1.4 Chapter 2

1.4.1 The Subspace Topology

1. For $x \in A$ let U_x be the open set containing x and is contained in A. Since for each $x \in A$, there is some U_x, we see that $A \subset \bigcup_{x\in A} U_x$. On the other hand since each U_x is contained in A the reverse inclusion also holds. Hence
$$A = \bigcup_{x\in A} U_x$$
Since the arbitrary union of open sets is open, A is open.
2. No
3. We see that ϕ and X are in $\mathcal{T}_C$ because $X - \phi$ is all of X and $X - X = \phi$ which is in particular countable.
Let $\{U_\lambda\}$ be some indexed collection of non-empty elements of $\mathcal{T}_C$. Then we compute that
$$X - \bigcup U_\lambda = \bigcap (X - U_\lambda)$$
The latter set is countable as it is a subset of a countable set.
Let $U_1, \dots, U_n$ be some non-empty elements of $\mathcal{T}_C$. Then
$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_1)$$
Since the finite union of countable sets is countable, the above set is countable. Hence we conclude that $\mathcal{T}_C$ is a topology.
4. (a) Suppose $\{\mathcal{T}_a\}$ is a family of topologies of X. They all contain ϕ and X, hence $\bigcap \mathcal{T}_a$ contains ϕ and X.
Suppose $\{U_\lambda\}$ is some indexed collection of elements of $\bigcap \mathcal{T}_a$. Then each U_λ is in each $\mathcal{T}_a$. Since each $\mathcal{T}_a$ is a topology, each $\mathcal{T}_a$ contains the union of all $\{U_\lambda\}$ and hence the union is in $\bigcap \mathcal{T}_a$. Identical argument for finite intersections.
However, $\bigcup \mathcal{T}_a$ is not necessarily a topology. Consider $\mathcal{T}_1 = \{\phi, \{a,b,c\}, \{a,b\}\}$ and $\mathcal{T}_2 = \{\phi, \{a,b,c\}, \{b,c\}\}$. We can easily check that both are topologies. Their union contains both $\{a,b\}$ and $\{b,c\}$ but not their intersection $\{b\}$ and hence cannot be a topology.
(b) $\bigcap \{\mathcal{T}_\alpha\}$ is the largest topology contained in all $\mathcal{T}_\alpha$. Suppose we had some other $\mathcal{T}' \subset \mathcal{T}_\alpha$ for all α. Then $T' \subset \bigcap \{\mathcal{T}_\alpha\}$.
Let $\{\mathcal{S}_\beta\}$ be such that each $\mathcal{S}_\beta$ is a topology satisfying $\mathcal{S}_\beta \supset \bigcup \{\mathcal{T}_\alpha\}$. Then $\bigcap \mathcal{S}_\beta$ is the smallest topology containing all $\mathcal{T}_\alpha$.
(c) If $\mathbb{R}_l$ is finer than $\mathbb{R}_K$ then there must exist
5. Let $\mathcal{A}$ be a basis for a topology on X. Let $\{\mathcal{T}_A\}$ be all the topologies on X that contain $\mathcal{A}$. Let $\mathcal{T} = \bigcap \mathcal{T}_A$ and $\mathcal{T}'$ be the topology generated by $\mathcal{A}$. We recall that $\mathcal{T}'$ is simply the collection of all unions of elements of $\mathcal{A}$. As each of $\mathcal{T}_A$ is a topology, they will contain each of the unions and hence the collection of all unions will be in $\mathcal{T}$. This implies that $\mathcal{T}' \subset \mathcal{T}$.
The reverse inclusion is clear because $\mathcal{T}'$ is a topology containing $\mathcal{A}$ and hence any element of $\bigcap \mathcal{T}_A$ can only contain elements from $\mathcal{T}'$.

1.4.2 Closed sets and limit points

6. (a) We recall that $\overline{C} = C \cup C'$. Thus all we need show is that $A' \subset B'$. Let $x \in A'$ and U be some neighbourhood containing x. We know that U intersects A at some point other than A. Thus it must also intersect B at some point other than x. Then $U \cap (B - \{x\})$ is non-empty so $x \in B'$.

(b) From the previous question, it is clear that $\overline{A}\cup\overline{B}\subset\overline{A\cup B}$. So we simply need show the reverse inclusion.
Note that $A\cup B\subset\overline{A}\cup\overline{B}$ since $A\subset\overline{A}$ and $B\subset\overline{B}$. Therefore by the previous question $\overline{A\cup B}\subset\overline{\overline{A}\cup\overline{B}}$. Note that $\overline{A}\cup\overline{B}$ is closed since the union of two closed sets is closed. Hence $\overline{A\cup B}\subset\overline{A}\cup\overline{B}$.

(c) Clearly $A_\alpha\subset\bigcup A_\alpha$ for each α. Therefore by part a, $\overline{A_\alpha}\subset\overline{\bigcup A_\alpha}$. Then $\bigcup\overline{A_\alpha}\subset\overline{\bigcup A_\alpha}$. In order to see that equality does not hold, we take $A_n=(0,1-\frac{1}{n}$. Then $\overline{A_n}=[0,1-\frac{1}{n}]$ so $\bigcup\overline{A_n}=[0,1)$. On the other hand $\overline{\bigcup A_n}=\overline{(0,1)}=[0,1]$.

7. It's possible that as we change our U, the neighbourhood intersects different A_α (see example above), preventing x from being in the closure of any given A_α.
8. (a) We simply use Q6a to infer that $\overline{A\cap B}\subset\overline{A}$ and $\overline{A\cap B}\subset\overline{B}$ to conclude that $\overline{A\cap B}\subset\overline{A}\cap\overline{B}$. However equality does not hold (consider $A=(0,1)$ and $B=(1,2)$ in the standard topology on $\mathbb{R}$).
   (b) Same as above to conclude that $\overline{\bigcap A_\alpha}\subset\bigcap\overline{A_\alpha}$ but not necessarily equal.
9. Suppose $x\in\overline{A}$ and $y\in\overline{B}$. Then $U\times V$ be an open set containing $x\times y$. Then U is a neighbourhood of x and V is a neighbourhood of y so their intersection with A and B respectively is non-empty. Thus the intersection of $U\times Y$ with $A\times B$ is non-empty. Hence $x\times y\in\overline{A\times B}$.
   Suppose $x\times y\in\overline{A\times B}$. If $x\in A$ we are done. The suppose $x\notin A$. Let U be any open set containing x. Then $U\times Y$ is a neighbourhood of $x\times y$ hence intersections $A\times B$. This means that U intersects A and any point in this intersection is a point different from x implying that $x\in A'\subset\overline{A}$. Arguing similarly for y, we conclude that $x\times y\in\overline{A}\times\overline{B}$.
10. Let X be a topological space with the order topology. Let $a,b\in X$ such that $a<b$ (in particular then a,b are distinct). Now consider the interval (a,b). If this interval is empty, we can take $U_1=(-\infty,b)$ and $U_2=(a,\infty)$ as disjoint neighbourhoods of a and b (the intersection of U_1 and U_2 is exactly (a,b)). If the interval is not empty, we take $x\in(a,b)$ and consider $U_1=(-\infty,x)$ and $U_2=(x,\infty)$ as neighbourhoods instead where disjointness is clear.
11. Let X and Y be two Hausdorff spaces. Let $x_1\times y_1$ and $x_2\times y_2$ be two distinct points in $X\times Y$ (equality of one component implies inequality of the other). Suppose x_1,x_2 are distinct and that y_1,y_2 are distinct. Let U_1,U_2 be disjoint neighbourhoods for x_1 and x_2 respectively and similarly let V_1,V_2 be disjoint neighbourhoods of y_1 and y_2. Then we claim that $U_1\times V_1$ is disjoint from $U_2\times V_2$. Indeed, suppose this were not the case the point $x'\times y'$ resided in their intersection. Then this would imply that x' is in the intersection of U_1 and U_2 and that y' is in the intersection of V_1 and V_2 leading to a contradiction.
    Suppose instead that $x_1=x_2$ but y_1,y_2 are still distinct. In this case, we let U be any neighbourhood containing x_1 and take V_1,V_2 to be disjoint neighbourhoods of y_1 and y_2 respectively. As before, we conclude that $U\times V_1$ and $U\times V_2$ are disjoint. The argument is essentially identical if $y_1=y_2$ while x_1 and x_2 are different.
12. Let x_1,x_2 be distinct points in some subspace A of X. We take disjoint open sets of X containing the respective points and intersect them with A to find disjoint open sets in A.
13. Suppose X is a Hausdorff space. We show that that the complement of Δ is open. First we note that the complement of Δ is $\{(x,y)\in X\times X : x\neq y\}$. By Hausdorff, we have that there exist disjoint open sets U_1 and U_2 that contain x and y respectively. Then $U_1\times U_2$ is open and does not intersect with Δ. This allows us to write the complement as a union of open sets, implying that Δ is closed.
    Now suppose X is not a Hausdorff space. We show that Δ is not closed as it does not contain all of its limit points. Since X is not Hausdorff, there are elements x,y in X that are distinct but who neighbourhoods always intersect. Now let W be a neighbourhood of $x\times y\in X\times X$. We recall that the basis for the product topology is of the form $U\times V$ where U and V are

open in X and that any open set is some union of the basis elements. Hence in particular this means we can find U and V open in X such that $x \times y \in U \times V$. Since X is not Hausdorff, we know that $U \cap V$ is non-empty, so let z be an element in the intersection. Then $z \times z \in U \times V$, so in particular $W \cap \Delta$ is non-empty. This implies that (x,y) is a limit point of Δ not contained in Δ concluding the proof.

19. (a) Let $x \in \text{Int}\, A$. By definition of $\text{Int}\, A$, there exists an open U such that $x \in U \subset A$. Clearly, U cannot intersect $X - A$. Hence we find a neighbourhood of x that does not intersect $X - A$ and so x is not a limit point of $X - A$ and hence not in $\text{Bd}\, A$.
We can write $\overline{A} = \text{Int}\, A \cup \text{Bd}\, A$ as $\overline{A} = \text{Int}\, A \cup (\overline{A} \cap \overline{X - A}) = (\text{Int}\, A \cup \overline{A}) \cap (\text{Int}\, A \cup \overline{X - A})$. The first term is clearly just $\overline{A}$ and I claim that the second term, $\text{Int}\, A \cup \overline{X - A}$, is just X. Indeed if this true, we see easily the equation is true as $\overline{A} \cap X = \overline{A}$. To verify the claim, we simply need to confirm that $x \in A - \text{Int}\, A$ are accounted for (all other elements are clearly in the union). If $x \in A - \text{Int}\, A$, then there does not exist a neighbourhood of x that is contained in A. In particular, this means that every neighbourhood of x must intersect $X - A$ and so $x \in \overline{X - A}$.
(b) Suppose $\text{Bd}\, A = \phi$. Let x be a limit point of A. By assumption, $x \notin \overline{(X - A)}$ and hence not in $X - A$. Then x must be in A implying that that A contains all of its limit points and is therefore closed. We can apply an identical argument to $X - A$ to conclude that $X - A$ is closed and so A is open. If we instead assume that A is open and closed then $\overline{A} = A$ and $\overline{(X - A)} = X - A$ their intersection is empty by definition. Hence $\text{Bd}\, A = \phi$.
(c) First we note that U is open if and only if $U = \text{Int}\, U$. Suppose $\text{Bd}\, U = \overline{U} - U$ and so $\overline{U} = \text{Bd}\, U \cup U$ From (a), we know that $\overline{U} = \text{Int}\, U \sqcup \text{Bd}\, U$. Using these two equalities and elementary set theory, we find that $\text{Int}\, U = U$ and hence U is open. On the other hand if U is open, we use that $\text{Int}\, U = U$ and the equality from (a) to obtain the desired result.

1.4.3 Continuous Functions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies the $\varepsilon - \delta$ definition of of continuity. Let $x \in \mathbb{R}$ be arbitrary and let V be some neighbourhood of $f(x)$. As V is open in $\mathbb{R}$, it is the union of some open intervals and at least one of these intervals must contain $f(x)$. Let J be one such interval. We choose $\varepsilon = \min\{\sup J - f(x), f(x) - \inf J\}$. Then by assumption, there exists a δ such that $f((x-\delta, x+\delta)) \subset J \subset V$. Hence taking $U = (x-\delta, x+\delta)$ we are done.
2. Consider $f : (0,1) \to \mathbb{R}$ where $f(x) = 69$ for all x. Then $f((0,1)) = \{69\}$ which does not have 69 as a limit point.
3. (a) Suppose i is continuous. Let U' be open in X. Then $i^{-1}(U) = U$ is open in X' by definition of continuity. This implies that $\mathcal{T}'$ is finer than $\mathcal{T}$.
On the other hand suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Let U be open in $\mathcal{T}'$. Then $i^{-1}(U) = U$ which must also be open in $\mathcal{T}$ (as this topology is coarser). Therefore i is continuous.
(b) i is a homeomorphism implies that $\mathcal{T}' \supset \mathcal{T}$ and i^{-1} continuous implies that $\mathcal{T} \supset \mathcal{T}'$ hence $\mathcal{T} = \mathcal{T}'$. If $\mathcal{T} = \mathcal{T}'$, then $i(U) = U$ is open so i^{-1} is continuous.
4. Clearly f is bijective. Let $f(X) = X \times \{y_0\}$. Let $U \times V \cap (X \times \{y_0\} = U \times (V \cap \{y_0\})$ be open in $X \times \{y_0\}$. Then $f^{-1}(U \times \{y_0\}) = U$ which is open implying that f is continuous. Additionally $f(U) = U \times \{y_0\}$ is open in $f(X)$, so f^{-1} is continuous.
5. We have $f : (a,b) \to (0,1)$ given by $f(x) = \frac{x-a}{b-a}$ and $g : [a,b] \to [0,1], g(x) = \frac{x-a}{b-a}$.
6. We define $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 0 \text{ otherwise} \end{cases}$$

which is only continuous at 0.

7. (a)
   (b) Suppose $f : \mathbb{R}_l \to \mathbb{R}_l$ is continuous. Let $x \in \mathbb{R}_l$ be arbitrary and let V be some neighbourhood of $f(x)$. We know there exists a neighbourhood U of x such that $f(U) \subset V$. Let $[y,z) \subset U$ be a basis element containing x. Then $[x,z) \subset [y,z) \subset U$ implying that $f([x,z)) \subset V$. Thus f must be continuous from the right.

   Suppose f is continuous from the right. This means for that all $x \in \mathbb{R}_l$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < y - x < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.
8.
9. (a) We simply use the pasting lemma inductively.
   (b) We define $f : [0,1] \to \mathbb{R}$, given by $f(x) = \frac{1}{x}$ if x is not 0 and $f(x) = 0$ if x is 0. Let $A_0 = \{0\}$ and $A_n = [\frac{1}{n}, 1]$. Then $\bigcup_{i=0}^{\infty} A_i = [0,1]$ and $f|_{A_i}$ for any i is continuous but clearly f itself is not continuous.
   (c) Let $x \in X$ and let U_x be a neighbourhood that only intersects finitely many A_α, say $A_{\alpha_1}, \ldots, A_{\alpha_n}$. Then
   $$U_x = \bigcup_{i=1}^{n} (U_x \cap A_{\alpha_i})$$
   Since $f|A_{\alpha_i}$ is continuous, $f|_{U_x \cap A_{\alpha_i}}$ is continuous. Note that $U_x \cap A_{\alpha_i}$ is closed in U_x for each α_i hence by part a we know that $f|_{U_x}$ is continuous. Then as $X = \bigcup U_x$ we are done by Theorem 18.2.
10.
11. Suppose $F : X \times Y \to Z$ is continuous. Let $h_{y_0} : X \to Y$ be the map given by $h_{y_0}(x) = x \times y_0$. Let $\iota_{y_0} : X \to X \times Y$ given by $\iota_{y_0}(x) = x \times y_0$ is continuous as shown previously. Then $h_{y_0} = F \circ \iota_{y_0}$ hence is continuous.
12.
13. Let $g : \overline{A} \to Y, h : \overline{A} \to Y$ be continuous maps such that $g(x) = h(x) = f(x)$ for all $x \in A$. We will show that $g(x) = h(x)$ for all $x \in \overline{A} - A$ as well.

    Suppose this is not the case. Let $x \in \overline{A} - A$ be such that $g(x) \neq h(x)$. Note that this in particular means that $x \in A'$. As Y is Hausdorff, we can find disjoint sets U and V containing $g(x)$ and $h(x)$ respectively. Then $g^{-1}(U) \cap h^{-1}(V)$ is a neighbourhood of x hence contains a point from A as x is a limit point. Let us denote this point as y. However, since g and h agree on A we must have that $g(y) = h(y)$ which contradict disjointness of U and V.

1.4.4 Product Topology

1. Let $(x_\alpha)_{\alpha \in J} \in \prod X_\alpha$. Then for each α there exists a $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha$. Thus $(x_\alpha)_{\alpha \in J} \in \prod B_\alpha$. Let $(x_\alpha)_{\alpha \in J} \in \prod B_\alpha \cap \prod B'_\alpha = \prod (B_\alpha \cap B'_\alpha)$. For each $B_\alpha \cap B'_\alpha$ we can find a C_α contained in the intersection that contains x_α. Then $(x_\alpha)_{\alpha \in J} \in \prod C_\alpha \subset \prod (B_\alpha \cap B'_\alpha)$.
2.
3.
4.
5.
6. Let $x_1, x_2, \ldots$ be a sequence in $\prod X_\alpha$ that converges to x. We will first show that for each α, we have $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converging to $\pi_\alpha(x)$ in both the product and box topology.

   Let U_α be a neighbourhood of $\pi_\alpha(x)$. Then $\pi_\alpha^{-1}(U_\alpha)$ is a neighbourhood of x (in both the product and box topology). Then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in \pi_\alpha^{-1}(U_\alpha)$. Thus for all $n \geq N$, $\pi_\alpha(x_n) \in U_\alpha$.

   Now let $x_1, x_2, \ldots$ be a sequence in $\prod X_\alpha$ such that for each α we have $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converging to some x_α. We will show that $x_1, x_2, \ldots$ converges to $x = (x_\alpha)$ in the product topology.

   Let $\prod U_\alpha$ be a basis element of the product topology containing x. Then there are $\alpha_1, \ldots, \alpha_k$

such that $U_\alpha = X_\alpha$ if $\alpha \notin \{\alpha_1, \dots, \alpha_k\}$. Each U_{α_i} is a neighbourhood of x_{α_i}, hence there exists some N_i such that for all $n \geq N_i$, we have $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$. Let $N = \max\{N_i\}$. Then for all $n \geq N$, we have $\pi_\alpha(x_n) \in U_\alpha$ for all α. Therefore for all such n, we get $x_n \in \prod U_\alpha$.
This unfortunately does not hold in the box topology. Consider the sequence

$$
\begin{aligned}
x_1 &= (1,1,1,1,\dots)\\
x_2 &= (0,2,2,2,\dots)\\
x_3 &= (0,0,3,3,\dots)\\
&\vdots
\end{aligned}
$$

Then each $\pi_i(x_n)$ converges to 0, however the open neighbourhood $\prod_{i\in\mathbb{N}}(-1,1)$ of 0 contains no elements of the sequence.

7. In the product topology, we claim $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$. Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\omega$ be arbitrary. Let $\prod U_n$ be some basis element of the product topology containing $(x_n)_{n\in\mathbb{N}}$. Recall that there must some $N \in \mathbb{N}$ such that $U_n = \mathbb{R}$ for all $n \geq N$. Additionally $x_n \in U_n$ for all $n \in \mathbb{N}$. Therefore $(x_1, \dots, x_N, 0, 0, \dots) \in \mathbb{R}^\infty \cap \prod U_n$.
In the box topology, we claim that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. We will show that $\mathbb{R}^\omega - \mathbb{R}^\infty$ is open.
Note that $\mathbb{R}^\omega - \mathbb{R}^\infty = \{(x_n \in \mathbb{R}^\omega) : \forall M \in \mathbb{N} \exists N \in \mathbb{N} x_N \neq 0\}$. Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^\omega - \mathbb{R}^\infty$. Then $(x_n)_{n\in\mathbb{N}} \in \prod U_i$ where $U_i = \mathbb{R}$ if $x_i = 0$ and $U_i = \mathbb{R} - \{0\}$ if $x_i \neq 0$.
8. It is clear that h is bijective with

$$h^{-1}((x_1, x_2, \dots)) = \left(\frac{1}{a_1}(x_1 - b_1), \frac{1}{a_2}(x_2 - b_2), \dots\right)$$

In order to see that h is continuous, we only need show that each h_i. Let $f_i : \mathbb{R} \to \mathbb{R}, f_i(x) = a_i x + b_i$. Then $h_i = f_i \circ \pi_i$. As the composition of two functions is continuous, h_i is continuous allowing us to conclude that h is continuous in the product topology. We can similarly conclude that h^{-1} is continuous proving that h is a homeomorphism.
9.
10. (a) For each α, there is a topology $\mathcal{T}_\alpha$ that makes f_α continuous (consider the topology generated by taking as subbasis the preimages of open sets). We have shown previously that given any family of topologies $\{\mathcal{T}_\alpha\}$ there is a unique smallest/coarsest topology that contains all $\mathcal{T}_\alpha$.
    (b) It is clear that $\mathcal{S}$ is a subbasis since it contains A itself. It is also clear that in the topology generated by $\mathcal{S}$, each f_α is continuous. We also shown previously that the topology generated by a subbasis is the coarsest topology containing the subbasis. Thus the topology generated by $\mathcal{S}$ is the coarsest topology relative to which each f_α is continuous. Thus the topology generated by $\mathcal{S}$ must be $\mathcal{T}$ by the uniqueness shown above.
    (c) Suppose $g : Y \to A$ is continuous. Then $f_\alpha \circ g$ is continuous for each α as the composition of continuous functions is continuous.
    Now suppose that $f_\alpha \circ g$ is continuous for each α. We only need show that the preimages of the subbasis of A under g is open.
    Let U_β be open X_β for some β. Then $f_\beta^{-1(U_\beta)} \in \mathcal{S}$. So $g^{-1}(f^{-1}(U_\beta)) = (f_\beta \circ g)^{-1}(U_\beta)$. By assumption $f_\beta \circ g$ is continuous so $(f_\beta \circ g)^{-1}(U_\beta)$ is open as desired.
    (d) We only need show that f maps basis elements to open sets since

$$f\left(\bigcup A_\alpha\right) = \bigcup f(A_\alpha)$$

Since we have a subbasis, we know that the basis elements are going to be finite intersections of these elements. Thus our basis elements are of the form

$$\bigcap_{i=1}^{n} f_{\beta_i}^{-1}(U_{\beta_i})$$

for some $\{\beta_1,\dots,\beta_n\} \subset J$ where U_{β_i} is open in X_{β_i}.
As an aside, note in principle, our basis could also consist of some more sets. For example if $U_1,\dots,U_n$ are open sets in some particular X_β, then $f_\beta^{-1}(U_1)\cap\cdots\cap f_\beta^{-1}(U_n)$ is also in our basis. However $f_\beta^{-1}(U_1)\cap\cdots\cap f_\beta^{-1}(U_n) = f_\beta^{-1}(U_1\cap\cdots\cap U_n)$, where $U_1\cap\cdots\cap U_n$ is open by properties of open sets. Thus without loss of generality, we can assume that we only have (at most) one open set from each space.
Given a basis element, we will show there exists an open set U in $\prod X_\alpha$ that contains the image of this basis element.
Let $\bigcap_{i=1}^{n} f_{\beta_i}^{-1}(U_{\beta_i})$ be a basis element and let a be an arbitrary element of this basic set. Then $f(a) \in \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})$. This holds for all a in the given basic set hence we are done.

Suppose $(x_k)_{k\in\mathbb{N}}$ converges to $y \in \mathbb{R}^\omega$ with the box topology. Consider the neighbourhoods y of the form

$$U_\varepsilon = \prod(y_n-\varepsilon, y_n+\varepsilon)$$

We know there exists some $N\in\mathbb{N}$ such that for all $k \geq N$, we have $x_k \in U_\varepsilon$.

1.4.5 Metric Topology

1.
2. We define the following metric

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} 1 \text{ if } x_1 \neq x_2 \\ |y_1-y_2| \text{ if } x_1 = x_2 \end{cases}$$

We recall from an earlier exercise that the dictionary order topology on $\mathbb{R}$ is the same as the product topology on $\mathbb{R}^d\times\mathbb{R}$ where $\mathbb{R}_d$ refers to $\mathbb{R}$ with the discrete topology and $\mathbb{R}$ refers to $\mathbb{R}$ with the standard topology. Thus we simply need show that the topology generated by the above metric is the same as the product topology on $\mathbb{R}_d\times\mathbb{R}$.
A basis element in $\mathbb{R}^d\times\mathbb{R}$ is of the form $\{x\}\times(a,b)$. Then given any $(x,y)\in\{x\}\times(a,b)$, we take $\delta = \min\{1, y-a, b-y\}$ and note that $B_d((x,y),\delta)\subset\{x\}\times(a,b)$.
Conversely, let $B_d((x,y),\delta)$ be given. If $\delta > 1$ then our ball is the whole space, so we are done. So suppose $\delta \leq 1$. Then $B_d((x,y),\delta) = \{x\}\times(y-\delta, y+\delta)$ which is a basis element itself in $\mathbb{R}_d\times\mathbb{R}$. Hence the spaces are equal.

1.4.6 The Fundamental Group

1. (a) A star shape where we take a_0 to be the center
   (b) Since every line connecting a_0 to a point lies in A we can construct the straight-line homotopy to contract all paths to the constant loop at a_0

2. Suppose $\gamma = \alpha * \beta$. Then $\widehat{\gamma}([f]) = [\overline{\gamma}] * [f] * [\gamma]$ Then

$$\begin{aligned}[\overline{\gamma}] * [f] * [\gamma] &= [\overline{\gamma} * f * \gamma] \\ &= [\overline{\beta} * \overline{\alpha} * f * \alpha * \beta] \\ &= [\overline{\beta}] * [\overline{\alpha} * f * \alpha] * [\beta] \\ &= \widehat{\beta}([\overline{\alpha} * f * \alpha]) \\ &= \widehat{\beta}(\widehat{\alpha}([f]))\end{aligned}$$

3. Suppose $\pi_1(X, x_0)$ is abelian. Let α, β be any two paths from x_0 to x_1. Then we see that $[\beta * \overline{\alpha}]$ is in $\pi_1(X, x_0)$. Let $[f] \in \pi_1(X, x_0)$ be arbitrary. Then we know that

$$[\beta * \overline{\alpha}] * [f] = [f] * [\beta * \overline{\alpha}]$$

which implies that

$$[\overline{\alpha}] * [f] * [\alpha] = [\overline{\beta}] * [f] * [\beta]$$

as desired.
Now suppose that $\widehat{\alpha} = \widehat{\beta}$ for all paths α, β from x_0 to x_1. This implies that

$$[\beta * \overline{\alpha}] * [f] = [f] * [\beta * \overline{\alpha}]$$

Let g be any loop based on x_0 and take $\beta = [g * \alpha]$. This gives us that $[g] * [f] = [f] * [g]$ as desired.