Definition 1.1 (Equicontinuous)**.** We say $E \subset C([a,b];\mathbb{R})$ (that is E is some set of real-valued functions from the closed interval $[a,b]$) is *equicontinuous at* $x \in [a,b]$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $y \in [a,b]$ satisfying $|x-y| < \delta$ we have $|f(x) - f(y)| < \epsilon$ for every $f \in E$. We say E is equicontinuous if E is continuous at x for every $x \in [a,b]$.

Theorem 1.2 (Ascoli-Arzela Theorem)**.** *Let $E \subset C([a,b];\mathbb{R})$. The E is compact if and only if E is closed, bounded and equicontinuous.*

Proof. Note: the topology on $C([a,b];\mathbb{R})$ is that induced by the supremum norm.

Suppose E is closed, bounded and equicontinuous. We wish to show that E is compact, which we will do by showing that it sequentially compact.

Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence in E. We wish to find a convergent subsequence. First as E is closed, if we do find a convergent subsequence, we know that the limit will be contained in E as well. So we only need find a Cauchy subsequence of (f_n). We know that E is bounded which means that there is some $M > 0$ such that $|f(x)| < M$ for all $x \in [a,b]$ and $f \in E$. In particular then $f(x)$ is always contained in $[-M, M]$ which is compact hence sequentially compact. This might inspire us to do the following: for every $x \in [a,b]$, $(f_n(x))_{n\in\mathbb{N}}$ form a sequence in $[-M,M]$ which we know has a convergent subsequence. This gives us a candidate for a convergent subsequence. Unfortunately, this is not (necessarily) it as although the the subsequence converges for some given x, the same subsequence of functions might not converge for some other $x' \in [a,b]$. However this is easily fixed.

Let us call the convergent subsequence $(f_{1,n})_{n\in\mathbb{N}}$ (1 for the number of points where we know convergence is assured). Now we choose some other x' and by the same logic as above we can find a convergent subsequence of $(f_{1,n})$ that converges on x' (we consider the sequence of real numbers $f_{1,n}(x')$ which is also contained in $[-M,M]$). We denote this new subsequence of functions as $(f_{2,n})_{n\in\mathbb{N}}$. We can continue this process to obtain convergence on any finite number of points in $[a,b]$. Let $(g_n)_{n\in\mathbb{N}}$ be the sequence of functions defined by $g_n = f_{n,n}$ and our hope is of course that as we obtain convergence on more and more points, we begin to approximate a continuous function on $[a,b]$ and indeed converge towards it. In order to show this, we need to ascertain this we need show that $(g_n(x))_{n\in\mathbb{N}}$ converges for very $x \in [a,b]$ and moreover that the convergence is uniform. This means we need show that for every $\epsilon > 0$ we can find some $N \in \mathbb{N}$ such that for every $x \in [a,b]$ we have

$$|g_n(x) - g_m(x)| < \epsilon$$

if $n, m \geq N$.

Let $X = x_1, x_2, \dots$ be the set of points on which we are asserting convergence (these would be the x and x' from above for example). We note now that there are two things we would like to be true, that would allow us to prove the desired inequality. First, we are trying to approximate the function using a countable set of points, so it would be nice if for every $x \in [a,b]$ we could always find some x_j in a neighbourhood of it. That is, we would like X to be a dense subset of

$[a,b]$ (some people may wish to take the rationals, but I am fairly certain that any dense subset should suffice). Second, we would like to be able to bound $|g_n(x) - g_n(x_j)|$ (where x_j is some sufficiently close term to x) somehow as

$$\begin{aligned}|g_n(x) - g_m(x)| &= |g_n(x) - g_n(x_j) + g_n(x_j) - g_m(x_j) + g_m(x_j) - g_m(x)| \\ &\leq |g_n(x) - g_n(x_j)| + |g_n(x_j) - g_m(x_j)| + |g_m(x_j) - g_m(x)|\end{aligned}$$

We can bound the central term using the fact that $(g_n(x_j))_{n\in\mathbb{N}}$ is a convergent sequence and we can simultaneously bound the other two terms using equicontinuity!

We take x to an arbitrary element of $[a,b]$ as before and suppose $\epsilon > 0$ is given. Let δ be such that $|x-y| < \delta$ implies that $|f(x) - f(y)| < \frac{\epsilon}{3}$ for all $x, y \in [a,b]$ and all $f \in E$. The density of X means that there is some $x_j \in X$ such that $|x - x_j| < \delta$. We know that $(g_n(x_j))_{n\in\mathbb{N}}$ is convergent so there exists some $N \in \mathbb{N}$ such that for $n, m \geq N$ we have

$$|g_n(x_j) - g_m(x_j)| < \frac{\epsilon}{3}$$

(Note: this may suggest that N depends on our choice of x_j, however we can easily remedy this. However, we can easily remedy this. We take δ as before and cover $[a,b]$ with $\delta-$balls centered at x_i for each $x_i \in X$. The compactness of $[a,b]$ reduces this to a finite cover of $\delta-$balls centered at $x_1, \ldots, x_k$. We have convergence of $(g_n(x_i)))_{n\in\mathbb{N}}$ for every $1 \leq i \leq k$ hence for every i we can find N_i such that

$$|g_n(x_i) - g_m(x_i)| < \frac{\epsilon}{3}$$

for every $n, m \geq N_i$. We can then take our N to be the maximum of these N_i.)

We have thus found an $N \in \mathbb{N}$ such that for $n, m \geq N$ we get

$$\begin{aligned}|g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(x_j)| + |g_n(x_j) - g_m(x_j)| + |g_m(x_j) - g_m(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

Now we show the converse. Suppose E is compact. We know it must be bounded (consider balls of increasing radius centered at 0 which forms a cover of M). As E is a compact subset of a Hausdorff space, it is closed. All that remains to show then, is that E is equicontinuous. That is, we wish to show that for every $x \in [a,b]$ and every $\epsilon > 0$ we can find a $\delta > 0$ such that for every $f \in E$ and every $y \in [a,b]$ we have that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Compactness means that all the functions in E are close to a finite subset of functions in E (more precisely, compactness guarantees the existence of a finite collection of functions, say $f_1, \ldots, f_n$, in E that can approximate any other function in E to some given degree of precision). We would like to say that we can

Let $\mathcal{A}$ be an open cover of E formed by $\frac{\epsilon}{3}-$balls centered at every $f \in E$. Compactness allows us to reduce this to a finite cover say $B_{\frac{\epsilon}{3}}(f_1), \ldots, B_{\frac{\epsilon}{3}}(f_n)$.

For each f_i, we can find a δ_i such that $|x-y| < \delta_i$ implies that $|f_i(x)-f_i(y)| < \frac{\epsilon}{3}$. We then take δ to be minimum of all these δ_i. Then for any $f \in E$, we can find some f_i such that $\|f - f_i\| < \frac{\epsilon}{3}$. Then given any $x, y \in [a, b]$ with $|x - y| < \delta$, we have

$$
\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}
$$

□