Nets, ahoy!

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Outline

- Sequences: Yay or nay

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- Nets: The what, the how and the why

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- *Joke*
- Filters

Sequences and me (colourised)

Source (emojis): OpenMoji

Sequences being neat

- Divergence of harmonic series

Sequences being neat

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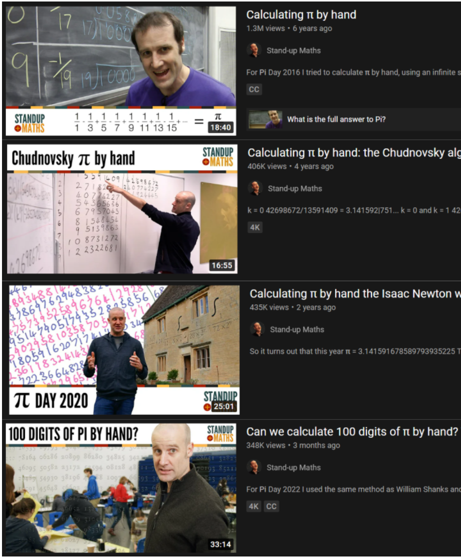
- Computing irrational numbers

Sequences being neat

- Divergence of harmonic series
- Computing irrational numbers



Sequences being neat

- Divergence of harmonic series
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- Taylor series

Sequences being neat

- Divergence of harmonic series
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(Riddle) Can you find uncountably many subsets of $\mathbb{N}$ *such that any pair has only finite intersection?*

Sequence properties

The set of convergent sequences in a space determines its topology.

Sequence properties

The set of convergent sequences in a metrizable space determines its topology.

Sequence properties

Theorem

Let X be a metrizable topological space. Then for all $A \subset X$, we have $x \in \overline{A}$ if and only if there exists a sequence in A that converges to x.

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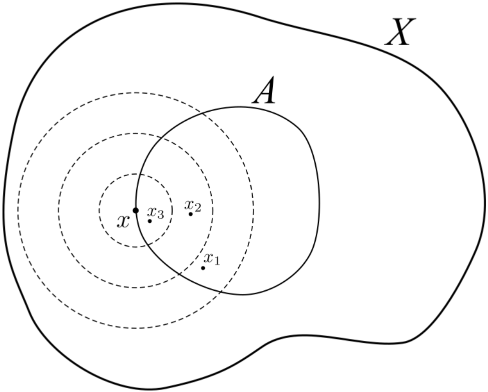

Sequence properties

Theorem

Let X be a metrizable topological space. Then for all $A \subset X$, we have $x \in \overline{A}$ if and only if there exists a sequence in A that converges to x.

Proof.

$\Leftarrow$ Let x_n be a sequence in A that converges to x. This means that every neighbourhood of x has non-trivial intersection with A so $x \in \overline{A}$.

$\Rightarrow$ Suppose $x \in \overline{A}$. Let B_n be (open) balls of radius $\frac{1}{n}$ centered at x. Since x is in the closure of A, $B_n \cap A$ is non-empty for all n. So we define x_n to be any element that lies in this intersection. This sequence (x_n) converges to x by construction. $\square$

Theorem

Let X, Y be metrizable topological spaces and $f : X \to Y$ a map between them. Then f is continuous if and only if $(x_n) \to x$ implies $f(x_n) \to f(x)$ for all convergent sequences (x_n).

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Proof.

$\Rightarrow$ Suppose f is continuous. Let V be a neighbourhood of $f(x)$. Then $f^{-1}(V)$ is a neighbourhood of x. So (x_n) eventually in $f^{-1}(V)$ hence $f(x_n)$ eventually in V.

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Proof.

$\Rightarrow$ Suppose f is continuous. Let V be a neighbourhood of $f(x)$. Then $f^{-1}(V)$ is a neighbourhood of x. So (x_n) eventually in $f^{-1}(V)$ hence $f(x_n)$ eventually in V.
$\Leftarrow$ Suppose $A \subset Y$ is closed. Let $(x_n) \to x$ be a convergent sequence in $f^{-1}(A)$. By assumption $f(x_n) \to f(x)$. Since A is closed, we know $f(x) \in A$ (by previous theorem) thus $x \in f^{-1}(A)$. Thus $f^{-1}(A)$ contains its limit points and is therefore closed (again by the previous theorem). □

But why...

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- Why not

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- Why not
- Equivalences are fun
- Convergence is what topology is all about!!

Prerequisite definitions

Definition (Preorder)

Let Λ be a set. Then a binary relation on Λ $\preceq$ is a preorder if it is

- reflexive and,
- transitive

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Definition (Directed set)

Let Λ be a set with $\preceq$ as a preorder. Then Λ is called an (upward) directed set if there is an upper bound for every pair of elements. In other words, for every $\lambda_1, \lambda_2 \in \Lambda$, we can find $\mu \in \Lambda$ such that $\lambda_1 \preceq \mu$ and $\lambda_2 \preceq \mu$.

Examples of directed sets

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- $\Lambda := \mathcal{P}(X)$; $\preceq := \subseteq$ or $\preceq := \supseteq$ for any set X
- $\Lambda := \mathbb{N}$; $x \preceq y \Leftrightarrow x//4 \leq y//4 \Leftrightarrow \lfloor \frac{x}{4} \rfloor \leq \lfloor \frac{y}{4} \rfloor$

Nets: The what

Definition (Net)

Let X be any set and Λ an upward directed set. Then a function $f : \Lambda \to X$ is called a *net* (in X). It is typically denoted $(x_\lambda)_{\lambda \in \Lambda}$ where $x_\lambda := f(\lambda)$.

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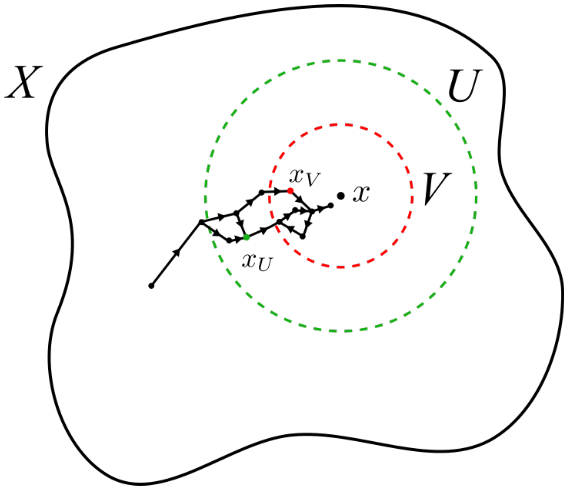
Definition (Net)

Let X be any set and Λ an upward directed set. Then a function $f : \Lambda \to X$ is called a *net* (in X). It is typically denoted $(x_\lambda)_{\lambda \in \Lambda}$ where $x_\lambda := f(\lambda)$.

Definition (Convergence of nets)

A net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space X converges to a point $x \in X$ if for every open neighbourhood U of x there exists $\lambda' \in \Lambda$ such that we have $x_\lambda \in U$ for all $\lambda' \preceq \lambda$.

Nets: The how



Nets: The why

Theorem

Let X be a topological space and A be some subset of X. Then $x \in \overline{A}$ if and only if there exists a net in A converging to x.

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$\Leftarrow$ Suppose there exists a net (x_λ) in A converging to some x. This implies every neighbourhood of x has non-trivial intersection with A. Thus $x \in \overline{A}$.

Nets: The why

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Proof.

$\Leftarrow$ Suppose there exists a net (x_λ) in A converging to some x. This implies every neighbourhood of x has non-trivial intersection with A. Thus $x \in \overline{A}$.
$\Rightarrow$ Suppose $x \in \overline{A}$. We define Λ to be the collection of open neighbourhoods of x and $\preceq := \supseteq$. For $U \in \Lambda$, we define x_U by choosing any point in $U \cap A$ which is non-empty by assumption. Then $x_U \to x$ by construction. □

Nets: The why

Theorem

Let $f : X \to Y$ be a map between topological spaces. Then f is continuous if and only if $(x_\lambda) \to x \in X$ implies $f(x_\lambda) \to f(x) \in Y$ for every convergent net (x_λ) in X.

Nets: The why

Theorem

Let $f : X \to Y$ be a map between topological spaces. Then f is continuous if and only if $(x_\lambda) \to x \in X$ implies $f(x_\lambda) \to f(x) \in Y$ for every convergent net (x_λ) in X.

Proof.

$\Rightarrow$ Let V be a neighbourhood of $f(x)$, implying $f^{-1}(V)$ is an open neighbourhood of x (by continuity if f). Therefore x_λ eventually in $f^{-1}(V)$ and hence $f(x_\lambda)$ eventually in V.
$\Leftarrow$ Suppose $A \subset Y$ is closed. Let x_λ be a convergent net in $f^{-1}(A)$ converging to x. Then by assumption $f(x_\lambda) \to f(x)$ so $f(x) \in A$ by previous theorem and closure of A. Therefore $x \in f^{-1}(A)$ and $f^{-1}(A)$ is closed (again by the previous theorem). $\square$

Joke

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Now that we're all *caught up* with nets...

Filters

Definition (Filter)

Let X be a set. Then a *filter* $\mathscr{F}$ is a collection of non-empty subsets of X such that

- For all $A, B \in \mathscr{F}$, $A \cap B \in \mathscr{F}$
- For all $A \in \mathscr{F}$, $A \subset B$ implies $B \in \mathscr{F}$

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Definition (Filter convergence)

A filter $\mathscr{F}$ in a topological space X converges to a point x if it contains all the neighbourhoods of x.

Examples of filters

Examples

Let X be a topological space and $x \in X$. Then we can have

- $\mathscr{F} := \{X\}$ (Trivial)
- $\mathscr{F} := \{A \subset X : x \in A\}$
- $\mathscr{F} := \{A \subset X : A \text{ is a neighbourhood of } x\}$ (Neighbourhood filter)

Nets vs Filters

Theorem

Let X be a topological space and (x_λ) a net in X. Then there exist a filter $\mathscr{F}$ in X such that (x_λ) converges to x if and only if $\mathscr{F}$ also converges to x.

Nets vs Filters

Theorem

Let X be a topological space and (x_λ) a net in X. Then there exist a filter $\mathscr{F}$ in X such that (x_λ) converges to x if and only if $\mathscr{F}$ also converges to x.

Proof.

We define $\mathscr{F} := \{A \subset X : x_\lambda \text{ eventually in } A\}$. Then if $x_\lambda \to x$ then $\mathscr{F}$ must contain the neighbourhood filter of x. Conversely if $\mathscr{F} \to x$ then $\mathscr{F}$ contains the neighbourhood filter of x (by definition of convergence of filters) so x_λ is eventually in every neighbourhood of x. □

Nets vs Filters

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Nets vs Filters

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Let X be a topological space and $\mathscr{F}$ a filter in X. Then there exists a net (x_λ) in X such that $\mathscr{F}$ converges to x if and only if (x_λ) converges to x.

Proof.

Define $\Lambda := \{(x, A) \in X \times \mathscr{F} : x \in A\}$ and $(x, A) \preceq (y, B) \Leftrightarrow A \supseteq B$. The net is simply the projection onto the first element, i.e., $f(x, A) := x$. If $\mathscr{F} \to x$, then every neighbourhood of x is in $\mathscr{F}$, so the net is eventually in every neighbourhood of x.

Suppose $x_{(y,A)}$ converges to x. Let U be a neighbourhood of x. There exists some (y, A) such that for all $(y, A) \preceq (z, B)$ we have $f(z, B) = z \in U$. This holds for all $z \in B$ so $B \subset U$. By assumption $B \in \mathscr{F}$ implying $U \in \mathscr{F}$. □

Nets vs Filters

Nets	**Filters**
Intuitive; complex definition	Unintuitive; simple definition
Requires a separate set	Uses only subsets of the space
Function	Set

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IT'S OBVIOUSLY FILTERS

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GOOGLE HOW COOL ULTRAFILTERS ARE

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NO, I MEAN IT. GOOGLE IT!

Exercises

0. Can you find uncountably many subsets of $\mathbb{N}$ such that any pair has only finite intersection?
1. Define what a subnet should be. (*Warmup: Give a rigorous definition for subsequences.*). Show that a topological space X is compact if and only if every net in X has a convergent subnet.
2. Show that $Y \subset X$ is open if and only if $Y \in \mathscr{F}$ for every filter $\mathscr{F}$ that converges to a point in Y.
3. Show that if $\mathscr{U}$ is an ultrafilter in a set X then for any $A \subset X$, you have $A \in \mathscr{U}$ or $X - A \in \mathscr{U}$.

Resources/References

- Gert Pederson, *Analysis Now*
- James Munkres, *Topology*
- Saitulaa Naranong, *Translating between Nets and Filters*,

Slides: `http://individual.utoronto.ca/rishibhp/notes/cumc22.pdf`

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