Function, Function on the Disk

What says this theorem of Pick's

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June 2023

Joke

Most people walk into bars, complex analysts walk into poles

(Source: First heard from Kunal Chawla)

Theorem Statement

Theorem (Nevanlinna–Pick Theorem)

Let D be the (open) unit disk in the complex plane. Let $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ all contained in D. Then there exists a holomorphic function $f : D \to D$ satisfying $f(z_j) = w_j$ for $j = 1, \ldots, n$ if and only if the quadratic form

$$Q_n(t_1, \ldots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} t_j \overline{t_k}$$

is non-negative

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$$Q_n = \left(\frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} \right)_{j,k}$$

is positive semi-definite

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$$Q_n = \left(\frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} \right)_{j,k}$$

has only non-negative eigenvalues

Basics

Definition

$f : \mathbb{C} \to \mathbb{C}$ is holomorphic at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

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exists

Theorem. f holomorphic $\Leftrightarrow$ f infinitely differentiable $\Leftrightarrow$ f analytic

Basics

Theorem (Schwarz's Lemma)

Suppose $f : D \to D$ *is a holomorphic function such that* $f(0) = 0$*. Then*

1. $|f(z)| \leq |z|$ *for all* $z \in D$
2. *If* $|f(z_0)| = |z_0|$ *for some non-zero* z_0*, then* $f(z) = \lambda z$ *where* $|\lambda| = 1$

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Proof.

Trust me bro □

Automorphisms of the Disk

Lemma

If $f : D \to D$ is an automorphism, then it is of the form

$$f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}$$

for $a \in D$.

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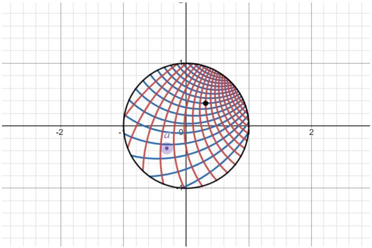


Figure: Desmos graph

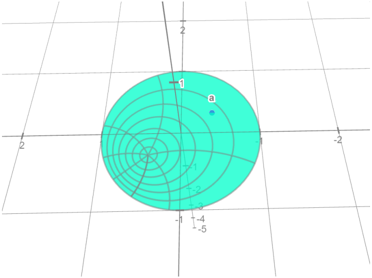


Figure: math3d graph

Proof of Lemma

$$
\begin{aligned}
(|z|^2 - 1)(|a|^2 - 1) &> 0 \\
|z|^2 |a|^2 - |z|^2 - |a|^2 + 1 &> 0 \\
|z|^2 |a|^2 + 1 &> |z|^2 + |a|^2 \\
|z|^2 |a|^2 - a\bar{z} - \bar{a}z + 1 &> |z|^2 - a\bar{z} - \bar{a}z + |a|^2 \\
(1 - \bar{a}z)(1 - a\bar{z}) &> (z - a)(\bar{z} - \bar{a}) \\
\frac{(z - a)(\bar{z} - \bar{a})}{(1 - \bar{a}z)(1 - a\bar{z})} &< 1 \\
\left| \frac{z - a}{1 - \bar{a}z} \right|^2 &< 1
\end{aligned}
$$

Proof of Lemma

Let f be an automorphism with $f(0) = a$. Define $g(z) = \frac{z+a}{1+\bar{a}z}$. Then $g^{-1} \circ f$ fixes the origin so $\left|(g^{-1} \circ f)(z)\right| \leq |z|$. The same thing holds for $f^{-1} \circ g$. Therefore $\left|(g^{-1} \circ f)(z)\right| = |z|$. Then by Schwarz's lemma

$$(g^{-1} \circ f)(z) = \lambda z$$

$$f(z) = g(\lambda z) = \lambda \frac{z + a\bar{\lambda}}{1 + \overline{a\bar{\lambda}}z}$$

Theorem

Suppose $f : D \to D$ *is any holomorphic map. Then for any* $z_1, z_2 \in D$ *we have*

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_2)} f(z_1)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right|$$

Equality for $z_1 \neq z_2$ *occurs if and only if* f *is a Möbius transform of the disk.*

Proof.
The map

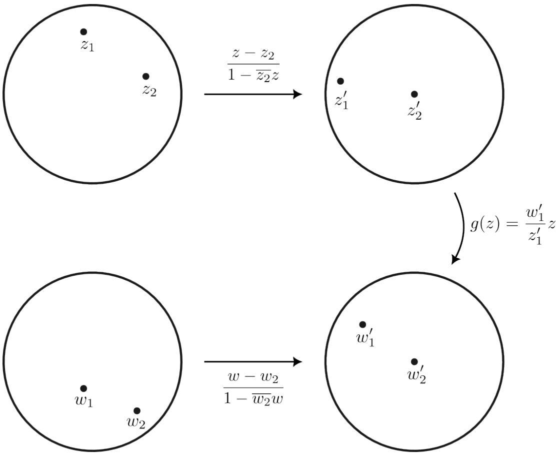
$$g(z) = \frac{f(z) - f(z_2)}{1 - \overline{f(z_2)}f(z)}$$

has a 0 at z_2. Then $g \circ h^{-1}$ is map that fixes the origin where

$$h(z) = \frac{z - z_2}{1 - \overline{z_2}z}$$

Therefore $\left|(g \circ h^{-1})(z)\right| \leq |z|$ so $|g(z)| \leq |h(z)|$. If equality occurs at some non-zero z, then g is a Möbius transform and hence so is f. □

$n = 2$ (Sufficient)



Theorem Statement

Theorem (Pick's Theorem)

Let D be the (open) unit disk in the complex plane. Let $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ all contained in D. Then there exists a holomorphic function $f : D \to D$ satisfying $f(z_j) = w_j$ for $j = 1, \ldots, n$ if and only if the matrix

$$Q_n = \left(\frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} \right)_{j,k}$$

is positive semi-definite

$n = 2$

- Consider the matrix $Q_2 = \begin{pmatrix} \frac{1-|w_1|^2}{1-|z_1|^2} & \frac{1-\overline{w_1}w_2}{1-\overline{z_1}z_2} \\ \frac{1-\overline{w_2}w_1}{1-\overline{z_2}z_1} & \frac{1-|w_2|^2}{1-|z_2|^2} \end{pmatrix}$
- Positive semidefinite if and only if

$$0 \le \det(Q_2) = \frac{1-|w_1|^2}{1-|z_1|^2}\frac{1-|w_2|^2}{1-|z_2|^2} - \frac{1-\overline{w_1}w_2}{1-\overline{z_1}z_2}\frac{1-\overline{w_2}w_1}{1-\overline{z_2}z_1}$$

$$\frac{(1-|w_1|^2)(1-|w_2|^2)}{|1-w_1\overline{w_2}|^2} \ge \frac{(1-|z_1|^2)(1-|z_2|^2)}{|1-z_1\overline{z_2}|^2}$$

- We have the following identity for $z\overline{w} \neq 1$

$$1 - \left|\frac{z-w}{1-z\overline{w}}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$$

so

$$\frac{(1-|w_1|^2)(1-|w_2|^2)}{|1-w_1\overline{w_2}|^2} \geq \frac{(1-|z_1|^2)(1-|z_2|^2)}{|1-z_1\overline{z_2}|^2}$$

$$\Leftrightarrow \left|\frac{w_1-w_2}{1-w_1\overline{w_2}}\right| \leq \left|\frac{z_1-z_2}{1-z_1\overline{z_2}}\right|$$

Proof of identity

$$
\begin{aligned}
1-\left|\frac{z-w}{1-z\overline{w}}\right|^2 &= \frac{|1-z\overline{w}|^2-|z-w|^2}{|1-z\overline{w}|^2}\\
&= \frac{(1-z\overline{w})(1-\overline{z}w)-(z-w)(\overline{z}-\overline{w})}{(1-z\overline{w})(1-\overline{z}w)}\\
&= \frac{(1-z\overline{w}-\overline{z}w+|z|^2|w|^2)-(|z|^2-z\overline{w}-w\overline{z}+|w|^2)}{|1-z\overline{w}|^2}\\
&= \frac{1-|z|^2|w|^2-|z|^2-|w|^2}{|1-z\overline{w}|^2}\\
&= \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}
\end{aligned}
$$

Proof

Suppose we are given $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ in the disk with the associated quadratic form

$$Q_n(t_1, \ldots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j \overline{w_k}}{1 - z_j \overline{z_k}} t_j \overline{t_k}$$

Define

$$z_j' = \frac{z_j - z_n}{1 - \overline{z_n} z}$$

and

$$w_j' = \frac{w_j - w_n}{1 - \overline{w_n} w}$$

Then

$$Q_n'(t_1, \ldots, t_n) = \sum_{j,k=1}^{n} \frac{1 - w_j' \overline{w_k'}}{1 - z_j' \overline{z_k'}} t_j \overline{t_k}$$

Proof

- We have

$$\frac{1 - z_j'\overline{z_k'}}{1 - z_j\overline{z_k}} = \frac{1 - |z_n|^2}{(1 - \overline{z_n}z_j)(1 - z_n\overline{z_k})} = \underbrace{\frac{1 - z_n}{1 - \overline{z_n}z_j}}_{\alpha_j} \cdot \underbrace{\frac{1 - \overline{z_n}}{1 - z_n\overline{z_k}}}_{\overline{\alpha_k}}$$

- Similarly

$$\frac{1 - w_j'\overline{w_k'}}{1 - w_j\overline{w_k}} = \underbrace{\frac{1 - w_n}{1 - \overline{w_n}w_j}}_{\beta_j} \cdot \underbrace{\frac{1 - \overline{w_n}}{1 - w_n\overline{w_k}}}_{\overline{\beta_k}}$$

- Therefore $Q_n'(t_1, \dots, t_n) = Q_n(\frac{\beta_1}{\alpha_1}t_1, \dots, \frac{\beta_n}{\alpha_n}t_n)$ so

$$Q_n \geq 0 \Leftrightarrow Q_n' \geq 0$$

Proof

- We want to show there exists f mapping $z'_1, \ldots, z'_{n-1}, 0$ to $w'_1, \ldots, w'_{n-1}, 0$ if and only if $Q'_n \geq 0$
- If f exists it has a power series representation around 0

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

  so $g(z) = f(z)/z$ is holomorphic and maps $z'_1, \ldots, z'_{n-1}$ to $w'_1/z'_1, \ldots, w'_{n-1}/z'_{n-1}$
- If $g(z)$ exists, we can take $f(z) = zg(z)$
- Let $\widetilde{Q}_{n-1}$ be the quadratic form

$$\widetilde{Q}_{n-1}(t_1, \ldots, t_{n-1}) = \sum_{j,k=1}^{n-1} \frac{1 - (w'_j/z'_j)\overline{(w'_k/z'_k)}}{1 - z'_j \overline{z'_k}} t_j \overline{t_k}$$

- We want to show $Q'_n \geq 0 \Leftrightarrow \widetilde{Q}_{n-1} \geq 0$

Proof

- Notice that since $z_n' = w_n' = 0$ we have

$$\begin{aligned} Q_n'(t_1, \ldots, t_n) &= |t_n|^2 + 2\mathrm{Re} \sum_{j=1}^{n-1} t_j \overline{t_n} + \sum_{j,k=1}^{n-1} \frac{1 - w_j' \overline{w_k'}}{1 - z_j' \overline{z_k'}} t_j \overline{t_k} \\ &= \left| t_n + \sum_{j=1}^{n-1} t_j \right|^2 - \left| \sum_{j=1}^{n-1} t_j \right|^2 + \sum_{j,k=1}^{n-1} \frac{1 - w_j' \overline{w_k'}}{1 - z_j' \overline{z_k'}} t_j \overline{t_k} \\ &= \left| \sum_{j=1}^{n} t_j \right|^2 + \sum_{j,k=1}^{n-1} \left(\frac{1 - w_j' \overline{w_k'}}{1 - z_j' \overline{z_k'}} - 1 \right) t_j \overline{t_k} \end{aligned}$$

Proof

- Then

$$\frac{1-w_j'\overline{w_k'}}{1-z_j'\overline{z_k'}}-1=\frac{1-w_j'\overline{w_k'}-1+z_j'\overline{z_k'}}{1-z_j'\overline{z_k'}}=\frac{1-(w_j'/z_j')\overline{(w_k'/z_k')}}{1-z_j'\overline{z_k'}}z_j'\overline{z_k'}$$

so

$$\begin{aligned}Q_n'(t_1,\ldots,t_n)&=\left|\sum_{j=1}^{n}t_j\right|^2+\sum_{j,k=1}^{n-1}\left(\frac{1-w_j'\overline{w_k'}}{1-z_j'\overline{z_k'}}-1\right)t_j\overline{t_k}\\&=\left|\sum_{j=1}^{n}t_j\right|^2+\sum_{j,k=1}^{n-1}\frac{1-(w_j'/z_j')\overline{(w_k'/z_k')}}{1-z_j'\overline{z_k'}}(z_j't_j)\overline{(z_k't_k)}\end{aligned}$$

Proof

- Finally we have

$$Q'_n(t_1, \ldots, t_n) = \left| \sum_{j=1}^{n} t_j \right|^2 + \widetilde{Q}_{n-1}(z_1 t_1, \ldots, z_{n-1} t_{n-1})$$

so

$$Q'_n \geq 0 \Leftrightarrow \widetilde{Q}_{n-1} \geq 0$$

Uniqueness

Lemma

Given $\{z_1, \ldots, z_n\}$ *and* $\{w_1, \ldots, w_n\}$*, there exists a unique* $f : D \to D$ *satisfying* $f(z_j) = w_j$ *for* $1 \leq j \leq n$ *if and only if* $\det(Q_n) = 0$.

$n = 2$

- Suppose we are given $\{z_1, z_2\}$ and $\{w_1, w_2\}$.
- Can assume $z_2 = w_2 = 0$
- Exists unique $f : D \to D$ with $f(z_1) = w_1$ and $f(0) = 0$ if and only if $|z_1| = |w_1|$.
  - $\Rightarrow$ Schwarz's lemma
  - $\Leftarrow$ if $|w_1| < |z_1|$ take $g : D \to D$ such that $g(z_1) = w_1/z_1$ and define $f(z) = zg(z)$
- By definition

$$Q_2 = \begin{pmatrix} \frac{1-|w_1|^2}{1-|z_1|^2} & 1 \\ 1 & 1 \end{pmatrix}$$

which has determinant 0 if and only if $|z_1| = |w_1|$

Inductive step

- Recall

$$Q_n(t_1, \ldots, t_n) = \left|\sum_{j=1}^{n} t_j\right|^2 + \sum_{j,k=1}^{n-1} \left(\frac{1 - w_j\overline{w_k}}{1 - z_j\overline{z_k}} - 1\right) t_j\overline{t_k}$$

$$= \sum_{j,k=1}^{n} t_j\overline{t_k} + \sum_{j,k=1}^{n-1} \underbrace{\frac{1 - (w_j/z_j)\overline{(w_k/z_k)}}{1 - z_j\overline{z_k}}}_{a_{j,k}} (z_j t_j)\overline{(z_k t_k)}$$

- Therefore if the entries of $\widetilde{Q}_{n-1}$ are $a_{j,k}$ then

$$Q_n = \begin{pmatrix} & & & 1 \\ & 1 + a_{j,k} z_j \overline{z_k} & & 1 \\ & & & \vdots \\ 1 & \cdots & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} & & & 0 \\ & a_{j,k} z_j \overline{z_k} & & 0 \\ & & & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Non-uniqueness

Lemma

Let $\{z_1, \ldots, z_n\}$ *and* $\{w_1, \ldots, w_n\}$ *be points in the unit disk such that* $\det(Q_n) > 0$*. Let*

$$\mathscr{S} := \{f : D \to D | f(z_j) = w_j, 1 \leq j \leq n\}$$

Fix $z_0 \neq z_j \in D$*. Then*

$$\mathscr{S}(z_0) := \{f(z_0) : f \in \mathscr{S}\}$$

is a closed disk in D*. Moreover given* $w \in \partial\mathscr{S}(z_0)$*, there exists a unique* $f \in \mathscr{S}$ *satisfying* $f(z_0) = w$*.*

$n = 1$

- By generalised Schwarz's lemma, $|f(z_0)|$ must satisfy

$$\left| \frac{f(z_0) - w_1}{1 - \overline{w_1} f(z_0)} \right| \leq \left| \frac{z_0 - z_1}{1 - \overline{z_1} z_0} \right|$$

- **Claim 1.** Set of points satisfying the inequality forms a closed disk
- **Claim 2.** Every point in this disk is achieved by some $f \in \mathscr{S}$
  - Follows from Nevanlinna–Pick Theorem with $n = 2$

Proof of Claim 1.

$$\frac{w - w_1}{1 - \overline{w_1} w} \cdot \frac{\overline{w} - \overline{w_1}}{1 - w_1 \overline{w}} \leq R^2$$

$$|w|^2 - 2\mathrm{Re}(\overline{w_1} w) + |w_1|^2 \leq R^2 (1 - 2\mathrm{Re}(\overline{w_1} w) + |w|^2 |w_1|^2)$$

$$|w|^2 - 2 \frac{1 - R^2}{1 - R^2 |w_1|^2} \mathrm{Re}(\overline{w_1} w) \leq \frac{R^2 - |w_1|^2}{1 - R^2 |w_1|^2}$$

$$\left| w - \frac{1 - R^2}{1 - R^2 |w_1|^2} w_1 \right|^2 \leq \frac{R^2 - |w_1|^2}{1 - R^2 |w_1|^2} + \left(\frac{1 - R^2}{1 - R^2 |w_1|^2} \right)^2 |w_1|^2$$

$$\left| w - \frac{1 - R^2}{1 - R^2 |w_1|^2} w_1 \right|^2 \leq R^2 \left(\frac{1 - |w_1|}{1 - R^2 |w_1|^2} \right)^2$$

Exercises

1. Translate the theorem to the upper half-plane.
2. What if we instead had $|w_j| \leq 1$? (some things become trivial but everything still works out!)
3. Show that
$$d(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|$$
forms a metric on D. What are the isometries in this metric? What are the 'straight lines'?

References

- Garnett, John. *Bounded Analytic Functions*

Slides:

http://individual.utoronto.ca/rishibhp/notes/cumc23.pdf

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