0 = (0,0,0,\dots) \in \overline{A}$. Let $\prod U_\alpha$ be a basis element containing 0. Then there exists some $\alpha_1,\dots,\alpha_n$ such that $U_\alpha = \mathbb{R}$ for all $\alpha \notin \{\alpha_1,\dots,\alpha_n\}$. Then the point $y = (y_\alpha)_{\alpha\in J}$ defined by $y_\alpha = 0$ if $\alpha \in \{\alpha_1,\dots,\alpha_n\}$ and 1 otherwise is in $\prod U_\alpha \cap A$.
Now we show that there is no sequence in A that converges to 0. Suppose not. Let $(a_n)_{n\in\mathbb{N}}$ be such a sequence. For each a_n, there exists $S_n := \{\alpha \in J : a_{n_\alpha} \neq 1\}$. By definition of A, we know that S_n is finite for all n. Hence $\bigcup_{n\in\mathbb{N}} S_n$ is countable so cannot be all of J. We choose some $\beta \in J - \bigcup_{n\in\mathbb{N}} S_n$ and consider $\pi_\beta^{-1}((-1,1))$ which is clearly a neighbourhood of 0 but cannot contain any element of $(a_n)_{n\in\mathbb{N}}$ as all elements of this sequence have 1 in the β coordinate. ■

Theorem 2.3.11 Let $f : X \to Y$ be a map between topological spaces X and Y. If f is continuous, then for every convergent sequence $x_n \to x$, we have $f(x_n) \to f(x)$. The converse holds if X is metrizable.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence that converges to some $x \in X$. Let U be some neighbourhood of $f(x)$. By continuity of f, we know that $f^{-1}(U)$ is a neighbourhood of x. Hence there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in f^{-1}(U)$. This means that for all such n, we have $f(x_n) \in U$.
Now suppose X is metrizable and we know that f is a map where $(x_n)_{n\in\mathbb{N}}$ converging to x implies that $f(x_n)$ converges to $f(x)$. We will show that f is continuous by showing that $f(\overline{A}) \subset \overline{f(A)}$ (see Theorem 2.1.2).
Let $A \subset X$ and let $x \in \overline{A}$. We know that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x. Then we know that $f(x_n)$ converges to $f(x)$. Since $f(x_n)$ is a sequence in $f(A)$, then $f(x) \in \overline{f(A)}$. ■

■ **Example 2.4** Counterexample: Let $\mathbb{R}_C$ be the countable complement topology on $\mathbb{R}$ and $\mathbb{R}$ be the standard topology on $\mathbb{R}$. Define $\iota : \mathbb{R}_C \to \mathbb{R}$ to be the identity map. Then every converging sequence in $\mathbb{R}_C$ (these are exactly the sequences that are eventually constant) is also convergent in $\mathbb{R}$. However $\iota^{-1}((0,1)) = (0,1)$ is not open in $\mathbb{R}_C$ hence ι is not continuous. ■

Exercise 2.6 Show that every convergent sequence in $\mathbb{R}_C$ is eventually constant. What other topology on $\mathbb{R}$ has the same set of convergent sequences? ■

Exercise 2.7 Let X be a set with the topologies $\mathcal{T}$ and $\mathcal{T}'$ where $(X,\mathcal{T})$ and $(X,\mathcal{T}')$ are both metrizable spaces. Suppose that a sequence converges in $\mathcal{T}$ if and only if it converges in $\mathcal{T}'$. Show that $\mathcal{T} = \mathcal{T}'$ (*Hint*: Recall the Sequence lemma). ■

Definition 2.3.6 — Uniform Convergence. Let $f_n : X \to Y$ be a sequence of functions from X to a metric space Y. Let d be the metric in Y. Then f_n converges to f uniformly if for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $d(f_n(x), f(x)) < \varepsilon$.

Theorem 2.3.12 Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a metric space Y. If f_n converges to f uniformly, then f is continuous.

Note now we have two ways of viewing convergence $\mathbb{R}^X$ (recall this is the set of all functions from X to $\mathbb{R}$). We can talk about uniform convergence of the functions or convergence in the uniform topology. These two notions of convergence are in fact the same.

2.4 Connectedness and Compactness

Definition 2.4.1 — Connected. A space X is connected if the only subsets of X that are open and closed in X are $\phi$and X. Equivalently we can say that a space X is connected if there do not exist disjoint open sets U and V such that $U \sqcup V = X$. If such a U and V do exist, we call them a separation of X.

Exercise 2.8 Verify that the definitions above are indeed equivalent. ■

Lemma 2.4.1 If Y is a subspace of X, then a separation of Y is a pair of disjoint non-empty subsets A, B of Y whose union is Y and neither of which contains the limit points of the other. If there is no separation then Y is connected.

Proof. Suppose A, B form a separation of Y. We will show that neither of them contains the limit point of the other.
Since A, B form a separation of Y then A is clopen in Y. This means that in particular A is closed in Y. Recall that the closure of A in Y is $\overline{A} \cap Y$ (where $\overline{A}$ denotes the closure of A in X). As A is closed in Y, we conclude that $A = \overline{A} \cap Y$. This means that

$$\phi = A \cap B = \overline{A} \cap Y \cap B = \overline{A} \cap B$$

As the closure of A contains its limit points, we conclude that B cannot contain the limit points of A. By symmetry one can conclude that the A cannot contain the limit points of B either.
Now we show that if we have a pair of disjoint non-empty subsets A, B of Y whose union is Y and neither of which contains the limit points of the other, then they form a separation of Y. We know that $\overline{A} \cap B = \phi$. Then

$$\overline{A} \cap Y = \overline{A} \cap (A \sqcup B) = (\overline{A} \cap A) \sqcup (\overline{A} \cap B) spi = A$$

Therefore the closure of A in Y is A implying that A is closed in Y. We similarly conclude that B is closed in Y. As $A = Y - B$ and $B = Y - A$, they must also be open in Y and thus form a separation. ■

■ **Example 2.5** The set $\{a, b\}$ with the trivial topology is connected. ■

■ **Example 2.6** As a subspace of $\mathbb{R}$, $[-1, 0) \cup (0, 1]$ is disconnected. ■

Lemma 2.4.2 If the sets C and D for a separation of a topological space X and Y is a connected subspace of X, then $Y \subset C$ or $Y \subset D$.

Proof. We know that C and D are both open in X so $C \cap Y$ and $D \cap Y$ are open in Y. If neither is empty, this contradicts the connectedness of Y. ■

Theorem 2.4.3 The arbitrary union of a collection of connected subsets of X that have a point in common is connected.

Theorem 2.4.4 Let A be a connected subspace of X. If B is also a subspace of X such that $A \subset B \subset \overline{A}$, then B is connected.

Proof. Suppose $B = C \cup D$ where C and D are disjoint sets that do contain each other's limit points. We will show that one of them must be empty. As A is connected, we know that either $A \subset C$ or $A \subset D$. Without loss of generality, assume $A \subset C$. Then $\overline{A} \subset \overline{C}$. Thus $B \cap D = \phi$. ■

In particular, this means that the closure of a connected set is always connected, as one may expect.

Theorem 2.4.5 The continuous image of a connected set is connected.

Proof. Let $f : X \to Y$ be a continuous map where X is a topological space. Let $Z = f(X)$ be the image of X, with the subspace topology. Suppose U, V are non-empty open sets whose union is Z. We will show that their intersection must be non-empty. This is clear as $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty and open in X, hence connectedness of X implies that their intersection is non-empty. But then the intersection of U and V is non-empty as well. ■

Theorem 2.4.6 A finite Cartesian product of connected spaces is connected.

2.4.1 Connected spaces of the Real Line

Definition 2.4.2 — Linear Continuum. Let L be a simply ordered set with more than 1 element. It is called a linear continuum if it satisfies:

1. The least upper bound property: every non-empty bounded subset, has a least upper bound
2. If $x < y$, then there exits a $z \in L$ such that $x < z < y$.

It is clear that $\mathbb{R}$ is a linear continuum.

Definition 2.4.3 — Convex Sets. Given an ordered set A, we say a subset T is convex, if given any $x, y \in T$, we have that $z \in T$ for all $x < z < y$.

Theorem 2.4.7 If L is a linear continuum in the order topology, then L is connected and so are intervals and rays.

Proof. We have shown previously that given a subset of an ordered set, we can endow it with the order topology and the subspace topology which in general will not be the same. It is something to be verified that these topologies do agree if the subset if convex.
We will thus show that all convex subsets of a linear continuum are connected. Let Y be a convex subset of L. Suppose it is not connected. Let U, V form a separation of Y. By assumption these two are non-empty, so we choose some $x \in U$ and $y \in V$. Consider the closed interval $[x, y]$ as a subspace of Y. We take $U_0 = U \cap [x, y]$ and $V_0 = V \cap [x, y]$. Note that U_0 is bounded from above, hence has a least upper bound which we will call c. We will show that c cannot be an element of U_0 or of V_0.
Suppose $c \in U_0$. As U_0 is open, there exists some $d \in L$ such that $[c, d) \subset U_0$. This contradicts c being an upper bound. We get a similar contradiction if $c \in V_0$. ■

Theorem 2.4.8 — Intermediate Value Theorem. Let $f : X \to Y$ be a continuous map where X is connected and Y is ordered. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. Consider $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. We know that A, B are non-empty, disjoint and open in $f(X)$. If there is no $c \in X$ such that $f(c) = r$ then we would have that $A \cup B = f(X)$. But then A and B are disjoint which contradicts the connectedness of $f(X)$. ■

2.4.2 Paths and Components

Definition 2.4.4 — Path. Given some x, y in a topological space X, a path from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$.

Definition 2.4.5 — Path-Connected. A topological space X is path connected if there exists a path between any two points in X.

■ **Example 2.7** $S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_1^2 + \dots x_n^2 = 1\}$ is connected and path connected ■

■ **Example 2.8** The punctured plane, $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected and path connected ■

■ **Example 2.9** $S := \{(x, \sin(1/x) | 0 < x \leq 1\}$ which is connected and path connected whereas $\overline{S} = S \cup (\{0\} \times [-1, 1])$ is only connected and *not* path connected. ■

Proposition 2.4.9 Path connectedness implies connectedness but the converse is not true.

Definition 2.4.6 — (Path-)Connected Components. Let X be a topological space. We define a relation on X, namely $x \sim y$ if and only if there exists a connected set containing x and y. The equivalence classes produced by this relation are called the connected components of X.
We can also define a relation where $x \sim_p y$ if and only if there exists a path from x to y. the equivalence classes produced by $\sim_p$ are called the path-connected components of X.

Exercise 2.9 Verify that the above are indeed equivalence relations. ■

2.5 Compactness

Definition 2.5.1 — Open cover. An open cover of a topological space X is a collection of open sets whose union is X. An open cover of a subset A of a topological space X is a collection of open sets whose union contains A.

Definition 2.5.2 — Compact. A topological space X is called compact if for every cover of X, there exists a finite subcover.

■ **Example 2.10** $\mathbb{R}$ is not compact as we can take our cover to be $\mathcal{U} = \{(-n, n)\}_{n \in \mathbb{N}}$ which clearly has no finite subcover. ■

■ **Example 2.11** $X = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$ is compact as any open set containing 0 will contain all but finitely many of the $\frac{1}{n}$. ■

Lemma 2.5.1 The closed subspace of a compact space is compact.

Proof. Let X be a compact space and let F be a closed subspace of X. Let $\mathcal{A}$ be an open cover of F. To this open cover, we add the set $X - F$ which we know is open. The compactness of X tells us we need only finitely many of these to cover X. From this subcover, we remove $X - F$ (if necessary) in order to get a finite subcover of F as desired. ■

R Compactness is a topological property. If two spaces are homeomorphic, then one is compact if and only if the other one is.

Lemma 2.5.2 Every compact subspace of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space and Y a compact subspace. We will show that Y is closed by showing that $X - Y$ is open.
Choose some $c \in X - Y$. For every $y \in Y$, there exist disjoint open sets U_y and V_y such that $c \in U_y$ and $y \in V_y$. Clearly the collection $\{V_y\}_{y \in Y}$ forms an open cover of Y. Compactness allows us to obtain a finite subcover, say $V_{y_1}, \dots, V_{y_n}$ of Y. Then $\bigcap_{i=1}^{n} U_{y_i}$ is an open neighbourhood of c contained in $X - Y$. ■

R Above we have in fact proven the following: given a compact subspace Y of a topological space and a point c not contained in Y, we can find disjoint open sets U and V such that $c \in U$ and $Y \subset V$. In other words the point c and compact subspace Y have 'disjoint neighbourhoods'.

■ **Example 2.12** $(a,b) \subset \mathbb{R}$ and $(a,b] \subset \mathbb{R}$ are not closed hence not compact. ■

■ **Example 2.13** If $\mathbb{R}$ is endowed with the finite complement topology, then every set is compact. ■

Lemma 2.5.3 The continuous image of a compact set is compact.

Proof. Let $f : X \to Y$ be a continuous map with X being a compact space. Let $\{U_\alpha\}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X. Hence the compactness of X means we only need finitely many of these to cover X, say $\{f^{-1}(U_{\alpha_i})\}_{i=1}^{n}$. Then $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $f(X)$. ■

Section 3 Introduction to Algebraic Topology

Having studied so much topology, we want to lift our gaze a bit and consider how many different topologies are possible which is to say, we would like to classify topological spaces, up to homeomorphism. This is a very difficult question that still has no complete answer. However one way we can begin is by associating topological spaces with certain groups in such way that if two topological spaces have two different groups then certainly the spaces are not homeomorphic. This is, broadly speaking, what one does in algebraic topology.

3.1 Fundamental Groups

The fundamental group is the set of all loops around a point, modulo homotopy. In the remainder of this section, we will often say things like 'the fundamental group of $\mathbb{R}^n$ is trivial' or that 'the fundamental group of S^1 is $\mathbb{Z}$'. Of course what we mean by that is that the fundamental group of $\mathbb{R}^n$ is isomorphic to the group of one element and that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$. Even more fundamentally, what we mean by this is that the method of 'combining' loops, on a circle for example, behaves the exact same way as adding two integers (for example, the integers are a cyclic group generated by 1 and -1. Similarly, the fundamental group for the circle is generated by one element and its inverse (try and guess what these might be)).

Admittedly, I am restating definitions (and possibly the obvious) again and again, however I do this because I think this is an important thing to keep in mind if you ever feel lost. We are associating an algebraic structure to our space and this structure is directly related to our space by telling us how we can combine loops. Remember that is our goal throughout.

3.2 Covering Spaces

Covering spaces are a common tool used in finding fundamental groups and such.

Definition 3.2.1 — Covering Space. Let E, B be topological spaces and $p : E \to B$ a continuous, onto map. An open set $U \subset B$ is said to be evenly covered by p if $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$ where $V_\alpha \subset E$ are disjoint open sets such that $p|_{V_\alpha}$ is a homeomorphism onto U.
If (E, B, p) are such that for every $x \in B$ there exists an open neighbourhood of x that is evenly covered by p, then E is said to be a covering space of B and p is called a covering map.

■ **Example 3.1** Let X be a topological space. Take $E = X \times \{1, \dots, n\}$ and define $p : E \to X$ by $p(x, i) = x$. Then E is a covering space of X with p as a covering map. ■

It is perhaps tempting to say, especially given the previous example, that E is disconnected as it can be expressed as the disjoint union of open sets. However note that this is only the case locally. A useful image is that of a spiral stair case which we take to be our covering space that covers the disk that lies in the the 'shadow' of the staircase. For every point in this disk we can find a small neighbourhood,

where the inverse projection looks like the disjoint union of a collection of open sets. However the staircase itself is connected (This is a deliberately imprecise formulation but what I find to be a useful image when trying to think of covering spaces).
A few remarks to be made about covering maps: covering maps are open maps (they map open sets to open sets) and the preimage of a single point under a covering map has the discrete topology. The staircase image should hopefully make both of these statements obvious. p is quite similar to being a projection and this is one way of asserting that. Furthermore in order to find the preimage of a point under p one can imagine drawing a vertical line that passes through the given point. Then the points in the preimage are exactly where the staircase intersects the line which is clearly just a discrete collection of points. See this demo for the staircase visualisation.

Lemma 3.2.1 Covering maps are open maps.

Proof. Let $A \subset E$ be open and choose some $x \in p(A)$. We wish to show that there is a neighbourhood of x contained in $p(A)$. Let U be a neighbourhood of x that is evenly covered by p. Then $p^{-1}(U) = \bigsqcup V_\alpha$ for some V_α open in E. Let $y \in A$ be such that $p(y) = x$. As the V_α are disjoint, there exists exactly one V_α containing y. Now we know that $A \cap V_\alpha$ is an open in V_α (and in particular contains y). Since $p|_{V_\alpha}$ is a homeomorphism, $p(A \cap V_\alpha)$ is open in U (the range of $p|V_\alpha$). As U is open in B, it follows that $p(A \cap V_\alpha)$ is open neighbourhood of x that is clearly contained in $p(A)$. ■

Lemma 3.2.2 The preimage of a single point under a covering map has the discrete topology (as a subspace of the covering space).

Proof. Let E be a covering space of some topological space B with $p : E \to B$ as a covering map. Let $b \in B$ be arbitrary. Let U be a neighbourhood of b that is evenly covered by p. Then $p^{-1}(U) = \bigsqcup V_\alpha$ where each V_α is homeomorphic to U. This means in each V_α there is exactly one e_α such that $p(e_\alpha) = b$. Hence the singleton sets $\{e_\alpha\}$ are open in $p^{-1}(b)$ as desired. ■

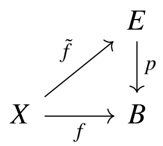
A useful and recurring example will be that of the circle being covered by $\mathbb{R}$.

Proposition 3.2.3 The map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos(2\pi x), \sin(2\pi x))$ is a covering map of S^1.

Proof. Let U_{x^+} be the set of points in S^1 with a positive x coordinate. We can similarly define $U_{x^-}, U_{y^+}, U_{y^-}$. It is clear that each of these sets is open and every point in S^1 must lie in at least one of the above sets (most will lie in two). Now note that $p^{-1}(U_{x^+}) = \bigsqcup_{n\in\mathbb{N}}(n - \frac{1}{4}, n + \frac{1}{4})$. This means that U_{x^+} is evenly covered by p. We can similarly argue the case for the other three sets to conclude that they are also evenly covered by p. Each forms a neighbourhood of some point in S^1 giving us the desired conclusion. ■

3.3 Lifts

Definition 3.3.1 Let $p : E \to B$ be a map (note we are not asserting that p is a covering map. Although we will be using lifts for covering maps, the idea is more general and is used elsewhere as well). If $f : X \to B$ is a continuous map (for some topological space X), then $\tilde{f} : X \to E$ is a lifting of f if $p \circ \tilde{f} = f$. In particular, $\tilde{f}$ is a lifting of f if the following diagram commutes:



■ **Example 3.2** Consider the covering map of S^1 from Proposition 3.2.3. We define $f(s) = (\cos(\pi s), \sin(\pi s))$ (note this a path from $(1,0)$ to $(-1,0)$). Then $\tilde{f}(s) = \frac{s}{2}$. ■

■ **Example 3.3** If $g(s) = (\cos(\pi s), -\sin(\pi s))$ (another path from $(1,0)$ to $(-1,0)$, then $\tilde{g}(s) = -\frac{s}{2}$. ■

■ **Example 3.4** If $h(s) = (\cos(4\pi s), -\sin(4\pi s))$ (looping twice around the circle), then $\tilde{h}(s) = 2s$. ■

As suggested by the above examples, an important property of covering maps is that a lifting of a path always exists. It is perhaps useful to return to our staircase analogy. Draw a path in the disk covered by the staircase. Intuitively, it is clear that we can draw paths on the staircase such that projecting them down gets us back the original path. In fact once we fix a starting point for the paths of this staircase, the lifting becomes unique. We formalise this intuition below.

Lemma 3.3.1 Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Then any path in B beginning at b_0 has a *unique* lifting to a path in E starting at e_0.

Proof. Let $f : I \to B$ be a path in B. We first construct the lifting and then show its uniqueness.
we know that for every $b \in B$ there is a neighbourhood U_b of b that is openly covered by p. This forms an open cover of B. Then $f^{-1}(U_b)$ forms an open cover of $I = [0,1]$. We can then apply the Lebesgue number lemma to partition I so that the image of each subinterval lies entirely within an open set evenly covered by p. In particular we find $s_0 < s_1 < \cdots < s_n$ such that $f([s_i, s_{i+1}] \subset U_b$ for some b.
We define $\tilde{f} : I \to E$ by first asserting that $\tilde{f}(0) = e_0$ (we want the path to start at e_0). Now suppose we have define $\tilde{f}(s)$ for all $0 \le s \le s_i$. We can then extend this to $\tilde{f}$ to be defined on $[s_i, s_{i+1}]$. We take $b = f(s_i)$ and note that U_b is evenly covered by p. Then $p^{-1}(U_b) = \bigsqcup V_\alpha$. We know that $\tilde{f}(s_i)$ is in some particular V_α. As $p|_{V_\alpha}$ is a homeomorphism, it is clear how to define $\tilde{f}(s)$, namely

$$\tilde{f}(s) = p|_{V_\alpha}^{-1}(f(s))$$

By the pasting lemma (see Theorem 2.1.3), defining $\tilde{f}$ in this piecewise manner still gives us a continuous function.
Now suppose $g : I \to E$ is another lifting of f that starts at e_0. Clearly $g(0) = 0 = \tilde{f}(0)$ Now suppose $g(s) = \tilde{f}(s)$ for $0 \le s \le s_i$ for some s_i. We will show that they must be equal on $[s_i, s_{i+1}]$. As before we know that $\tilde{f}(s_i)$ of some particular V_α where V_α is one of the disjoint open sets in $U_{f(s_i)}$. As $[s_i, s_{i+1}]$ is connected, we know that $g(s) \in V_\alpha$ for all $s \in [s_i, s_{i+1}]$. Additionally we know by definition of lifts that $p(g(s)) = f(s)$. For $s \in [s_i, s_{i+1}]$ we can just as easily say that $p|_{V_\alpha}(g(s)) = f(s)$. However $p|_{V_\alpha}$ is a homeomorphism which means that $g(s) = p|_{V_\alpha}^{-1}(f(s)) = \tilde{f}(s)$. ■

Now we wish to show that path homotopies can be lifted as well. This would mean that if two paths are homotopic in B then they would be homotopic in the covering space as well.

Lemma 3.3.2 Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $F : I \times I \to B$ a continuous map with $F(0,0) = b_0$. Then there is a unique lifting of F to $\tilde{F} : I \times I \to E$ satisfying $\tilde{F}(0,0) = e_0$. If F is a path homotopy then $\tilde{F}$ is also a path homotopy.

Proof. The gist of the proof is the same as before hence we do not provide full details. We first subdivide $I \times I$ into sub rectangles the images of each which under F lie in an open set that is evenly covered by p. We can construct $\tilde{F}$ piece wise as before by starting at the bottom left corner and working our way to the top right.

Now we need show that if F is a path homotopy then $\tilde{F}$ is a path homotopy. In particular we need to show that $\tilde{F}(0 \times I)$ and $\tilde{F}(1 \times I)$ are both one-point sets. We know that $F(0 \times I)$ and $F(1 \times I)$ are one-point sets as F is a path homotopy. Then $p^{-1}(F(0 \times I))$ has the discrete topology. The continuity of $\tilde{F}$ and connectedness of $0 \times I$ implies then that $\tilde{F}(0 \times I)$ must be a one-point set. ■

Note that two paths with the same starting point and ending point need not to be lifted to paths with the same starting and ending points. We have already seen an example of this (where?). You may also wish to use the staircase analogy to visualise why this not need be the case. However if the two paths are homotopic, this is no longer an issue.

Theorem 3.3.3 Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Suppose f, g are two path from b_0 to b_1 in B with their respective liftings being $\tilde{f}$ and $\tilde{g}$. Then if f and g are homotopic then $\tilde{f}$ and $\tilde{g}$ end at the same point and are homotopic.

Proof. Let F be the path homotopy between f and g. It is easy to see that $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$ (we know that $\tilde{F}$ is a path homotopy between two paths and by considering $\tilde{F}(s,0)$ and $\tilde{F}(s,1)$ we can find the paths). ■

Definition 3.3.2 — Lifting Correspondence. Let $p : E \to B$ be a covering map and let $b_0 \in B$. Choose $e_0 \in E$ so that $p(e_0) = b_0$. Then given $[f] \in \pi_1(B, b_0)$ we can lift it to be path $\tilde{f}$ in E (starting at e_0). By the above, $\tilde{f}(1)$ is independent of the choice of representatives. Hence the map

$$\begin{aligned} \Phi : \pi_1(B, b_0) &\to p^{-1}(b_0) \\ [f] &\mapsto \tilde{f}(1) \end{aligned}$$

is well defined. We call Φ the lifting correspondence derived from the covering map p (note that it depends on our choice of e_0).

Theorem 3.3.4 Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. If E is path-connected, then Φ is onto. If E is simply connected, then Φ is a bijection.

Proof. Suppose E is path-connected. Suppose $e_1 \in p^{-1}(b_0)$ and we wish to show that there is some $[f]$ in $\pi_1(B, b_0)$ such that $\tilde{f}(1) = e_1$. Let f be a path from e_0 to e_1. Then $p \circ f$ is a loop in B based at b_0 (since $p(0) = p(1) = b_0$) which lifts to f by construction and ends at e_1. Therefore if E is path-connected, Φ is onto.

Now suppose E is simply connected. We wish to show that Φ is injective, i.e. we wish to show that $\Phi([f]) = \Phi([g])$ then $[f] = [g]$. $\Phi([f]) = \Phi([g])$ implies that $\tilde{f}(1) = \tilde{g}(1)$. Thus $\tilde{f}$ and $\tilde{g}$ are paths in E that start and end at the same point. Since E is simply connected, this implies that the two path are homotopic. Composing p with the homotopy gets us a homotopy between f and g implying that $[f] = [g]$ as desired. ■

Theorem 3.3.5 The fundamental group of S^1 is isomorphic to the additive group of the integers.

3.3.1 Retractions and Fixed Points

Definition 3.3.3 — Retractions. If A is a subset of a topological space X, then a retraction of X onto A is a continuous map $r : X \to A$ such that $r|_A$ is the identity. If such a map r exists, then A is a retract of X.

Clearly a loop in A is also a loop in X. The existence of a retraction tells us that multiplying loops in A is compatible with multiplying them in X. Thus we see that the fundamental group of A must be contained in the fundamental group of X. We formalise this intuition below.

Lemma 3.3.6 If A is a retract of X, then the inclusion map $j : A \to X$ induces an injective group homomorphism between the fundamental groups.

Proof. Let r be a retraction of X onto A. Then $r \circ j$ is the identity on A so $(r \circ j)_* = r_* \circ j_*$ is the identity homomorphism. This means that j_* must be injective. ∎

Lemma 3.3.7 Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent:

- h is nullhomotopic
- h extends to a continuous map $k : B^2 \to X$
- $h*$ is the trivial homomorphism

Proof. ($1 \Rightarrow 2$) Suppose h is nullhomotopic. Let H be homotopy between h and a constant map. In particular then H is map from $S^1 \times I$ to X. We define a quotient map as follows $p : S^1 \times I \to B^2, p(x,t) = (1-t)x$. Then as H is constant on $S^1 \times \{1\}$, it continuous map k from B^2 to X.
Let us unpack the above proof to be a bit more visual. Recall that $S^1 \times I$ is just a cylinder. The quotient map allows us identity every point on the cylinder without its top rim with every point on the disk without the origin. Finally we identify the top rim with the origin to complete the map. Now given any map from $S^1 \times I$ we can easily interpret it as a map on the disk instead, provided that the top rim of the cylinder (that is $S^1 \times \{1\}$) is mapped to one point. If it's mapped to more than one point then we don't know where to map the origin on the disk.

($2 \Rightarrow 3$) Suppose h extends to a continuous $k : B^2 \to X$. Let $j : S^1 \to B^2$ be the inclusion map. Then $h = k \circ j$ (k is an extension is an extension of h so in particular they must agree on the domain of h which is what j identifies for us). Then $h_* = k_* \circ j_*$. As the fundamental group of S^1 is not trivial which the fundamental group of B^2 is, it must be true that j_* is trivial (i.e. maps everything to the identity). Therefore h_* is also trivial.
We translate this proof into the language of loops once again, with less formal precision. Suppose f is a loop in S^1. Then f is also a loop in B^2. As h is a continuous map from S^1 to X, we can 'push' f to a loop in X via h. However we can extend h to k which is a map on all of B^2. As h and k agree on S^1, pushing f via h is entirely equivalent to pushing it via k. However, we can contract f to a constant loop before pushing it which means we can contract the loop after its been pushed as well. This means every loop we push to X via h is homotopic to a constant map as desired.

($3 \Rightarrow 1$) Suppose h_* is trivial. We wish to define a homotopy $G : S^1 \times I \to X$ such that $G(x,0) = h(x)$ and $G(x,1) = c$ for some constant $c \in X$. Let $p_0 : I \to S^1$ be given by $p_0(x) = (\cos 2\pi x, \sin 2\pi x)$. Then

$h \circ p_0$ is homotopic to a constant map. Let H be this homotopy. In particular, H is a continuous map from $I \times I$ to X. Now we note that $p_0 \times id_I : I \times I \to S^1 \times I$ is a quotient map. We wish to define $G(x,t) = H((p_0 \times id_I)^{-1}(x,t))$. Note that p_0 is *nearly* bijective so the above readily makes sense almost everywhere. The only issue occurs at b_0 as $p_0^{-1}(b_0) = \{0,1\}$. Hence $(p_0 \times id_I)^{-1}(b_0,t) = \{(0,t),(1,t)\}$. However as H is a path homotopy $H(0,t) = H(1,t)$ for all $t \in [0,1]$. This means that G is indeed well-defined. It is then easily verified that G is a homotopy between h and a constant map. ■

3.4 Deformation Retracts and Homotopy Type

Lemma 3.4.1 Let $h,k : (X,x_0) \to (Y,y_0)$ be continuous maps. If h and k are homotopic such that the image of x_0 remains fixed at y_0 during the homotopy then h_* and k_* are equal.

Proof. Let f be a loop in X based at x_0. We wish to show that $h \circ f$ and $k \circ f$ are homotopic. However the homotopy between h and k gives us this exactly.
We provide some more details here. Let H be the homotopy from h to k that keeps the image of x_0 fixed at y_0. Then $H : X \times I \to Y$ is a continuous map such that $H(x,0) = h(x)$ and $H(x,1) = k(x)$ and $H(x_0,t) = y_0$ for all $t \in I$. Now define $\tilde{H} : I \times I$ given by $\tilde{H}(s,t) = H(f(s),t)$. Then $\tilde{H}(s,0) = H(f(s),0) = h(f(s)$ and $\tilde{H}(s,1) = H(f(s),1) = k(f(s))$ as desired. ■

Consider a loop in $\mathbb{R}^2 - \{0\}$. It is clear that any loop in this space is going to be homotopic to some loop on the unit circle as we simply need to normalise all the points along the loop. Thus if we wish to study the loops on this space, it suffices consider loops on the circle. We now formalise this idea and generalise it to $\mathbb{R}^n - \{0\}$.

Theorem 3.4.2 The inclusion map $j : S^n \to \mathbb{R}^{n+1}$ induces an isomorphism of fundamental groups.

Proof. We wish to show that j_* is invertible so in particular has an inverse. Let r be the retraction of $\mathbb{R}^{n+1} - \{0\}$ to S^n. Recall that this is given by

$$r(x) = \frac{x}{||x||}$$

We claim that r_* is the desired inverse.
First clearly $r \circ j$ is the identity on S^n hence $r_* \circ j_*$ is the identity homomorphism. This confirms that r_* is at least a left-inverse.
Now we show that although $j \circ r$ is not the identity, it is in fact homotopic to the identity. The map

$$H(x,t) = (1-t)x + t\frac{x}{||x||}$$

gives a homotopy from the identity on $\mathbb{R}^{n+1} - \{0\}$ to $j \circ r$. We verify that $H(x,t)$ is never 0 since $||H(x,t)|| = (1 - t(1 - \frac{1}{||x||})\,||x||$ which is never 0.
Additionally note that $b_0 = (1,0,\dots,0)$ remains fixed during the homotopy. Then applying the previous lemma, we see that $j_* \circ r_*$ must be the identity homomorphism proving that r_* is also the right inverse of j_* as desired. ■

The above idea generalises to what are called deformation retracts. If X is a topological space, then we say that $A \subset X$ is a deformation retract of X if there exists a continuous map $H : X \times I \to X$ such that $H|_{A \times I}$ is the identity and $H(x,1) \in A$ for all $x \in X$. The map H is called a deformation retraction.

(a) Figure-8

(b) Glasses

Figure 3.1: Two homotopically equivalent spaces

We see that $r(x) = H(x,1)$ is a retraction of X to A. We can use the above proof almost exactly to conclude that given a deformation retraction, r_* and j_* are inverses of one another (where as usual j is the inclusion map from A to X).

Theorem 3.4.3 Let A be a deformation retract of X and $x_0 \in A$. Then the inclusion map

$$j : (A, x_0) \to (X, x_0)$$

induces an isomorphism of fundamental groups.

With a deformation retract, we are in some sense removing the extra space that doesn't matter, in the sense that it doesn't introduce any new, distinct loops. For example, consider a loop on the punctured plane that goes around the origin. It is clear that such a loop is homotopic to the unit circle. Hence any of the space that is not in the unit circle isn't actually of any interest (at least if we're only focusing on the loops). That is what motivates deformation retractions and with a bit of practice, you will be able to easily spot what a space should deform retract to.

Next we want to talk about homotopy equivalences. Recall, that we say two maps are essentially the "same" if we can continuously deform one to the other (this of course made concrete with the idea of homotopies). We want to use a similar idea to characterise spaces. Namely we should be able to distort a space in some ways without that affecting the loop structure of the space.

Consider the example of glasses and the figure-8 as shown in Figure ??. We instantly see that neither space is the deformation retract of the other. However, we can intuit that the fundamental group of the two spaces must be the same since, from the perspective of loops, having a single connecting point or having a single connecting line is the same thing. This means that the connecting line hasn't really added any meaningful structure to the space, when viewed from the perspective of loops (this is similar to the discussion above of how deformation retracts allow us to remove what may be considered redundant space). This is exactly what homotopy equivalence aims to capture.

Definition 3.4.1 — Homotopy Equivalence. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that $g \circ f : X \to X$ is homotopic to the identity on X and $f \circ g : Y \to Y$ is homotopic to the identity on Y. In this case, f and g are called homotopy equivalences and are said to be homotopy inverses of one another. Two spaces X and Y are said to be homotopy equivalent if there exists a pair of homotopy equivalences between them.

As one may suspect, homotopy equivalence is indeed an equivalence relation. Two spaces that are homotopy equivalent are also said to be of the same **homotopy type**.

We would like to prove that spaces of the same homotopy type have isomorphic fundamental groups.

However, first we need to make a few statements about homotopies that do *not* preserve the basepoint during the homotopy. In particular, we want to say that allowing a loop to move along a path starting at the basepoint, doesn't get us a different loop.

Lemma 3.4.4 Let $h,k : X \to Y$ be continuous maps with $h(x_0) = y_0$ and $k(x_0) = y_1$. Let α be the path from y_0 to y_1 given by $\alpha(t) = H(x_0,t)$ where $H : X \times I \to Y$ is the homotopy from h to k. Then

$$k_* = \widehat{\alpha} \circ h_*$$

In other words, the following diagram commutes

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{h_*} & \pi_1(Y,y_0) \\ & \searrow^{k_*} & \downarrow \widehat{\alpha} \\ & & \pi_1(Y,y_1) \end{array}$$

Theorem 3.4.5 $\pi_1(X \times Y, (x_0,y_0))$ is isomorphic to $\pi_1(X,x_0) \times \pi_1(Y,y_0)$.

Proof. We wish to create an isomorphism between the space of loops on $X \times Y$ and the space of loops on X and Y respectively (or more precisely on the product of the latter two spaces). The projection maps gives a natural way to translate from one space to the other so it seems clear that they provide the desired isomorphism. Let us write this more formally.
Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. We claim that the map

$$\begin{aligned} \Phi : \pi_1(X \times Y, (x_0,y_0)) &\to \pi_1(X,x_0) \times \pi_1(Y,y_0) \\ [f] &\mapsto p_*([f]) \times q_*([f]) \end{aligned}$$

is a group isomorphism.
The fact that Φ is a group homomorphism follows from the fact that p_* and q_* are group homomorphisms. We then verify the surjectivity of Φ, so let $[g]$ and $[h]$ be elements of $\pi_1(X,x_0)$ and $\pi_1(Y,y_0)$ respectively. Then the loop f given by $f(s) = (g(s),h(s))$ is such that $\Phi([f]) = [g] \times [h]$. In order to verify injectivity, we show that that the kernel is trivial. So suppose $\Phi([f]) = \text{id}$ (the identity on $\pi_1(X,x_0) \times \pi_1(Y,y_0)$ is of course the product of the identities on $\pi_1(X,x_0)$ and $\pi_1(Y,y_0)$ respectively). Then, by definition of Φ and the identity elements of fundamental groups, we know that $p \circ f$ and $q \circ f$ are homotopic to the constant maps c_{x_0} and c_{y_0} respectively. Let G and H be the respective homotopies. Then $F(s,t) = (G(s,t),H(s,t))$ gives us a homotopy from f to $c_{(x_0,y_0)}$ as desired. ■

The following theorem is a special case of the Siefert-Van Kampen Theorem which will be discussed later.

Theorem 3.4.6 Suppose $X = U \cup V$ where U and V are open sets of X and $U \cap V$ is path connected. Let $x_0 \in U \cap V$. Let i,j be the inclusion mappings for U,V respectively. Then the images of i_* and j_* generate $\pi_1(X,x_0)$.

Proof. Let $f : [0,1] \to X$ be a loop in X. We wish to express f as some product of loops in U and V respectively. We first find a partition of $[0,1]$ such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ where each $f([t_i,t_{i+1}])$ is contained entirely within either U or V (the existence of such t_i is guaranteed by the

Lebesgue number lemma and compactness of $[0,1]$). We would also like that each $f(t_i)$ lies in $U \cap V$. We can get this by asserting that $f([t_i,t_{i+1}])$ and $f([t_{i+1},t_{i+2}])$ lie in separate sets (i.e. one lies in U and the other in V). If this is not the case, we can simply remove t_{i+1} from the partition.
Let f_i denote the path corresponding to $f([t_i,t_{i+1}])$, rescaled to have domain $[0,1]$. Then

$$[f] = [f_1 * f_2 * \cdots * f_{n-1}]$$

We know that $f(t_i)$ always lies in $U \cap V$ which is path-connected. Then let σ_i denote the path from $f(t_i)$ to x_0. Let σ_0 and σ_n both denote the constant path at x_0. Then

$$\begin{aligned}[f] &= [\sigma_0 * f_1 * \sigma_1 * \overline{\sigma}_1 * f_2 * \sigma_2 * \ldots \overline{\sigma}_{n-1} * f_{n-1} * \sigma_n] \\ &= [\sigma_0 * f_1 * \sigma_1] * [\overline{\sigma}_1 * f_2 * \sigma_2] * \ldots [\overline{\sigma}_{n-1} * f_{n-1} * \sigma_n]\end{aligned}$$

Each $\sigma_{i-1} * f_i * \sigma_i$ is a loop in U or a loop in V based at x_0, as desired. ■

Corollary 3.4.7 Suppose $X = U \cup V$ where U,V are open and $U \cap V$ is path-connected. If U and V are simply connected (i.e. trivial fundamental group), then X is also simply connected.

Let use this corollary to prove something we may have suspected all along.

Theorem 3.4.8 If $n \geq 2$, then S^n is simply connected

Proof. Let p,q be two distinct points on S^n (often taken to be the north and south poles). Let $U = S^n - \{p\}$ and $V = S^n - \{q\}$. Then $U \cap V = S^n - \{p,q\}$, which is path connected (given than $n \geq 2$). Each of U and V are homeomorphic to R^n hence are simply connected. This implies that S^n is also simply connected, by the previous corollary. ■

(R) A common homeomorphism from $S^n - \{N\}$ to $\mathbb{R}^n$ where N is the North pole is stereographic projection, given by

$$f(x_1,\ldots,x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1,\ldots,x_n)$$

We now define a new and important space in algebraic topology: the (real) projective plane, $\mathbb{R}P^2$.

Definition 3.4.2 — Projective Plane. The real projective plane is the set of all lines that pass through the origin in $\mathbb{R}^3$.

Although perhaps a seemingly bizzare space (points in this space are after all lines), this space (and its related cousins) turn out to be quite important and have some lovely properties, especially when considered as cell complexes (a story for another time, you can read up on these Allan Hatcher's freely available textbook *Algebraic Topology*). For now let us simply try and understand this space a bit better. Every line can be characterised by 2 points hence every line passing through the origin in $\mathbb{R}^3$ can be identified by the pair of points it intersects on the unit sphere, S^2. These are of course going to be antipodal points. Hence we can equivalently describe $\mathbb{R}P^2$ as the sphere where we identify antipodal points. The quotient map in this case is also a covering map. The preimage of a point in $\mathbb{R}P^2$ has exactly two points in S^2 (the pair of antipodal points). Moreover the covering space, S^2 is simply connected, hence by lifting correspondence, we know that the order of the fundamental group is 2.

There is only one group (up to isomorphism) of order 2, so we know that the fundamental group of $\mathbb{R}P^2$ is the cyclic group of 2 elements.
There is a useful way of drawing $\mathbb{R}P^2$. Recall that $\mathbb{R}P^2$ may be defined as the sphere with antipodal points identified. Forgetting the equator for a minute, we can choose the open Northern hemisphere to be the representatives of each equivalence class. The open hemisphere is homeomorphic to the open disk. Now we fit the forgotten equator on the boundary of this disk and mark it to remind ourselves that antipodal points on the boundary are identified with each other. This can be a useful illustration of the projective plane.

Exercise 3.1 Venture a guess as to what a space with a fundamental group of 3 elements may look like. If you are currently unable, to verify your answer the following theorem should help. ■

Exercise 3.2 You might think that we skipped over something by jumping to lines in $\mathbb{R}^3$ when lines in $\mathbb{R}^2$ seem like a more natural starting step. Indeed one can identically define a space called $\mathbb{R}P$, known as the real projective line, which is the set of all lines in the plane that pass through the origin (or equivalently define it as the circle with antipodal points identified). As it turns out this is a space we are already intimately familiar with. What is it? ■

3.5 Siefert-Van Kampen Theorem

It's useful to start with an example that isn't going to be exactly a demonstration of the theorem but helps motivate the idea of the theorem.

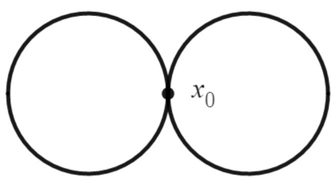


Figure 3.2: Figure-8

We consider the example of the figure-8. We are trying to find the distinct loops on this space. We will take our basepoint to be the point the circles have in common. One loop we could have is going around the left circle, in a clockwise path say. Let us call this loop a. Then a generates all the loops we could have on the left circle (we infer this from the case of the circle). Similarly we can have a loop that goes around the right circle once, which we call b, which generates all the loops on the right circle. Intuition suggests that a and b are not path homotopic so are indeed distinct (one can imagine that if such a path homotopy were to exist it would have to go through the space within or outside the circles which is of course not allowed). Then any string of a and b (and a^{-1} and b^{-1}) is also a valid, and distinct, loop on this space. This is known as the free group of two generators and is indeed the fundamental group of the figure-8 (I like to think of the free group as taking a set and throwing everything into it that you need in order to get a group with the operation being concatenation).
Note how we were able to infer the fundamental group of the figure-8, using the fact that we knew the fundamental group of the different parts of it (namely the two circles). This is what the Siefert-Van Kampen Theorem codifies for us.

There are several ways of stating the theorem. We provide two here.

Theorem 3.5.1 — Siefert-Van Kampen. Let $X = U \cup V$ where U, V are open in X. Assume that U, V and $U \cap V$ are path connected. Let x_0 be in $U \cap V$ and let $j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphism of the free product that extends the homomorphisms $\varphi_{1_*}, \varphi_{2_*}$. Then j is surjective and its kernel is the least normal subgroup that contains all elements represented by $\langle \varphi_{1_*}(g)^{-1} \varphi_{2_*}(g) \rangle$ for $g \in \pi_1(U \cap V, x_0)$.

Theorem 3.5.2 Suppose $X = U \cup V$ where U, V are open in X with U, V and $U \cap V$ path connected. Suppose we have the following presentations

$$\pi_1(U \cap V, x_0) = \langle S : R \rangle$$
$$\pi_1(U, x_0) = \langle S_1 : R_1 \rangle$$
$$\pi_1(V, x_0) = \langle S_2 : R_2 \rangle$$

Define $R_S = \{\langle \varphi_{1_*}(s) \rangle = \langle \varphi_{2_*}(s) \rangle\}$. Then

$$\pi_1(X, x_0) = \langle S_1 \cup S_2 : R_1 \cup R_2 \cup R_S \rangle$$

Let's try and unpack the first statement a little bit. The loops in X are of course going to be given by loops in U and V since $X = U \cup V$. We can perform any number of loops in U, followed by any number of loops in V, repeated any number of times. This is why the domain of j is the free product of $\pi_1(U, x_0)$ with $\pi_1(V, x_0)$; we know $\pi_1(X, x_0)$ is 'at most that big'. The problem is that, in general, this will be an over count the loops (for example loops that lie entirely in $U \cap V$ will be seen in both), hence why j is surjective. The over count occurs from the loops in $U \cap V$. Suppose g is a loop $\pi_1(U \cap V, x_0)$. Then viewing g as a loop just in U or a loop just in V should be the same, so $\phi_{1_*}(g) = \phi_{2_*}(g)$. This is equivalent to $\phi_{1_*}(g)^{-1} \phi_{2_*}(g) = e$ hence why this lies in the kernel of j. A similar breakdown of the second phrasal of the theorem can be done and is recommended. You should be able to convince yourself that the two statements really are the same.

This is the informal intuition behind the theorem, a formal proof is not provided but can be found in almost any algebraic topology textbook.

■ **Example 3.5** Let's try finding the fundamental group of the figure-8 using the Siefert-Van Kampen Theorem. So let X be the figure-8. Let p be some point on left circle and q some point on the right circle (both distinct from x_0, the common point of the two circles). We define $U = X - \{p\}$ and $V = X - \{q\}$. Note that U, V are open and that $U \cap V$ is path-connected.

Consider U. The punctured (left) circle can be deformation retracted to x_0 leaving just the one circle. Therefore $\pi_1(U, x_0) = \langle a : \phi \rangle$. Analogously, $\pi_1(V, x_0) = \langle b : \phi \rangle$. We see that $U \cap V$ be deformation retracted to x_0, hence $\pi_1(U \cap V, x_0) = \langle \phi : \phi \rangle \, \{e\}$. Therefore the kernel of j must be trivial as well (or $R_S = \phi$), therefore $\pi_1(X, x_0)$ is $\pi_1(U, x_0) * \pi_1(V, x_0)$ (or $\pi_1(X, x_0) = \{a, b : \phi\}$) which is the free group of two generators as claimed earlier. ■

■ **Example 3.6** Let's try a more interesting example and try to find the fundamental group of the torus using the Siefert-Van Kampen Theorem.

We take U to be the interior of the space and V to be $X - \{p\}$ where p is the point in the center. It is clear that $U \cap V$ is path-connected (being a punctured disk). Then U is homeomorphic to the open disk therefore $\pi_1(U, x_0) = \langle \phi : \phi \rangle$. We deformation retract V to its boundary. The boundary is the wedge of two circles (one for the horizontal edges and one for the vertical edges). Therefore $\pi_1(V,x_0)$ is the free group of two generators. One the generators is going to correspond to the the horizontal edges and one to the vertical edges. Of course x_0 itself is not on the boundary so its not exactly as simple as that but not much more complicated either. We let σ denote a path from x_0 to the point that the wedge of circles has in common. Then $\pi_1(V,x_0) = \langle [\sigma * a * \overline{\sigma}], [\sigma * b * \overline{\sigma}] : \phi \rangle$, where a and b are paths travelling along the horizontal and vertical edges respectively. We define $A = [\sigma * a * \overline{\sigma}]$ and $B = [\sigma * b * \overline{\sigma}]$ for the sake of brevity.

As we mentioned, $U \cap V$ is the punctured, open disk. Therefore $\pi_1(U \cap V, x_0) = \langle s : \phi \rangle$, where s is a loop that goes around the puncture. Then we only need to interpret s as words (or loops) in $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$. As a loop in U, this loop is trivial (as indeed all loops are). So in particular $\varphi_{1_*}(s) = e$. However as a loop in V, it is not. One can imagine expanding s until it lies exactly on the boundary. We then see that $\varphi_{2_*}(s) = ABA^{-1}B^{-1}$. Therefore $\pi_1(X,x_0) = \left\langle A, B : ABA^{-1}B^{-1} = e \right\rangle$. This is the abelian free group of two generators, or simply $Z \times Z$.

■

II Exercises

part.1chapter.1section.1.1section.1.2section.1.3subsection.1.3.1s