On the Classification of UHF-algebras

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February 2022

1 Introduction

A Uniformly Hyper-Finite C^*-algebra, or UHF-algebra, is one which is (isomorphic) to the inductive limit of the sequence

$$M_{k_1} \xrightarrow{\varphi_1} M_{k_2} \xrightarrow{\varphi_2} M_{k_3} \xrightarrow{\varphi_3} \cdots \tag{1.1}$$

where the k_i are natural numbers and φ_i are unital $*-$homomorphisms (in other words they are homomorphisms that preserve the star operation and map the unit to the unit). It is a delightful fact in C^*-algebra theory that we have a complete classification of these algebras and moreover, it is given by a single parameter! This parameter is called a supernatural number. To be precise a supernatural number is a formal product of the form

$$2^{n_1}3^{n_2}5^{n_3}\cdots$$

where $n_i \in \{0, 1, \ldots, \infty\}$. If all $n_i < \infty$ and all but finitely many of them are 0, then we have a regular natural number. Thus these supernatural numbers form a generalisation of sorts of natural numbers (in fact, Dixmier originally called these 'generalised integers'). What we will find is that to every supernatural number we can associate a (unique) UHF-algebra and vice-versa.

2 Supernatural numbers to UHF-algebras

The first thing to note is that a unital homomorphism $\varphi : M_m \to M_n$ exists if and only if $m|n$ (this mean m divides n). In this case the homomorphism is given by

$$a \mapsto diag(a, \ldots, a)$$

where a is repeated $\frac{n}{m}$ times. Thus for every k_i it must be true that $k_i|k_{i+1}$.

Suppose we have a supernatural number n whose sequences of exponents is $\{n_1, n_2, n_3, \ldots\}$. In order to find the corresponding UHF-algebra, we will need to define a sequence of algebras first. We will define

$$k_i = \prod_{j=1}^{i} p_j^{\min\{n_j, i\}}$$

where p_j denotes the j-th prime number. Then clearly $k_i|k_{i+1}$. Thus the UHF-algebra we associate to n is the direct limit of

$$M_{k_1} \to M_{k_2} \to \cdots$$

We will call this direct limit M_n. In the case when n is a natural number, the two seemingly different definitions of M_n agree since in this case the sequence $\{k_i\}$ eventually becomes constant at n.

3 UHF-Algebras to Supernatural numbers

The more interesting question is how can we associate a UHF-algebra A with a supernatural number. Suppose A is the direct limit of

$$M_{k_1} \to M_{k_2} \to \dots$$

Essentially by definition, A is the smallest algebra that contains all the M_{k_i} (the φ_i are injective thus each map is an inclusion of sorts). Thus we want the 'smallest' supernatural number n such that the corresponding algebra M_n contains all the M_{k_i}. By previous discussion, we know this must mean that n is a multiple of all the k_i. From number theory, we already know then what n must be: the LCM or lowest common multiple of all the k_i. If we write the k_i in their formal prime decomposition, we can even give an explicit construction of n. Suppose the prime factorisation of each k_i is given by

$$k_i = \prod_{j=1}^{\infty} p_j^{a_{i,j}}$$

As with natural numbers, the lowest common multiple of all the k_i can be found by taking the largest exponent for each prime. In particular we get that

$$n = \prod_{j=1}^{\infty} p_j^{\sup_i\{a_{i,j}\}}$$

We claim that $A \cong M_n(\mathbb{C})$. First we consider the idea of finding the direct limit of a 'subsequence' of matrix algebras. Suppose B is direct limit of

$$M_{l_1} \to M_{l_2} \to \cdots$$

Suppose now we take a subsequence of $\{l_i\}$ which we will call $\{l'_i\}$. Then it must be true that if

$$M_{l'_1} \to M_{l'_2} \to \cdots \to B'$$

then $B' \cong B$. This is because each matrix algebra in the sequence of M_{l_i} contains the previous ones. Thus by removing the algebras that are in-between, we lose no information.

Since the M_{k_i}, which is used to construct A, form a subsequence of the sequence that is used to construct M_n, we get that $A \cong M_n$.

As an aside, we can recover the k_i even if we have somehow forgotten the sequence from which A arose (or perhaps we met the algebra in the wild and it neglected to share its sequence in the introduction). We apply the normalised trace to all one-dimensional projections and look at the collection of numbers that arises. This is going to be a collection of the so-called unit fractions (i.e. fractions whose numerator is 1). The denominators of these fractions are exactly the k_i. This is because the (standard) trace of any one-dimensional projection in M_k is 1 so the normalised trace will be $\frac{1}{k}$. One might wonder why use the normalised trace rather than the standard trace. This is because the normalised trace extends to the higher matrix algebras. To be precise suppose τ_i is the normalised trace on M_{k_i}. Then $\tau_i = \tau_{i+1} \circ \varphi_i$.

Suppose $a \in M_{k_i}$. Then

$$\begin{aligned}
(\tau_{i+1} \circ \varphi_i)(a) &= \tau_{i+1}(diag(a, \dots, a)) \\
&= \frac{1}{k_{i+1}} \mathrm{Tr}(diag(a, \dots, a)) \\
&= \frac{1}{k_{i+1}} \frac{k_{i+1}}{k_i} \mathrm{Tr}(a) \\
&= \tau_i(a)
\end{aligned}$$

This means that the normalised trace will map a to the same value, regardless what space we think of a inhabiting.

4 The Hidden K_0

We are so close to talking about K_0, it seems a shame to not give it some of the spotlight. Consider our previous set of unit fractions that told us the relevant k_i (recall this was done be applying the normalised trace to one-dimensional projections). Let us apply normalised trace to *all* projections (not just the one-dimensional ones). This gets us a new set of rational numbers which we call Q. We can use the unit fractions in Q again to find all the M_k that our contained in our algebra, not just those used to define the limit.

More importantly, however, let us consider the subgroup of rational numbers generated by Q, which denote $\widehat{Q}$. We claim that this group is simply

$$\frac{1}{n}\mathbb{Z}$$

This makes sense if n is a natural number but how should be interpret in the general case of a supernatural number? As before, it is direct limits that come to the rescue. Suppose n is a supernatural number and let $\{k_i\}$ be as defined in section 2. Then

$$\frac{1}{k_1}\mathbb{Z} \to \frac{1}{k_2}\mathbb{Z} \to \cdots \to \frac{1}{n}\mathbb{Z}$$

where each arrow is the (group) homomorphism that maps $\frac{1}{k_i}$ to $\frac{1}{k_{i+1}}$. As before, if n is a true natural number, we get the result we expect. With this, it is clear why $\widehat{Q} = \frac{1}{n}\mathbb{Z}$ (for each M_{k_i} the corresponding group is generated by $\frac{1}{k_i}$ since this is what the one-dimensional projections are mapped to. In particular the numbers in Q are going to be integer multiples of $\frac{1}{k_i}$ so they generate the claimed group).

Thus to every UHF-algebra, we can associate an abelian group. This is exactly K_0! In fact we can do a similar classification for the irrational rotation algebras, but that is a story for another time.

5 Categoric Viewpoint

Before we wrap everything up, one final thing to mention is that we can express everything above in the language of category theory. In particular we have the category of supernatural numbers where $n \to m$ if $n|m$ and the category of UHF-algebras where the arrows are given by the unital $*-$homomorphisms. Then the above arguments give functors in both directions that are in fact inverses of one another.